Project M – Uniform transience of spherically symmetric graphs

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Introduction and outline

The phenomenon of transience was introduced in the course (Lecture 9, Chapter 9). This phenomenon is studied for $c = 0$ since, otherwise, some of the conditions for transience are always true. Furthermore, transience was characterized for weakly spherically symmetric graphs in Lecture 13.

The goal of this talk is to recall the relevant material from the course and then strengthen some of the equivalent characterizations of transience to give a notion of uniform transience. We then present a criterion for uniform transience of weakly spherically symmetric graphs and clarify the role of c for uniform transience.

Definitions

Definition ($C_c(X)$)

The set $C_c(X)$ denotes all functions $f \in C(X)$ with finite support.

Definition ($\mathcal{D}_0$)

The space $\mathcal{D}_0$ is defined as the set of $f \in \mathcal{D}$ for which there exists a sequence (φ_n) in $C_c(X)$ such that $\varphi_n \to f$ pointwise and $Q(f - \varphi_n) \overset{n\to\infty}{\longrightarrow} 0$.

Definition ($C_0(X)$)

The set $C_0(X) = \overline{C_c(X)}^{\|\cdot\|_\infty}$ denotes the functions vanishing at infinity.

Definitions

Definition (Green's function, transience)

Let b be a connected graph over (X, m). The function

$$G : X \times X \to [0, \infty], (x, y) \mapsto \int_0^\infty e^{-tL} 1_y(x) dt$$

is called the *Green's function*. A graph is called *transient* if $G(x, y) < \infty$ for one (all) $x, y \in X$. Otherwise, the graph is called *recurrent*.

Definition (Capacity)

The *capacity* at $x \in X$ is defined as

$$\text{cap}(x) := \inf\{Q(\varphi) \mid \varphi \in C_c(X),\ \varphi(x) = 1\}.$$

Characterisation of transience

Theorem (Lecture 10, Theorem 9.7/Theorem 6.1 in the book)

Let b be a connected graph over X. The following statements are equivalent:

(i) $1 \notin \mathcal{D}_0$.

(ii) *There exists a $w \geq 0, w \neq 0$ s.t.*

$$Q(\varphi) \geq \sum_{x \in X} w(x)\varphi^2(x)$$

holds for all $\varphi \in C_c(X)$.

(iii) $G(x, y) < \infty$ *for all* $x, y \in X$.

(iv) $\operatorname{cap}(x) > 0$ *for some (all)* $x \in X$.

The intuition behind transience

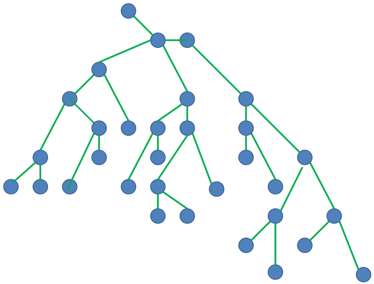


Figure: Finite graphs are recurrent.

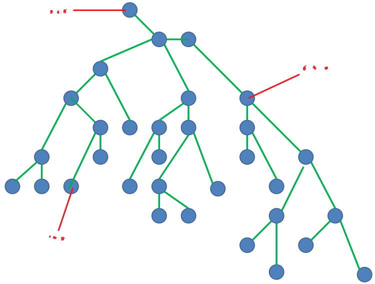


Figure: Infinite graph

Characterisation of uniform transience

Theorem (Keller/Lenz/Schmidt/Wojciechowski and Keller/Pinchover/Pogorzelski)

Let (b, c) be a connected graph over (X, m). The following statements are equivalent:

(i) $\mathcal{D}_0 \subseteq C_0(X)$.

(ii) *There exists $C \geq 0$ with $Q(\varphi) \geq C\|\varphi\|_\infty^2$ for all $\varphi \in C_c(X)$.*

(iii) $D(Q_m^{(D)}) \subseteq C_0(X)$ *holds for any measure on X with full support.*

(iv) $\inf_{x \in X} \operatorname{cap}(x) > 0$.

The previous theorem was shown in the context of graphs by Matthias Keller, Daniel Lenz, Marcel Schmidt and Radosław Wojciechowski in Note on uniformly transient graphs and in the more general context of Schrödinger operators on graphs by Matthias Keller, Yehuda Pinchover and Felix Pogorzelski in Criticality theory for Schrödinger operators on graphs.

