A probabilistic perspective of recurrence and transience

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Outline

Simple Random Walk on $\mathbb{Z}^d$

In every step one takes a step in a random of the $2d$ directions (e.g. left, right, up, down in $d = 2$) with the same probability $p = \frac{1}{2d}$.

Theorem [Recurrence for Simple Random Walks] (Polya 1921)

The following assertions hold if and only if $d \leq 2$:

- one almost surely returns to the starting point in finite time
- one almost surely reaches any point at some point in time
- one almost surely reaches any point infinitely many times,

but the expected time for those events is infinity.

"A drunk man will find his way home, but a drunk bird may get lost forever" (S. Kakutani)

Reminder Probability Theory: Random Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E countable and discrete.

Definition [Random Variable]

A **random variable** is a measurable function

$$Y : \Omega \to E.$$

Then the probability that X is in the state $a \in E$ is given by

$$\mathbb{P}(Y = a) := \mathbb{P}(Y^{-1}(\{a\})).$$

For a function $f : E \to \mathbb{R}$, $f(Y) = f \circ Y$ is a real valued random variable.

The space Ω is usually not explicitly given, and may vary, as long as the probabilities stay the same.

Reminder Probability Theory: Expected Value

Definition [Expected Value]

Let Y be a real valued random variable and Y integrable or non-negative, then the **expected value** is given by

$$\mathbb{E}[Y] := \int_\Omega Y \, \mathrm{d}\mathbb{P}.$$

Example

The throw of two dice is simulated by $\Omega = \{1, ..., 6\}^2$ with $\mathbb{P}(A) = \frac{1}{36}\#A$. Then $Y_i(x_1, x_2) := x_i$ for $i = 1, 2$ and $S := Y_1 + Y_2$ defines three random variables. Then

$$\mathbb{E}[S] = 7.$$

Reminder Probability Theory: Conditional Probability

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition [Conditional Probability]

Fix two events $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$.
Then the probability of A under the **condition** B is

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Even if $\mathbb{P}(B) = 0$ we will use

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B).$$

Example

Going back to the two dice:

$$\mathbb{P}(S = 8 \mid Y_1 = 3) = \frac{\mathbb{P}(S = 8, Y_1 = 3)}{\mathbb{P}(Y_1 = 3)} = \frac{\mathbb{P}(Y_1 = 3, Y_2 = 5)}{\mathbb{P}(Y_1 = 3)} = \frac{1}{6}$$

Reminder Probability Theory: Independent Random Variables

Fix two random variables $Y, Z : \Omega \to E$.

Definition [Independent Random Variables]

The random variables Y, Z are **independent** if for all $a, b \in E$:

$$\mathbb{P}(Y = a, Z = b) = \mathbb{P}(Y = a)\,\mathbb{P}(Z = b).$$

Especially if $\mathbb{P}(Z = b) > 0$:

$$\mathbb{P}(Y = a \mid Z = b) = \mathbb{P}(Y = a).$$

Example

In the example with the two dice, Y_1, Y_2 are independent, while Y_1, S are not.

Transition Probability on Graph

Let (b, c) be a graph on (X, m). Let $x_\infty \notin X$ be the **graveyard**, and $\bar{X} := X \cup \{x_\infty\}$.

Definition [Transition Probability]

The **transition probability** $p(x, y)$ denotes the probability of jumping from x to y. It is defined by

$$p(x,y) := \begin{cases} \dfrac{b(x,y)}{\deg(x)} & x, y \in X \\ \dfrac{c(x)}{\deg(x)} & x \in X,\ y = x_\infty \\ 1 & x = y = x_\infty \\ 0 & x = x_\infty,\ y \in X \end{cases}$$

for $x, y \in \bar{X}$, with $\deg(x) = \sum_{y \in X} b(x,y) + c(x)$, and assume $\deg(x) > 0$.

