The Neumann semigroup via exhaustion

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Introduction

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But an equivalent question can be asked about Neumann boundary conditions. Namely, what happens with the semigroup when restricting to compact subspaces and imposing Neumann boundary conditions? Does the semigroup converge?

In this talk, we will review the basic setup and some known results and then discuss what happens in the Neumann case. We will see that there are connections to both stochastic completeness and form uniqueness.

Introduction, continued

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This talk is based on ongoing work with Matthias Keller (University of Potsdam) and Florentin Münch (Max Planck Institute for Mathematics in the Sciences Leipzig).

Graphs

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We let $C(X) = \{f\colon X \longrightarrow \mathbb{R}\}$ be the space of real-valued functions on X, $C_c(X) = \{f \in C(X) \mid f \text{ has finite support}\}$ and let

$$\ell^p(X, m) = \{f \in C(X) \mid \sum_{x \in X} |f(x)|^p m(x) < \infty\}$$

for $p \in [1, \infty)$ with norm $\|f\|_p = \sum_{x \in X} |f(x)|^p m(x)$. When $p = 2$, we will write $\|f\|$ for $\|f\|_2$. We will write $\|f\|_\infty = \sup_{x \in X} |f(x)|$ for $f \in \ell^\infty(X)$, the space of bounded functions.

Forms associated to graphs

We let

$$\mathcal{D} = \{f \in C(X) \mid \sum_{x,y \in X} b(x,y)(f(x)-f(y))^2 + \sum_{x \in X} c(x) f^2(x) < \infty\}$$

and for $f, g \in \mathcal{D}$, we let

$$\mathcal{Q}(f,g) = \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x \in X} c(x) f(x) g(x).$$

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We let $Q^{(D)}$ be the restriction of $\mathcal{Q}$ to $D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$ where $\|f\|_{\mathcal{Q}} = \left(\|f\|^2 + \mathcal{Q}(f)\right)^{1/2}$ with associated operator $L = L^{(D)}$. We will refer to this operator as the Laplacian or the Dirichlet Laplacian. In some sense, this is the minimal operator associated to a graph.

We now go outside of the course and let $Q^{(N)}$ be the restriction of $\mathcal{Q}$ to $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$ with associated operator $L^{(N)}$. We will refer to $Q^{(N)}$ as the *Neumann form* and $L^{(N)}$ as the *Neumann Laplacian*.

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In general, it is known that $Q^{(D)}$ and $Q^{(N)}$ are not equivalent. If $Q^{(D)} = Q^{(N)}$, we say that the graph satisfies *form uniqueness*. We refer to Theorem 3.2 in the book for some general characterizations for this property. In particular, in this case all α-harmonic functions in $D(Q^{(N)})$ for $\alpha > 0$ are trivial. We will return to this later.

Laplacians

We now let $\mathcal{F} = \{f \in C(X) \mid \sum_{y \in X} b(x,y)|f(y)| < \infty\}$ and for $f \in \mathcal{F}$ and $x \in X$ let

$$\mathcal{L}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x)$$

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denote the *formal Laplacian* associated to the graph. As discussed in the course, $\mathcal{D} \subseteq \mathcal{F}$ and for $f \in \mathcal{D}$ and $\varphi \in C_c(X)$ we have the following Green's formula:

$$\mathcal{Q}(f,\varphi) = \sum_{x \in X} \mathcal{L}f(x)\varphi(x)m(x) = \sum_{x \in X} f(x)\mathcal{L}\varphi(x)m(x).$$

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This implies, in particular, that both $L^{(D)}$ and $L^{(N)}$ are restrictions of $\mathcal{L}$.

Recurrence and stochastic completeness

We now focus on the case when $c = 0$, i.e., when there is no killing term, and let $L = L^{(D)}$ with semigroup e^{-tL} for $t \geq 0$.