A graph satisfying one of those equivalent conditions is called *uniformly transient*.

Sketch of proof (i) $\Rightarrow$ (ii)

(i) $\Rightarrow$ (ii)

$\mathcal{D}_0 \subseteq C_0(X)$
$\Rightarrow$ There exists $C \geq 0$ with $\|\varphi\|_\infty \leq CQ^{\frac{1}{2}}(\varphi)$ for all $\varphi \in C_c(X)$.

By the closed graph theorem, there exists a $C_1 \geq 0$, s.t.

$$\|f\|_\infty \leq C_1 \|f\|_o \overset{\text{to show}}{\leq} C_1 C_2 Q^{\frac{1}{2}}(\varphi).$$

$$\begin{aligned} \text{Assume: } & \|\varphi_n\|_o > nQ^{\frac{1}{2}}(\varphi_n),\ \|\varphi_n\|_0 = 1 \\ & \Rightarrow Q(\varphi_n) \overset{n\to\infty}{\longrightarrow} 0 \Rightarrow |\varphi_n| \overset{n\to\infty}{\longrightarrow} 1 \text{ pointwise} \\ & \Rightarrow Q(1) \leq \lim_{n\to\infty} Q(|\varphi_n|) = 0 \\ & \Rightarrow 1 \in D, c \equiv 0 \Rightarrow 1 \in D_0 \Rightarrow 1 \in C_0(X). \end{aligned}$$

Sketch of proof (i) $\Leftarrow$ (ii)

(i) $\Leftarrow$ (ii)

$\mathcal{D}_0 \subseteq C_0(X)$
$\Leftarrow$ There exists $C \geq 0$ with $\|\varphi\|_\infty \leq C Q^{\frac{1}{2}}(\varphi)$ for all $\varphi \in C_c(X)$.

Let $f \in \mathcal{D}_0$. Thus, there exists a sequence (φ_n) in $C_c(X)$, such that $\varphi_n \overset{n\to\infty}{\longrightarrow} f$ pointwise and $Q(\varphi_n - f) \overset{n\to\infty}{\longrightarrow} 0$. Hence,

$$\|\varphi_n - f\|_\infty \leq C Q^{\frac{1}{2}}(\varphi_n - f) \overset{n\to\infty}{\longrightarrow} 0,$$

i.e., $f \in \overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(X)$.

Sketch of proof (i) $\Leftrightarrow$ (iii)

(i) $\Leftrightarrow$ (iii)

$\mathcal{D}_0 \subseteq C_0(X)$.
$\Leftrightarrow D(Q_m^{(D)}) \subseteq C_0(X)$ holds for any measure on X with full support.

Theorem 1.19 in the book yields $D(Q_m^{(D)}) = \mathcal{D}_0 \cap \ell^2(X, m)$.
$\Rightarrow$) If $\mathcal{D}_0 \subseteq C_0(X)$, then

$$\mathcal{D}_0 \subseteq C_0(X) \Rightarrow \mathcal{D}_0 \cap \ell^2(X,m) \subseteq C_0(X) \cap \ell^2(X,m) \subseteq C_0(X)$$
$$\Rightarrow D(Q_m^{(D)}) \subseteq C_0(X).$$

$\Leftarrow$) For any $f \in \mathcal{D}_0$ one can find a measure m such that $f \in \ell^2(X, m)$. Then, the statement follows.

Sketch of proof (ii)$\Rightarrow$ (iv)

(ii) $\Rightarrow$ (iv)

There exists $C \geq 0$ with $\|\varphi\|_\infty \leq CQ^{\frac{1}{2}}(\varphi)$ for all $\varphi \in C_c(X)$.
$\Rightarrow \inf_{x\in X} \operatorname{cap}(x) > 0$.

$\operatorname{cap}(x) := \inf\{Q(\varphi) \mid \varphi \in C_c(X), \varphi(x) = 1\}$
Let $x \in X$ be arbitrary, then $\|\varphi\|_\infty \geq 1$ for all $\varphi \in C_c(X)$ with $\varphi(x) = 1$, thus,

$$\|\varphi\|_\infty \leq CQ^{\frac{1}{2}}(\varphi),\ \varphi(x) = 1 \Rightarrow Q(\varphi) \geq \frac{1}{C^2}\|\varphi\|_\infty^2 \geq \frac{1}{C^2},$$

thus, $\inf_{x\in X} \operatorname{cap}(x) \geq \frac{1}{C^2}$.