Random Walk on Graph

Definition [Random Walk on Graph]

The **random walk on** (b, c) is a Markov chain given by a sequence of random variables $(Y_n)_{n\in\mathbb{N}_0}$ on $\bar{X}$, with $\mathbb{P}(Y_0 = x) > 0$ for $x \in \bar{X}$ and

$$\mathbb{P}(Y_{n+1} = x_{n+1} \mid Y_0 = x_0, ..., Y_n = x_n) = p(x_n, x_{n+1})$$

for $x_0, ..., x_{n+1} \in \bar{X}$ with $\mathbb{P}(Y_0 = x_0, ..., Y_n = x_n) > 0$.
For any **starting state** $x \in \bar{X}$ and event A define

$$\mathbb{P}_x(A) := \mathbb{P}(A \mid Y_0 = x).$$

The measures $(\mathbb{P}_x)_{x\in\bar{X}}$ are unique for the graph (b, c).

Holding Times

Definition [Holding Times]

Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables, i.e. for $t \in [0, \infty)$

$$\mathbb{P}(\xi_n > t) = \int_t^\infty e^{-s}\,\mathrm{d}s = e^{-t}.$$

The **holding times** $(S_n)_{n\in\mathbb{N}}$ are random variables defined by

$$S_n := \begin{cases} \frac{1}{\operatorname{Deg}(Y_{n-1})}\xi_n & Y_{n-1} \in X \\ 0 & Y_{n-1} = x_\infty \end{cases}$$

with $\operatorname{Deg}(x) = \frac{\deg(x)}{m(x)}$.

Jump Process on Graph

Definition [Jump Process on Graph]

The **jumping times** $(J_n)_{n\in\mathbb{N}}$ are random variables defined by

$$J_n := \sum_{k=1}^{n} S_k.$$

Then the jump process $(\mathbb{X}_t)_{t\geq 0}$ is the time continuous Markov process defined by

$$\mathbb{X}_t := \begin{cases} Y_n & t \in [J_n, J_{n+1}) \\ x_\infty & \text{otherwise} \end{cases}$$

There are two ways the process $\mathbb{X}$ can enter the graveyard:

- due to the killing term after a finite number of jumps
- after performing infinite many jumps in finite time.

Lifetime

Definition [Lifetime]

Also define the **lifetime**

$$\zeta := \inf\{t \geq 0 \mid \mathbb{X}_t = x_\infty\} = \sup_{n \in \mathbb{N}} J_n$$

as well as the number of jumps at time t by

$$N(t) := \sup\{n \in \mathbb{N} \mid J_n \leq t\}$$

Note that $N(t) = \infty$ if and only if $\mathbb{X}_t = x_\infty$.

Random Walk and Heat Semigroup on Graphs

Let L be the Dirichlet Laplacian of the graph (b, c) on (X, m).

Theorem

For $f \in \ell^2(X)$ it holds that

$$e^{-tL} f(x) = \mathbb{E}_x[1_{\{t<\zeta\}} f(\mathbb{X}_t)] \qquad \forall x \in X,\ t \in [0, \infty).$$

Corollary

For $x, y \in X$ and $t \in [0, \infty)$:

$$\mathbb{P}_x(\mathbb{X}_t = y) = \mathbb{E}_x[1_{\{t<\zeta\}} 1_y(\mathbb{X}_t)] = e^{-tL} 1_y(x) = m(y) e^{-tL} \frac{1}{m(x)} 1_x(y)$$

Proof for Finite Graphs

Lemma

Assume X is **finite**. Then for $f \in \ell^2(X)$ it holds that

$$e^{-tL} f(x) = \mathbb{E}_x[1_{\{t<\zeta\}} f(\mathbb{X}_t)] \qquad \forall x \in X,\ t \in [0, \infty).$$

For simplicity we introduce the notation $f(x_\infty) = 0$ for all $f \in \ell^2(X)$. Thus we want to prove that

$$e^{-tL} f(x) = \mathbb{E}_x[f(\mathbb{X}_t)].$$

Define an operator semigroup on $\ell^2(X)$ by

$$T(t)f(x) := \mathbb{E}_x[f(\mathbb{X}_t)] \qquad \forall x \in X,\ t \in [0, \infty),\ f \in \ell^2(X).$$