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We say that the graph is *recurrent* if

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We say that the graph is *stochastically complete* if

$$\sum_{y \in X} e^{-tL} 1_y(x) = 1$$

for some (equivalently, all) $x \in X$ and $t \geq 0$. Otherwise, the graph is called *stochastically incomplete*.

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We call a function $v \in \mathcal{F}$ *α-harmonic* if v is α-sub and α-super harmonic. If $\alpha = 0$, then we call v (sub/super) harmonic.

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2. Stochastic completeness $\Longleftrightarrow$ Every α-harmonic function in $\ell^\infty(X)$ for $\alpha > 0$ is trivial.
3. Form uniqueness $\Longleftrightarrow$ Every α-harmonic function in $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$ for $\alpha > 0$ is trivial.

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The characterization for recurrence is part of Theorems 6.1 in the book (Theorem 9.7 in the lecture notes), the one for stochastic completeness is Theorem 7.2 in the book (Theorem 10.22 in the lecture notes) and the characterization for form uniqueness can be found in Theorem 3.2 of the book.

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We first recall the Dirichlet restrictions to the finite subsets X_k. That is, if $i_k \colon C(X_k) \hookrightarrow C(X)$ denotes extension by 0, then we let $Q_k^{(D)}$ be a form on $\ell^2(X_k, m)$ via

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The corresponding operator is denoted by $L_k^{(D)}$ and satisfies

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Dirichlet and Neumann restrictions

In particular, for any vertex $x \in X_k$ we have

$$L_k^{(D)} f(x) = \frac{1}{m(x)} \sum_{y \in X_k} b(x,y)(f(x) - f(y)) + \frac{f(x)}{m(x)} \left(\sum_{y \notin X_k} b(x,y) + c(x) \right).$$

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Thus, the Dirichlet restriction sees both inside of X_k and how much is going outside of X_k.

If we take away the connection to the outside, then we get a Laplacian on just a finite graph. That is, we can restrict b to $X_k \times X_k$; and c and m to X_k to get a finite graph and then consider the graph Laplacian which is denoted by $L_k^{(N)}$, i.e.,

$$L_k^{(N)} f(x) = \frac{1}{m(x)} \sum_{y \in X_k} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).$$

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If we want to view $L_k^{(N)}$ in terms of the formal Laplacian we note that

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We will refer to the semigroups $e^{-tL_k^{(D)}}$ and $e^{-tL_k^{(N)}}$ acting on $\ell^2(X_k, m)$ as the *restricted Dirichlet and Neumann semigroups*.

A first difference: stochastic completeness

We now assume that $c = 0$ and that X is infinite and connected. In this case, the semigroup associated to $L_k^{(N)}$ has no killing term and is thus stochastically complete, i.e., $e^{-tL_k^{(N)}}1 = 1$; however, as $L_k^{(D)}$ has a killing term (given by the edges leaving X_k), the semigroup associated to $L_k^{(D)}$ is stochastically incomplete, i.e., $e^{-tL_k^{(D)}}1 < 1$.

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This follows rather directly as $L_k^{(N)}1 = 0$ while $L_k^{(D)}1 \neq 0$ so that $\lambda_0(L_k^{(D)}) > 0$ and thus the semigroup associated to $L_k^{(D)}$ decays exponentially, see Theorem 0.65 in the book for more details.

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In general, for $f \geq 0$, we have that

$$e^{-tL_k^{(D)}}f \leq e^{-tL_k^{(N)}}f.$$

Properties of the Dirichlet exhaustion

By using the minimum principle, it is not hard to show that

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for any $f \geq 0$ and $k \in \mathbb{N}$.

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Namely, if $L = L^{(D)}$, then

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- If $f \geq 0$, then $e^{-tL} f$ is the minimal positive solution to the heat equation with initial conditions given by f.
- If $f \geq 0$, then $(L + \alpha)^{-1} f$ is the minimal positive solution to the equation $(\mathcal{L} + \alpha)u = 0$.

Properties of the Dirichlet exhaustion

As mentioned in the introduction, the domain monotonicity in the case of Riemannian manifolds was first utilized to establish the existence of solutions of the heat equation on Riemannian manifolds by Dodziuk in 1983.