Sketch of proof (ii)$\Leftarrow$(iv)

(ii)$\Leftarrow$ (iv)

There exists $C \geq 0$ with $\|\varphi\|_\infty \leq CQ^{\frac{1}{2}}(\varphi)$ for all $\varphi \in C_c(X)$.
$\Leftarrow \inf_{x\in X} \operatorname{cap}(x) > 0$.

$\operatorname{cap}(x) := \inf\{Q(\varphi) \mid \varphi \in C_c(X), \varphi(x) = 1\}$
Let $C := \inf_{x\in X} \inf\{Q(\varphi) \mid \varphi \in C_c(X), \varphi(x) = 1\}$ and $\varphi \in C_c(X)$. Consider $\varphi \neq 0$ and $x \in X$ with $\varphi(x) \neq 0$. Then,

$$Q\left(\frac{\varphi}{\varphi(x)}\right) = Q(\varphi) \cdot \frac{1}{\varphi(x)^2} \geq C \Rightarrow Q(\varphi) \geq C\varphi(x)^2.$$

As x was arbitrary, $Q(\varphi) \geq C\|\varphi\|_\infty^2$ holds. $\square$

A further characterisation

Theorem (Keller/Pinchover/Pogorzelski)

Let (b, c) be a connected graph over X. Then, (b, c) is uniformly transient iff there exists a $C > 0$ such that $G(x, x) \leq C$ for all $x \in X$.

This will be needed later to show a criterion for uniform transience in the spherically symmetric case. We note that we assume that $m = 1$ here.

A further characterisation

A reminder of one of the basic properties of Green's function:

Theorem (Lecture 10, Theorem 9.1)

Let (b, c) be a connected graph over (X, m). Then, for all sequences (K_n) of finite subsets of X with $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$ we have

$$G(x, y) = \lim_{n\to\infty} (L_{K_n}^{(D)})^{-1} 1_y(x).$$

A further characterisation

Proof of characterisation.
$\Rightarrow$) Assume (b,c) is uniformly transient. Let $x \in X$ be given and let (K_n) be an increasing sequence of finite subsets of X with $\bigcup_n K_n = X$. For $g_n := (L_{K_n})^{-1} 1_x$ we have $g_n \in C_c(X)$ and $g_n \to G(\cdot, x)$ pointwise. Green's formula and (ii) from the characterisation of uniform transience yield

$$g_n(x) = \langle 1_x, g_n \rangle = \langle \mathcal{L} g_n, g_n \rangle \overset{GF}{=} Q(g_n) \overset{(ii)}{\geq} \frac{1}{C^2} \|g_n\|_\infty^2 \geq \frac{1}{C^2} g_n(x)^2$$

and therefore $g_n(x) \leq C^2$. Taking the limit $n \to \infty$ we get $G(x,x) \leq C^2$.

A further characterisation

The other implication relies on facts from GaDDS.

$\Leftarrow$) Assume $G(x,x) \leq C$ for all $x \in X$. Let $x \in X$ be given. Theorem 6.30 and Lemma 6.27 from the book yield

$$\operatorname{cap}(x) \overset{6.30}{=} Q\left(\frac{G(\cdot,x)}{G(x,x)}\right) \overset{6.27}{=} \frac{1}{G(x,x)} \geq \frac{1}{C}.$$

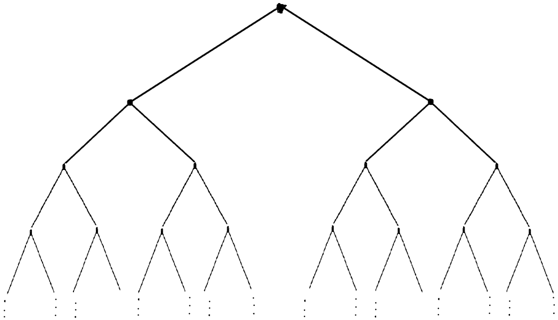
This implies uniform transience. $\square$