Proof that T is a Semigroup

For $x, y \in X$ and $s, t \in [0, \infty)$:

$$
\begin{aligned}
& T(s+t)1_y(x) && = && \mathbb{E}_x[1_y(\mathbb{X}_{s+t})] = \mathbb{P}_x(\mathbb{X}_{s+t} = y) \\
= \quad & \sum_{z \in X} \mathbb{P}_x(\mathbb{X}_{s+t} = y, \mathbb{X}_s = z) && = && \sum_{z \in X} \mathbb{P}_x(\mathbb{X}_s = z)\mathbb{P}_x(\mathbb{X}_{s+t} = y | \mathbb{X}_s = z) \\
= \quad & \sum_{z \in X} \mathbb{P}_x(\mathbb{X}_s = z)\mathbb{P}_z(\mathbb{X}_t = y) && = && \sum_{z \in X} \mathbb{E}_x[1_z(\mathbb{X}_s)\mathbb{E}_z[1_y(\mathbb{X}_t)]] \\
= \quad & \mathbb{E}_x[\mathbb{E}_{\mathbb{X}_s}[1_y(\mathbb{X}_t)]] && = && T(s)T(t)1_y(x).
\end{aligned}
$$

Markov Property

For $x, y \in X$, $t \in [0, \infty)$ and measurable $A \in X^{[0,\infty)}$, with $\mathbb{P}_x(\mathbb{X}_t = y) > 0$:

$$\mathbb{P}_x((\mathbb{X}_{s+t})_{s \geq 0} \in A \mid \mathbb{X}_t = y) = \mathbb{P}_y((\mathbb{X}_s)_{s \geq 0} \in A).$$

Proof that T is Generated by $-L$

For $x, y \in X$:

$$\frac{T(t)1_y(x) - 1_y(x)}{t} = \frac{\mathbb{E}_x[1_y(\mathbb{X}_t)] - 1_y(x)}{t} = \frac{\mathbb{P}_x(\mathbb{X}_t = y) - 1_y(x)}{t}$$

$$\begin{aligned}
&= \quad \frac{1}{t}\left(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)\right) && \xrightarrow[t \searrow 0]{} \quad -\operatorname{Deg}(x)1_y(x) \\
&+ \quad \frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) && \xrightarrow[t \searrow 0]{} \quad \frac{b(x,y)}{m(x)} \\
&+ \quad \frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) && \xrightarrow[t \searrow 0]{} \quad 0.
\end{aligned}$$

From those we get:

$$\lim_{t \searrow \infty} \frac{T(t)1_y(x) - 1_y(x)}{t} = -\operatorname{Deg}(x)1_y(x) + \frac{b(x,y)}{m(x)} = -L1_y(x).$$

Proof of Convergence

Proof Structure

1. $\frac{1}{t}\mathbb{P}_x(S_1 < t) \to_{t\searrow 0} \mathrm{Deg}(x)$
2. $\frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) \to_{t\searrow 0} 0$
3. $\frac{1}{t}(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)) \to_{t\searrow 0} -\mathrm{Deg}(x)1_y(x)$
4. $\frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) \to_{t\searrow 0} \frac{b(x,y)}{m(x)}$
5. $\frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) \to_{t\searrow 0} 0.$

$$
\begin{aligned}
\frac{1}{t}\mathbb{P}_x(S_1 < t) \quad &= \quad \frac{1}{t}\mathbb{P}_x\left(\frac{1}{\mathrm{Deg}(x)}\xi_1 < t\right) \\
= \quad \frac{1}{t}\mathbb{P}_x(\xi_1 < \mathrm{Deg}(x)t) \quad &= \quad \frac{1}{t}\int_0^{\mathrm{Deg}(x)t} e^{-s}\,\mathrm{d}s \\
= \quad -\frac{e^{-\mathrm{Deg}(x)t} - 1}{t} \quad &\to_{t\searrow 0} \quad \mathrm{Deg}(x)
\end{aligned}
$$