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For the graph case, we refer to the work of Keller/Lenz from 2012 as well as Proposition 1.20 and Lemmas 1.21, 1.23, and 1.24 in the book (and Lemmas 5.2, 5.3, 5.8, 5.9 in the lecture notes).

Properties of the Neumann exhaustion

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A second question is when (and in what sense) do the restricted Neumann semigroups converge to the Dirichlet semigroup. We will see that this ties in with form uniqueness and stochastic completeness.

Convergence of Neumann semigroups in $\ell^2(X, m)$

Theorem

Let $f \in \ell^2(X, m)$. Then,

$$e^{-tL_k^{(N)}} f \to e^{-tL^{(N)}} f$$

as $k \to \infty$ in $\ell^2(X, m)$.

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As for the Dirichlet case, one can pass to the resolvent and use the characterization of the resolvent as the minimizer of the energy. More specifically, for $\alpha > 0$, we let $R_{\alpha,k}^{(N)} = (L_k^{(N)} + \alpha)^{-1}$ denote the resolvent and recall that $u = R_{\alpha,k}^{(N)} f$ is the unique minimizer of the functional

$$v \mapsto Q_k^{(N)}(v) + \alpha \| v - \frac{1}{\alpha} f \|^2$$

Sketch of proof continued

where

$$Q_k^{(N)}(v) = \frac{1}{2} \sum_{x,y \in X_k} b(x,y)(v(x) - v(y))^2 + \sum_{x \in X_k} c(x) v(x)^2$$

is the Neumann form which we think of as acting on $D(Q^{(N)})$.

Sketch of proof continued

where

$$Q_k^{(N)}(v) = \frac{1}{2} \sum_{x,y \in X_k} b(x,y)(v(x) - v(y))^2 + \sum_{x \in X_k} c(x)v(x)^2$$

is the Neumann form which we think of as acting on $D(Q^{(N)})$. We note that if $u = R_{\alpha,k}^{(N)} f = (L_k^{(N)} + \alpha)^{-1} f$, then $L_k^{(N)} u = f - \alpha u$ which leads to the following calculation:

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$$Q_k^{(N)}(u) + \alpha \| u - \frac{1}{\alpha} f \|^2 = \left\langle L_k^{(N)} u, u \right\rangle + \alpha \left\langle u - \frac{1}{\alpha} f, u - \frac{1}{\alpha} f \right\rangle$$

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$$\begin{aligned} Q_k^{(N)}(u) + \alpha\|u - \frac{1}{\alpha}f\|^2 &= \left\langle L_k^{(N)} u, u \right\rangle + \alpha \left\langle u - \frac{1}{\alpha}f, u - \frac{1}{\alpha}f \right\rangle \\ &= \langle f - \alpha u, u\rangle - \left\langle f - \alpha u, u - \frac{1}{\alpha}f \right\rangle \\ &= \left\langle f - \alpha u, \frac{1}{\alpha}f \right\rangle = \frac{1}{\alpha}\|f\|^2 - \left\langle R_{\alpha,k}^{(N)} f, f \right\rangle. \end{aligned}$$

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A further argument using the characterization of the resolvent $R_\alpha^{(N)} = (L^{(N)} + \alpha)^{-1}$ given above and Fatou's lemma establishes that $R_\alpha^{(N)} f(x) = \lim_{k \to \infty} R_{\alpha,k}^{(N)} f(x)$ for all $f \in D(Q^{(N)})$ and all $x \in X$ and a further argument gives convergence in $\ell^2(X, m)$.

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We remark that the result above can also be obtain by using Mosco convergence.

Connection to form uniqueness

Given the above, we know that the Neumann restrictions always converge to the Neumann semigroup and the Dirichlet restrictions to the Dirichlet semigroup. What is the connection between the two?