Comparison to transience

Transience	**Uniform transience**
For one (all) $x, y \in X$, $G(x,y) < \infty$	There exists a $C > 0$ with $G(x,x) \leq C$ for all $x \in X$
$1 \notin \mathcal{D}_0$	$\mathcal{D}_0 \subseteq C_0(X)$
$Q(\varphi) \geq \sum_{x \in X} w(x)\varphi(x)^2$ for some $0 \neq w \geq 0$, $\varphi \in C_c(X)$	$Q(\varphi) \geq C\|\varphi\|_\infty^2$, $C \geq 0$, $\varphi \in C_c(X)$
$\mathrm{cap}(x) > 0$ for some (all) $x \in X$	$\inf_{x \in X} \mathrm{cap}(x) > 0$

Transience and uniform transience are not the same

The following graph is transient, but not uniformly transient.



Transience and uniform transience are not the same

This graph (tree with root x_0) with $b(x,y) = b(r,r+1)$ for $x \in S_r(x_0)$, $y \in S_{r+1}(x_0)$, and $b(r,r+1) \to 0$ as $r \to \infty$ not too rapidly (e.g. $b(r,r+1) = 1/(r+1)$) is transient, but not uniformly transient. Transience follows from material to be presented. That the graph is not uniformly transient follows from taking a look at the capacity:

$$\begin{aligned}\operatorname{cap}(x) &\leq Q(1_x) \\ &= \frac{1}{2} \sum_{y,z \in X} b(y,z)(1_x(y) - 1_x(z))^2 \\ &= 2b(r,r+1) + b(r-1,r) \to 0\end{aligned}$$

as $r \to \infty$.

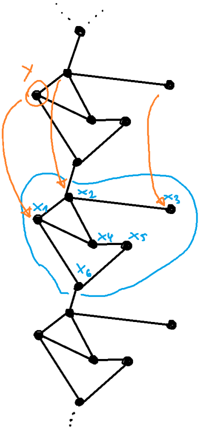
Transience and uniform transience are the same (sometimes)

Definition

(b, c) is called *quasi-vertex-transitive* iff there exists an $n \in \mathbb{N}$ and $x_1, ..., x_n \in X$ such that for all y there is a bijective

$$
\begin{aligned}
&h_y : X \to X, \\
&h_y(y) = x_j, \\
&c(h_y(z)) = c(z), \\
&b(h_y(v), h_y(w)) = b(v, w)
\end{aligned}
$$

for all $v, w, z \in X$ and some $j \in \{1, ..., n\}$.



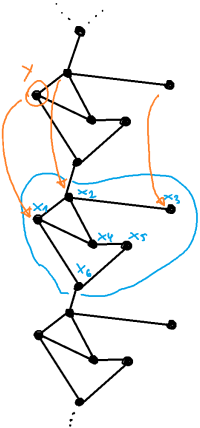
Transience and uniform transience are the same (sometimes)

Theorem (KLSW)

A quasi-vertex-transitive graph which is transient is also uniformly transient.

Idea of proof: Given the quasi-vertex-transitivity, there are only finitely many capacity types all of which are positive. Hence, the infimum is positive as well.

As an example, keep in mind regular trees with edge weights 1.



Recap

Definition (Terminology)

- $S_r(o) := \{x \in X \mid d(o,x) = r\}$,
- $B_r(o) := \{x \in X \mid d(o,x) \leq r\}$,
- $k_\pm(x) := \frac{1}{m(x)} \sum_{y \in S_{r\pm 1}(o)} b(x,y)$,
- $q(x) := \frac{c(x)}{m(x)}$

Definition (Spherically symmetric functions)

A function $f \in C(X)$ is called *spherically symmetric*, if it is constant on $S_r(o)$ for all $r \geq 0$.

Recap

Definition (Weakly spherically symmetric graphs (WSSG))

A locally finite, connected graph (b, c) over (X, m) is called *weakly spherically symmetric*, if $k_{\pm}$ and q are spherically symmetric functions.