Proof of Convergence

Proof Structure

1. $\frac{1}{t}\mathbb{P}_x(S_1 < t) \to_{t\searrow 0} \mathrm{Deg}(x)$
2. $\frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) \to_{t\searrow 0} 0$
3. $\frac{1}{t}(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)) \to_{t\searrow 0} -\mathrm{Deg}(x)1_y(x)$
4. $\frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) \to_{t\searrow 0} \frac{b(x,y)}{m(x)}$
5. $\frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) \to_{t\searrow 0} 0.$

$$
\begin{aligned}
\frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) &\leq \frac{1}{t}\mathbb{P}_x\left(\frac{\xi_1}{\mathrm{Deg}(x)} < t, \frac{\xi_2}{\mathrm{Deg}(Y_1)} < t, Y_1 \neq x_\infty\right) \\
\leq \frac{1}{t}\mathbb{P}_x(\xi_1 < Ct, \xi_2 < Ct) &= \frac{1}{t}\mathbb{P}_x(\xi_1 < Ct)\mathbb{P}_x(\xi_2 < Ct) \\
= t\left(\frac{\mathbb{P}_x(\xi_1 < Ct)}{t}\right)^2 &\to_{t\searrow 0} 0
\end{aligned}
$$

Proof of Convergence

Proof Structure

1. $\frac{1}{t}\mathbb{P}_x(S_1 < t) \to_{t\searrow 0} \mathrm{Deg}(x)$
2. $\frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) \to_{t\searrow 0} 0$
3. $\frac{1}{t}(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)) \to_{t\searrow 0} -\mathrm{Deg}(x)1_y(x)$
4. $\frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) \to_{t\searrow 0} \frac{b(x,y)}{m(x)}$
5. $\frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) \to_{t\searrow 0} 0.$

$$
\begin{aligned}
& \frac{1}{t}(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)) && = && \frac{1}{t}(\mathbb{P}_x(N(t) = 0) - 1)1_y(x) \\
= \quad & -\frac{1}{t}\mathbb{P}_x(N(t) > 0)1_y(x) && = && -\frac{1}{t}\mathbb{P}_x(S_1 < t)1_y(x) \\
\to_{t\searrow 0} \quad & -\mathrm{Deg}(x)1_y(x)
\end{aligned}
$$

Proof of Convergence

Proof Structure

1. $\frac{1}{t}\mathbb{P}_x(S_1 < t) \to_{t\searrow 0} \mathrm{Deg}(x)$
2. $\frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) \to_{t\searrow 0} 0$
3. $\frac{1}{t}(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)) \to_{t\searrow 0} -\mathrm{Deg}(x)1_y(x)$
4. $\frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) \to_{t\searrow 0} \frac{b(x,y)}{m(x)}$
5. $\frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) \to_{t\searrow 0} 0.$

$$\begin{aligned}
& \frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) \\
= \quad & \frac{1}{t}\mathbb{P}_x(N(t) \geq 1, Y_1 = y) - \frac{1}{t}\mathbb{P}_x(N(t) \geq 2, Y_1 = y) \\
= \quad & \frac{1}{t}\mathbb{P}_x(S_1 < t)\,\mathbb{P}_x(Y_1 = y) - \frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 = y) \\
\to_{t\searrow 0} \quad & \mathrm{Deg}(x)b(x,y)/\deg(x) \; = \; b(x,y)/m(x)
\end{aligned}$$