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Theorem

Let $\varphi \in C_c(X)$ with $\varphi \geq 0$ and $\varphi \neq 0$. Let $L = L^{(D)}$. The following statements are equivalent:

(i) $Q^{(D)} = Q^{(N)}$.

(ii) $e^{-tL_k^{(N)}}\varphi \to e^{-tL}\varphi$ *as* $k \to \infty$ *in* $\ell^2(X, m)$.

(iii) $e^{-tL_k^{(N)}}\varphi(x) \to e^{-tL}\varphi(x)$ *as* $k \to \infty$ *for every* $x \in X$.

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Proof.

That (i) implies (ii) follows from the previous result. That (ii) implies (iii) is obvious. That (iii) implies (i) follows from positivity improving properties of semigroups. □

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Theorem

Let $\varphi \in C_c(X)$ with $\varphi \geq 0$ and $\varphi \neq 0$. Let $L = L^{(D)}$. The following statements are equivalent:

(i) *The graph is stochastically complete.*

(ii) $e^{-tL_k^{(N)}}\varphi \to e^{-tL}\varphi$ *as* $k \to \infty$ *in* $\ell^1(X, m)$.

Sketch of proof

Proof.

To show that (i) implies (ii) we note that

$$\begin{aligned}\|e^{-tL}\varphi - e^{-tL_k^{(N)}}\varphi\|_1 \leq \|e^{-tL}\varphi - e^{-tL_k^{(D)}}\varphi\|_1 \\ + \|e^{-tL_k^{(N)}}\varphi - e^{-tL_k^{(D)}}\varphi\|_1\end{aligned}$$

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&\qquad + \|e^{-tL_k^{(N)}}\varphi - e^{-tL_k^{(D)}}\varphi\|_1 \\
&= \|e^{-tL}\varphi\|_1 - \|e^{-tL_k^{(D)}}\varphi\|_1 + \|e^{-tL_k^{(N)}}\varphi\|_1 - \|e^{-tL_k^{(D)}}\varphi\|_1 \\
&= 2\left(\|\varphi\|_1 - \|e^{-tL_k^{(D)}}\varphi\|_1\right)
\end{aligned}$$

which can be made arbitrarily small due to stochastic completeness.

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which can be made arbitrarily small due to stochastic completeness.

Conversely, if the graph is stochastically incomplete, then

$$\|e^{-tL}\varphi - e^{-tL_k^{(N)}}\varphi\|_1 \geq \|e^{-tL}\varphi\|_1 - \|e^{-tL_k^{(N)}}\varphi\|_1 = \|e^{-tL}\varphi\|_1 - \|\varphi\|_1$$

which is bounded from below due to stochastic incompleteness. □

The Feller property

We now introduce another property which has not been mentioned in the book or the course.

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We say that a semigroup is *Feller* or C_0*-conservative* if the semigroup maps $C_c(X)$ to $C_0(X)$.

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As for exhaustion, this property has been previously studied for the Dirichlet semigroup. As it is sufficient to consider positive functions in the above and $e^{-tL^{(N)}} \geq e^{-tL}$ for such functions, it follows that if the Neumann semigroup is Feller, then so is the Dirichlet semigroup. However, we can say even more.

The Feller property and form uniqueness

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The following statements are equivalent:

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Clearly (ii) implies (i).

Conversely, if (i) holds, then clearly e^{-tL} is Feller where $L = L^{(D)}$. Now, suppose that $Q^{(D)} \neq Q^{(N)}$ so that $e^{-tL} \neq e^{-tL^{(N)}}$. Let $x \in X$ and let $u_t = \left(e^{-tL^{(N)}} - e^{-tL}\right) 1_x$.

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Now, as $e^{-tL^{(N)}}$ is Feller, there exists a finite set K containing x such that $e^{-TL^{(N)}}1_x < \varepsilon$ on

$$\partial W = \{y \in W \mid b(y,z) > 0 \text{ for some } z \notin W\}.$$

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which gives a contradiction. Thus, $Q^{(D)} = Q^{(N)}$. $\square$

The Feller property and α-harmonic functions

We can also approach the Feller property for $e^{-tL^{(N)}}$ in terms of α-harmonic functions.

