Project N: Infinite trees over non-archimedean ordered fields

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Mathematics of the Levi-Civita field and eigenvalue problems

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Project N

Contents

Historical aspects

- Discovered by Levi-Civita in 1892 and 1898. He proved the Cauchy completness and introduced the order.
- Rediscovered by Ostrowski in 1935, Neder 1943 and Laugwitz 1968.
- Modern approaches can be found in the book by Lightstone and Robinson 1975
- Berz and Shamseddine extend previous work and formulate a workable analysis in the 90'ies.

Left finite sets and the Levi-Civita field

Definition

1. $Q \subset \mathbb{Q}$ is *left-finite*. if for every $r \in \mathbb{Q}$ there are only finitely many $q \in Q$ such that $q \leq r$.

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w.l.o.g. $q_1 < q_2 < \dots$

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2. The set

$$\mathcal{R} = \left\{ x = \sum_{i=1}^{\infty} x_i \varepsilon^{q_i} \,\middle|\, x_i \in \mathbb{R},\ q_i \in Q \text{ with } Q \text{ left-finite} \right\},$$

is a real closed field called the *Levi-Civita field*. Additionally, $x\colon Q \to \mathbb{R}$ with $q_i \mapsto x_i$. The addition and multiplication is defined as for formal power series. $\mathbb{R}$ is embedded in $\mathcal{R}$ and contains the rational functions.

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Example

Consider $\frac{1}{3-4\varepsilon+\varepsilon^2}$ and calculate the correspondig element in $\mathcal{R}$!

- Find the element $x = \sum_{i=1}^{\infty} x_i \varepsilon^{q_i}$ such that $(3 - 4\varepsilon + \varepsilon^2)x = 1$
- Compare coefficients and obtain: $q_1 = 0$, $x_1 = 1/3$ and $q_2 = 1$, $x_2 = 4/9$.
- Obtain the recurrence relations: $q_i = q_{i-1} + 1$ and $3x_i - 4x_{i-1} + x_{i-2} = 0$ for $i > 2$.
- Solving the relations gives: $x_i = -\frac{1}{2\cdot 3^i} + \frac{1}{2}$

Consequently,

$$\frac{1}{3 - 4\varepsilon + \varepsilon^2} = \sum_{i=1}^{\infty} \left(-\frac{1}{2 \cdot 3^i} + \frac{1}{2} \right) \varepsilon^{i-1}.$$

An order on $\mathcal{R}$ and the infinitesimal ε

Definition/Theorem

1. Let $x \neq y$ be in $\mathcal{R}$ and $z = x - y$ with $z = \sum_{i=1}^{\infty} z_i \varepsilon^{q_i}$. We say $x > y$ if $z_1 > 0$.

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5. $\varepsilon^q \gg 1$ if $q < 0$.

Absolute value on $\mathcal{R}$

Definition

1. Let $x, y \in \mathcal{R}$. The absolute value $|\cdot| : \mathcal{R} \to \mathcal{R}$ is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & \text{else} \end{cases}$$

   and has the properties $|x| = 0$ iff $x = 0$; $|x \cdot y| = |x| \cdot |y|$; $|x + y| \leq |x| + |y|$.

2. A norm for $x = (x_1, \ldots, x_n) \in \mathcal{R}^n$ is given by $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$.

Strong convergence and Cauchy completness

Definition

Let (a_n) be a sequence in $\mathcal{R}$.

1. We say (a_n) *converges strongly* to $a \in \mathcal{R}$ iff for all $\delta > 0$, $\delta \in \mathcal{R}$ there is a $N \in \mathbb{N}$ such that $|a_n - a| < \delta$ for all $n > N$.

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Theorem

1. *Every strongly convergent sequence is regular.*
2. *$\mathcal{R}$ is Cauchy complete.*

A semi-norm on $\mathcal{R}$ and weak convergence

Definition

1. A semi-norm $\|\cdot\|_r : \mathcal{R} \to \mathbb{R}$ is defined by

$$\|x\|_r = \sup_{q \le r}\{|x[q]|\}.$$

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3. Strong convergence implies weak convergence.

Note that $\mathcal{R}$ with the weak convergence is not Cauchy complete. There is also an unbounded null sequence $a_n = \varepsilon^{-n}/n$.

Convergence criterion for weak convergence

For a better understanding of weak convergence, we have the following theorem:

Theorem

1. *Let the sequence (a_n) converge weakly to the limit $a \in \mathcal{R}$. Then $(a_n[q]) \longrightarrow a[q]$ for each q and convergence is uniform on every set which is bounded above.*

Recall: (a_n) is regular if $\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n)$ is left-finite

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2. *If* (a_n) *is regular and* $a_n[q] \longrightarrow a[q]$ *converges for each* q, *then the sequence* (a_n) *converges weakly.*

Recall: (a_n) is regular if $\bigcup_{n=0}^{\infty} \text{supp}(a_n)$ is left-finite

A graph over the Levi-Civita field

Definition

A *graph over* $\mathcal{R}$ is a couple (V, b) where V is the set of vertices and $b\colon V \times V \to \mathcal{R}$ satisfying

1. $b(x, y) \geq 0$ for any $x, y \in V$,

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Note that edges, paths, connectedness, completeness, subgraphs etc. are analogously defined as in the classical case.
We work with finite graphs in this talk.

Laplace operator

Definition

Define the set of functions $\mathfrak{F}_V = \{f \mid f\colon V \to \mathcal{R}\}$ with an analogue of inner product

$$\langle f, g\rangle = \sum_{x\in V} f(x)g(x)b(x).$$

where $b(x) = \sum_{y\in V} b(x,y)$.