Proof of Convergence

Proof Structure

1. $\frac{1}{t}\mathbb{P}_x(S_1 < t) \to_{t\searrow 0} \mathrm{Deg}(x)$
2. $\frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) \to_{t\searrow 0} 0$
3. $\frac{1}{t}(\mathbb{P}_x(N(t) = 0, Y_0 = y) - 1_y(x)) \to_{t\searrow 0} -\mathrm{Deg}(x)1_y(x)$
4. $\frac{1}{t}\mathbb{P}_x(N(t) = 1, Y_1 = y) \to_{t\searrow 0} \frac{b(x,y)}{m(x)}$
5. $\frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) \to_{t\searrow 0} 0.$

$$\frac{1}{t}\mathbb{P}_x(N(t) \geq 2, \mathbb{X}_t = y) \leq \frac{1}{t}\mathbb{P}_x(J_2 < t, Y_1 \neq x_\infty) \to_{t\searrow 0} 0$$

For Infinite Graphs

Theorem

For $f \in \ell^2(X)$, $x \in X$, $t \in [0, \infty)$:

$$e^{-tL} f(x) = \mathbb{E}_x[1_{\{t<\zeta\}} f(\mathbb{X}_t)]$$

Fix $(K_n)_{n \in \mathbb{N}}$, an exhausting increasing sequence of finite subsets of X,

$$b_n(x, y) := b(x, y) \quad c_n(x) := c(x) + \sum_{y \in X \setminus K_n} b(x, y) \qquad \forall x, y \in K_n.$$

Let L_n be the Dirichlet Laplacian on K_n, and $(\mathbb{X}_t^n)_{t \in [0,\infty)}$ the jump process on K_n. Define the stopping time of $\mathbb{X}_t$ first leaving K_n by

$$T_n := \inf\{t \in [0, \infty); \mathbb{X}_t \notin K_n\}.$$

Then we get:

$$e^{-tL} f(x) = \lim_{n \to \infty} e^{-tL_n} f(x) = \lim_{n \to \infty} \mathbb{E}_x[1_{\{t<\zeta_n\}} f(\mathbb{X}_t^n)]$$

$$= \lim_{n \to \infty} \mathbb{E}_x[1_{\{t<T_n\}} f(\mathbb{X}_t)] = \mathbb{E}_x[1_{\{t<\zeta\}} f(\mathbb{X}_t)]$$

Feynman-Kac Formula

Theorem [Feynman-Kac Formula]

Fix a graph (b, c) on (X, m).
For $f \in \ell^2(X)$, $x \in X$ and $t \in [0, \infty)$:

$$e^{-tL} f(x) = \mathbb{E}_x \left[1_{\{t<\zeta\}} e^{-\int_0^t (\frac{c}{m})(\mathbb{X}_t^b)} f(\mathbb{X}_t^b) \right]$$

where $\mathbb{X}_t^b$ is the jump process on $(b, 0)$.

Recurrence for General Markov Chains

Fix any Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ on X. Define the time T_y^k of the k-th arrival at $y \in X$ by

$$T_y^0 = 0, \qquad T_y^{k+1} := \inf\{n > T_y^k ; Y_n = y\}.$$

Those random variables are **stopping times**, i.e. their values are times (values in $\mathbb{N}$) and $\{T_y^k = n\}$ only depends on Y_m for $m \leq n$.

Lemma

For $x, y \in X$ and $k \in \mathbb{N}$:

$$\mathbb{P}_x(T_y^k < \infty) = \mathbb{P}_x(T_y^1 < \infty)\, \mathbb{P}_y(T_y^1 < \infty)^{k-1}.$$

Proof Lemma

For $x, y \in X$ and $k \in \mathbb{N}$:

$$\begin{aligned}
\mathbb{P}_x(T_y^k < \infty) &= \mathbb{P}_x(T_y^{k-1} < \infty)\,\mathbb{P}_x(T_y^k < \infty \mid T_y^{k-1} < \infty) \\
&= \mathbb{P}_x(T_y^{k-1} < \infty)\,\mathbb{P}_x(\exists n > T_y^{k-1} : Y_n = y \mid T_y^{k-1} < \infty) \\
&= \mathbb{P}_x(T_y^{k-1} < \infty)\,\mathbb{P}_y(\exists n > 0 : Y_n = y) \\
&= \mathbb{P}_x(T_y^{k-1} < \infty)\,\mathbb{P}_y(T_y^1 < \infty) \\
&= \mathbb{P}_x(T_y^1 < \infty)\,\mathbb{P}_y(T_y^1 < \infty)^{k-1}
\end{aligned}$$