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If all positive α-superharmonic functions in $\ell^p(X, m)$ for some $p \in [1, \infty)$ are trivial, then $e^{-tL^{(N)}}$ is Feller. If the graph is locally finite, then one can consider α-harmonic functions.

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Proof.

If $e^{-tL^{(N)}}$ is not Feller, then either $Q^{(D)} \neq Q^{(N)}$ or e^{-tL} is not Feller by the previous result.

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Proof.

If $e^{-tL^{(N)}}$ is not Feller, then either $Q^{(D)} \neq Q^{(N)}$ or e^{-tL} is not Feller by the previous result. If $Q^{(D)} \neq Q^{(N)}$, then there exists a non-trivial α-harmonic function u in any $\ell^p(X, m)$ space by general theory (e.g. Theorem 3.2 in the book).

The Feller property and α-harmonic functions, continued

If e^{-tL} is not Feller, then there exists a vertex $x \in X$ and a sequence $x_n \in X$ going to infinity such that $(L+\alpha)^{-1}1_{x_n}(x) > \varepsilon$ for all $n \in \mathbb{N}$.

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If e^{-tL} is not Feller, then there exists a vertex $x \in X$ and a sequence $x_n \in X$ going to infinity such that $(L+\alpha)^{-1}1_{x_n}(x) > \varepsilon$ for all $n \in \mathbb{N}$. As the resolvent extends to all $\ell^p(X,m)$ one can then create a sequence of normalized α-superharmonic functions which converge to a positive superharmonic function, see Corollary 4.5 in the book. By Fatou's lemma, the function is in $\ell^p(X,m)$ for $p \in [1,\infty)$.

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We remark that the implication in the preceding does not reverse in general as can be seen from examples.

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We remark that the implication in the preceding does not reverse in general as can be seen from examples. However, we will now show that it does in the case of birth-death chains, i.e., graphs over $X = \mathbb{N}$ with subsequent natural numbers connected and α-harmonic functions in $\ell^1(X,m)$.

Birth-death chains

We call a graph a *birth-death chain* if $X = \mathbb{N}$ and $b(x, y) > 0$ if and only if $|x - y| = 1$ and $c = 0$.

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(iii) *Either* $m(X) = \infty$ *or*

$$\sum_{r=0}^{\infty} \frac{1}{b(r, r+1)} = \infty = \sum_{r=0}^{\infty} \frac{m(B_r^c)}{b(r, r+1)}.$$

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On a birth-death chain, the following statements are equivalent:

(i) $e^{-tL^{(N)}}$ *is Feller.*

(ii) *All positive* α*-superharmonic functions in* $\ell^1(X,m)$ *are trivial.*

(iii) *Either* $m(X) = \infty$ *or*

$$\sum_{r=0}^{\infty} \frac{1}{b(r,r+1)} = \infty = \sum_{r=0}^{\infty} \frac{m(B_r^c)}{b(r,r+1)}.$$

Proof.

That (ii) implies (i) follows from the previous theorem.

The equivalence between (i) and (iii) follows from known characterizations of form uniqueness and the Feller property for e^{-tL}.

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Birth-death chains, continued

The equivalence between (i) and (iii) follows from known characterizations of form uniqueness and the Feller property for e^{-tL}. That (iii) implies (ii) can be shown by analyzing an α-harmonic function in this case and using some estimates. $\square$

It would be nice to extend the result above to weakly spherically symmetric graphs. Given the above, it would suffice to characterize form uniqueness since a characterization of the Feller property for the Dirichlet semigroup is known in this case.

A comment on maximality

We recall that if $f \geq 0$, then $e^{-tL} f$ is the minimal positive solution to the heat equation with initial condition f and $(L+\alpha)^{-1} f$ is the minimal positive solution to the Poisson equation.

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Thanks

Thank you to the organizers for the invitation and thank you for your attention!