Definition (Boundary of a set)

For $W \subseteq X$, define ∂W as

$$\partial W = (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

Recap

Lemma (Area of the boundary of spheres in WSSG)

$$b(\partial B_r(o)) = 2k_+(r)m(S_r(o)).$$

Theorem (Lecture 13, Remark 11.18)

Let b be a WSSG over (X, m). Then for $x \in S_r(o)$, we have

$$G(x, o) = m(o) \sum_{n=r}^{\infty} \frac{1}{\frac{1}{2} b(\partial B_r(o))}.$$

Recap

Definition (Averaging operator)

For $x \in S_r(o)$ and $f \in C(X)$, define

$$\mathcal{A}f(x) = \frac{1}{m(S_r(o))} \sum_{y \in S_r(o)} f(y)m(y)$$

and $A = \mathcal{A}|_{\ell^2(X,m)}$.

Note that a function f is spherically symmetric iff $\mathcal{A}f = f$.

Theorem (Lecture 13, Lemmas 11.8 and 11.9)

The graph (b, c) is weakly spherically symmetric iff $Ae^{-tL} = e^{-tL}A$ on $\ell^2(X, m)$ for all $t \geq 0$.

Characterization of transience for WSSG

Theorem (Lecture 13, Theorem 11.17)

Let b be a locally finite WSSG over X. Then, b is transient if and only if $\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(o))} < \infty$.

Ideally, we would expect to find a similar characterization for the case of uniform transience. However, there is no characterization as of now, only a criterion.

Uniform transience on WSSG

Theorem (Keller/Schmidt/Wojciechowski???)

Let b be a weakly spherically symmetric graph over (X, m). If for all $r \in \mathbb{N}_0$ we have

$$\#S_r(o) \sum_{k=r}^{\infty} \frac{1}{\frac{1}{2} b(\partial B_k(o))} \leq C,$$

then b is uniformly transient.

Preliminaries for the proof

We need the following definition and theorem from Wojciechowski.

Definition

A graph is called *Feller*, if

$$e^{-tL} : C_0(X) \to C_0(X)$$

for all $t \geq 0$.

Theorem

A weakly spherically symmetric graph is Feller if it is transient.

Preliminaries for the proof

We also need the following lemma.

Lemma

Let b be a weakly spherically symmetric graph over (X, m). If $f \in D$ satisfies $\mathcal{A}f(x) = 1$ for $x \in S_r(o)$ and $\mathcal{A}f \in C_0(X)$, then

$$Q(f) \geq \left(\sum_{k=r}^{\infty} \frac{1}{\frac{1}{2} b(\partial B_k(o))} \right)^{-1}$$

We will omit the proof of this since it is just a long calculation using standard techniques. It can be found e.g. in a paper of Huang.

Proof of the criterion

Since uniform transience is independent of the measure, we can assume $m \equiv 1$.
For $\alpha > 0$ and $r \in \mathbb{N}_0$, let

$$g_\alpha = (L+\alpha)^{-1} 1_{S_r(o)}.$$

Since b is spherically symmetric, A commutes with the semigroup and thus with the resolvent. Therefore,

$$A g_\alpha = A(L+\alpha)^{-1} 1_{S_r(o)} = (L+\alpha)^{-1} A 1_{S_r(o)} = g_\alpha$$

which means that g_α is spherically symmetric. Hence, for $x \in S_r(o)$, we have

$$A\left(\frac{g_\alpha}{g_\alpha(r)}\right)(x) = 1.$$

Proof of the criterion

Since the condition from the criterion also implies transience, the graph is Feller by the theorem above.
Therefore, $e^{-tL}(C_0(X)) \subseteq C_0(X)$ which is equivalent to $(L+\alpha)^{-1}(C_0(X)) \subseteq C_0(X)$ by general principles and which in turn implies $g_\alpha \in C_0(X)$. We can then apply the lemma from above to get

$$\left(\sum_{k=r}^{\infty} \frac{1}{\frac{1}{2}b(\partial B_k(o))}\right)^{-1} \leq Q\left(\frac{g_\alpha}{g_\alpha(r)}\right) = \frac{1}{g_\alpha(r)^2} Q(g_\alpha).$$