Strong Markov Property

For $x, y \in X$, a stopping time T and $A \in X^{\mathbb{N}}$ measurable and $\mathbb{P}_x(T < \infty, Y_T = y) > 0$:

$$\mathbb{P}_x((Y_{n+T})_{n\in\mathbb{N}_0} \in A \mid T < \infty, Y_T = y) = \mathbb{P}_y((Y_n)_{n\in\mathbb{N}_0} \in A)$$

Recurrence for general Markov Chains

Also assume the Markov chain is irreducible, i.e. for all $x, y \in X$ there exists some $n \in \mathbb{N}$ such that $\mathbb{P}_x(Y_n = y) > 0$.
In the case of graphs this is equivalent to the graph being connected.

Definition/Theorem [Recurrence]

The following assertions are equivalent:

(i) for some $y \in X$: $\mathbb{P}_y(T_y^1 < \infty) = 1$

(ii) for all $x, y \in X$: $\mathbb{P}_x(T_y^1 < \infty) = 1$

(iii) for some/all $x, y \in X$: $\mathbb{P}_x(T_y^k < \infty \; \forall k \in \mathbb{N}) = 1$

(iv) for some/all $x, y \in X$: $\mathbb{E}_x[\#\{n \in \mathbb{N};\; Y_n = y\}] = \infty$

(v) for some/all $x, y \in X$: $\sum_{n \in \mathbb{N}} \mathbb{P}_x(Y_n = y) = \infty$.

In this case the Markov chain is called recurrent.

(iv) and (v) are the same since for $x, y \in X$:

$$\mathbb{E}_x[\#\{n \in \mathbb{N};\; Y_n = y\}] = \sum_{n \in \mathbb{N}} \mathbb{E}_x[1_y(Y_n)] = \sum_{n \in \mathbb{N}} \mathbb{P}_x(Y_n = y).$$

Recurrence Proof I

Theorem [Recurrence]

$\mathbb{P}_z(T_z^1 < \infty) = 1$ for some $z \in X \Rightarrow \mathbb{P}_x(T_y^1 < \infty) = 1$ for all $x, y \in X$.

For $k \in \mathbb{N}$: $\mathbb{P}_z(T_z^k < \infty) = \mathbb{P}_z(T_z^1)^k = 1$.
For $x \in X$ there is some $n \in \mathbb{N}$ such that $\mathbb{P}_z(Y_n = x) > 0$ and:

$$\begin{aligned} \mathbb{P}_x(T_z^1 < \infty) &= \mathbb{P}_x(\exists m > 0 : Y_m = z) \\ &= \mathbb{P}_z(\exists m > n : Y_m = z \mid Y_n = x) \quad = \quad 1. \end{aligned}$$

For $y \in X$ there is some $n \in \mathbb{N}$ such that $\mathbb{P}_z(Y_n = y) > 0$, so there is a possible path from z to y. Since we revisit z infinitely many times, we almost surely take that path at some time, thus $\mathbb{P}_z(T_y^1 < \infty) = 1$.
For $x, y \in X$:

$$\begin{aligned} \mathbb{P}_x(T_y^1 < \infty) &\geq \mathbb{P}_x(\exists m > T_z^1 : Y_m = y \mid T_z^1 < \infty) \\ &= \mathbb{P}_z(\exists m > 0 : Y_m = y) \quad = \quad 1. \end{aligned}$$

Recurrence Proof II

Theorem [Recurrence]

(ii) for all $x, y \in X$: $\mathbb{P}_x(T_y^1 < \infty) = 1$

(iii) for all $x, y \in X$: $\mathbb{P}_x(T_y^k < \infty \ \forall k \in \mathbb{N}) = 1$