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where $b(x) = \sum_{y \in V} b(x, y)$. The *Laplace operator* is defined as

$$\mathcal{L}f(x) = \sum_{y \in V} (f(x) - f(y))p(x, y).$$

for the *normalised weight* on edges $p(x, y) = \frac{b(x,y)}{b(x)}$ for any $x, y \in V$

Corresponding matrix

The corresponding matrix of the Laplace operator is given by

$$\begin{pmatrix} 1 & -p_{x_1x_2} & \dots & -p_{x_1x_n} \\ -p_{x_2x_1} & 1 & \dots & -p_{x_2x_n} \\ \vdots & & \ddots & \vdots \\ -p_{x_nx_1} & -p_{x_nx_2} & \dots & 1 \end{pmatrix}$$

where $p_{xy} = p(x,y)$ and $0 \leq p_{xy} \leq 1$.
The powers of ε are positive in power series.
For the eigenvalues hold: $0 \leq \lambda \leq 2$.

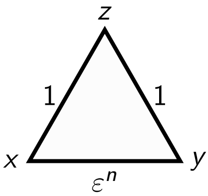
Probability Operator

We can define the operator $\mathcal{P} = I - \mathcal{L}$

$$\begin{pmatrix} 0 & p_{x_1x_2} & \dots & p_{x_1x_n} \\ p_{x_2x_1} & 0 & \dots & p_{x_2x_n} \\ \vdots & & \ddots & \vdots \\ p_{x_nx_1} & p_{x_nx_2} & \dots & 0 \end{pmatrix}.$$

As an interpretation, the matrix above contains the transition probabilities from one vertex to another.
The eigenvalues satisfy. $-1 \leq \lambda \leq 1$.

Example of a graph



The corresponding matrix of the Laplacian:

$$\begin{pmatrix} 1 & -\frac{\varepsilon^n}{1+\varepsilon^n} & -\frac{1}{1+\varepsilon^n} \\ -\frac{\varepsilon^n}{1+\varepsilon^n} & 1 & -\frac{1}{1+\varepsilon^n} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

The eigenvectors are $v_1^n = (-1, 1, 0)$, $v_2^n = \left(-\frac{1}{1+\varepsilon^n}, -\frac{1}{1+\varepsilon^n}, 1\right)$ and $v_3 = (1, 1, 1)$.

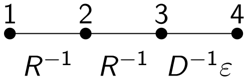
The corresponding eigenvalues are: $\lambda_1^n = \frac{1+2\varepsilon^n}{1+\varepsilon^n}$, $\lambda_2^n = \frac{2+\varepsilon^n}{1+\varepsilon^n}$ and $\lambda_3 = 0$. So

$\lambda_1^n = 1 + \varepsilon^n - \varepsilon^{2n} + \varepsilon^{3n} + o(\varepsilon^{3n})$ and

$\lambda_2^n = 2 - \varepsilon^n + \varepsilon^{2n} - \varepsilon^{3n} + o(\varepsilon^{3n})$.

Example of a graph

Let $D, R \in \mathbb{R}_{>0}$.



The corresponding matrix of the Probability operator is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{D}{D+R\varepsilon} & 0 & \frac{R\varepsilon}{D+R\varepsilon} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with the eigenvalues $\lambda_{1,2} = \pm 1$ and $\lambda_{3,4} = \pm\sqrt{\frac{R\varepsilon}{2D+2R\varepsilon}}$.

So $\lambda_{3,4} = \pm \frac{1}{\sqrt{2}} \left(\frac{R}{D}\right)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \mp \frac{1}{2\sqrt{2}} \left(\frac{R}{D}\right)^{\frac{3}{2}} \varepsilon^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}})$.

The von Mises algorithm (Power method)

Let $A \in \mathbb{M}_n(\mathcal{R})$ be a matrix. Let $\|\cdot\|$ be a norm on $\mathcal{R}^n$ and x_0 be a starting vector such that $\|x_0\| = 1$. Then the sequence z^k given by

$$z^{k+1} := A\tilde{x}^k, \quad \tilde{x}^{k+1} := \frac{z^{k+1}}{\|z^{k+1}\|}$$

converges to the eigenvector corresponding to the largest eigenvalue in absolute value.

Graphs and von Mises

We proved von Mises for the case where all eigenvalues are infinitesimal despite the largest eigenvalue in absolute value. We can use strong convergence in this case.
For the other cases, where the largest eigenvalue differs not only by infinitesimals, we have proof ideas by using weak convergence.

Outlook

- Prove von Mises with our ideas regarding weak convergence.
- Calculate the Cheeger constant numerically.
- Develop further concepts for infinite graphs (requires non-standard analysis).

References

- Khodr Shamseddine, Martin Berz. *Analysis on the Levi-Civita Field: A Brief Overview*, Contemporary Mathematics, 508 pp 215-237 (2010)
- Martin Berz, Christian Bischof, George Corliss, Andreas Griewank, Andreas. *Computational Differentiation: Techniques, Applications, and Tools*. Chapter 2: Calculus and Numerics on Levi-Civita Fields. eds., SIAM, 1996.
- Anna Muranova. *Effective impedance over ordered fields*. Journal of mathematical physics 62, 033502(2021)
- Anna Muranova *Laplace and probability operators on graphs over ordered fields*, preprint
- ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

ε^{-n} thanks for some $n \in \mathbb{N}$!