Proof of the criterion

A calculation yields

$$
\begin{aligned}
g_\alpha(r)^2 \left(\sum_{k=r}^{\infty} \frac{1}{\frac{1}{2}\partial b(B_k(o))} \right)^{-1} &\leq Q(g_\alpha) \\
&= \langle L g_\alpha, g_\alpha \rangle \\
&\leq \langle (L+\alpha) g_\alpha, g_\alpha \rangle \\
&= \sum_{y \in S_r} g_\alpha(y) = \# S_r(o) g_\alpha(r).
\end{aligned}
$$

Proof of the criterion

For $x \in S_r(o)$, we now have

$$\begin{aligned}(L+\alpha)^{-1}1_x(x) &\leq (L+\alpha)^{-1}1_{S_r(o)}(x)\\ &= g_\alpha(r)\\ &\leq \#S_r(o)\sum_{k=r}^{\infty}\frac{1}{\frac{1}{2}b(\partial B_k(o))} \leq C,\end{aligned}$$

which gives the conclusion by letting $\alpha \to 0^+$ since $(L+\alpha)^{-1}1_x(x) \to G(x,x)$ by Theorem 9.1 in the Lecture Notes. $\square$

The role of c

Theorem

If $(b,0)$ is uniformly transient, then (b,c) is uniformly transient.

Proof.
Because $(b,0)$ is uniformly transient, we have

$$\|\varphi\|_\infty \leq C Q_{b,0}^{1/2}(\varphi).$$

The statement then follows since $Q_{b,0} \leq Q_{b,c}$. □

This shows that c can't break uniform transience. In the following theorem we will see that c can help a graph become uniformly transient.

The role of c

Theorem

If $\inf_{x \in X} c(x) > 0$, then (b, c) is uniformly transient.

Proof.
Note that for $\varphi \in C_c(X)$ with $\varphi(a) = 1$, we have

$$Q(\varphi) \geq \sum_{x \in X} c(x)\varphi(x)^2 \geq c(a).$$

Since $\operatorname{cap}(a) = \inf\{Q(\varphi) \mid \varphi \in C_c(X),\ \varphi(a) = 1\}$, we get

$$\operatorname{cap}(a) \geq c(a)$$

and therefore

$$\inf_{x \in X} \operatorname{cap}(x) \geq \inf_{x \in X} c(x) > 0.$$

The role of c

While c can cause uniform transience, it is not always the case as the following example shows.

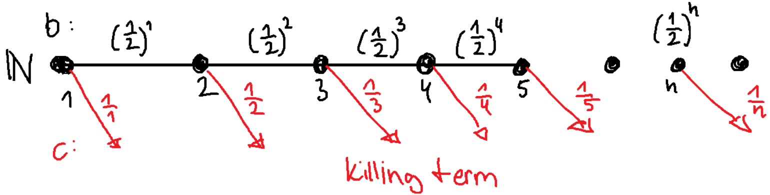


Figure: Graph with $c > 0$ that is not uniformly transient

The role of c

Let $X = \mathbb{N}$ with $b(n, n+1) = (1/2)^n$, 0 everywhere else and $c(n) = 1/n$ for all $n \in \mathbb{N}$. A calculation yields that

$$Q(1_n) = b(n, n-1) + b(n, n+1) + c(n) \to 0$$

and since $1_n \in C_c(X)$ with $1_n(n) = 1$, we get

$$\mathrm{cap}(n) \leq Q(1_n) \to 0.$$

Thus, (b, c) is not uniformly transient.

References

1. M. Keller, D. Lenz, M. Schmidt, C. Seifert, *26th Internet Seminar on Evolution Equations–Graphs and Discrete Dirichlet Spaces*, Lecture Notes, 2023.
2. M. Keller, D. Lenz, M. Schmidt, and R. K. Wojciechowski, *Note on uniformly transient graphs*, Rev. Mat. Iberoam. **33** (2017), no. 3, 831–860.
3. M. Keller, D. Lenz, R. K. Wojciechowski, *Graphs and discrete Dirichlet spaces*, Grundlehren der mathematischen Wissenschaften, vol. 358, Springer, 2021, 682 pp.
4. M. Keller, Y. Pinchover, F. Pogorzelski, *Criticality theory for Schrödinger operators on graphs*, J. Spectr. Theory **10** (2020), no. 1, 73–114.
5. R. K. Wojciechowski, *The Feller property for graphs*, Trans. Amer. Math. Soc. **369** (2017), no. 6, 4415–4431.