(iv) for all $x, y \in X$: $\mathbb{E}_x[\#\{n \in \mathbb{N};\ Y_n = y\}] = \infty$

(ii)$\Rightarrow$(iii): For $x, y \in X$ and all $k \in \mathbb{N}$:

$$\mathbb{P}_x(T_y^k < \infty) = \mathbb{P}_x(T_y^1 < \infty)\, \mathbb{P}_y(T_y^1 < \infty)^{k-1} = 1,$$

thus $\mathbb{P}_x(T_y^k < \infty \ \forall k \in \mathbb{N}) = 1$.

(iii)$\Rightarrow$(iv): For all $x, y \in X$

$$\mathbb{P}_x(T_y^k < \infty \ \forall k \in \mathbb{N}) = \mathbb{P}_x(\#\{n \in \mathbb{N}; Y_n = y\} = \infty) = 1$$

implies $\mathbb{E}_x[\#\{n \in \mathbb{N};\ Y_n = y\}] = \infty$.

Recurrence Proof III

Theorem [Recurrence]

(i) for some $y \in X$: $\mathbb{P}_y(T_y^1 < \infty) = 1$

(ii) for all $x, y \in X$: $\mathbb{P}_x(T_y^1 < \infty) = 1$

(iii) for some $x, y \in X$: $\mathbb{P}_x(T_y^k < \infty \ \forall k \in \mathbb{N}) = 1$

(iv) for some $x, y \in X$: $\mathbb{E}_x[\#\{n \in \mathbb{N};\ Y_n = y\}] = \infty$

(iii)$\Rightarrow$(i): Follows from

$$\mathbb{P}_x(T_y^2 < \infty) = \mathbb{P}_x(T_y^1 < \infty)\, \mathbb{P}_y(T_y^1 < \infty) = 1.$$

(iv)$\Rightarrow$(i): Follows from

$$\begin{aligned} & \mathbb{E}_x[\#\{n \in \mathbb{N};\ Y_n = y\}] && = && \mathbb{E}_x[\#\{k \in \mathbb{N}; T_y^k < \infty\}] \\ = \quad & \sum_{k \in \mathbb{N}} \mathbb{E}_x[1_{\{T_y^k < \infty\}}] && = && \sum_{n \in \mathbb{N}} \mathbb{P}_x(T_y^k < \infty) \\ = \quad & \mathbb{P}_x(T_y^1 < \infty) \sum_{k \in \mathbb{N}} \mathbb{P}_y(T_y^1 < \infty)^{k-1} && = && \infty \end{aligned}$$

Recurrence on Graphs

Theorem [Random walk perspective on recurrence]

Let (b, c) be a connected graph over (X, m). Then for all $x, y \in X$

$$G(x, y) = \int_0^\infty e^{-tL} 1_y(x) \, \mathrm{d}t = \frac{1}{\mathrm{Deg}(y)} \sum_{n=0}^{\infty} \mathbb{P}_x(Y_n = y)$$

especially (b, c) is recurrent if and only if $G(x, y) = \infty$ for some/all $x, y \in X$.

Here, $\frac{1}{\mathrm{Deg}(y)}$ is the expected holding time at the vertex y.
$G(x, y)$ is the expected total time that a particle stays at the vertex y if it starts at x, and $\sum_n \mathbb{P}_x(Y_n = y)$ ("discrete Green's function") specifies how often the particle visits the vertex y.

Recurrence on Graphs Proof

For $x, y \in X$:

$$\begin{aligned}
G(x,y) &= \int_0^\infty e^{-tL} 1_y(x)\ \mathrm{d}t \\
&= \int_0^\infty \mathbb{E}_x[1_{\{t<\zeta\}} 1_y(\mathbb{X}_t)]\ \mathrm{d}t \\
&= \mathbb{E}_x\left[\int_0^\zeta 1_y(\mathbb{X}_t)\ \mathrm{d}t\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}_x[S_n 1_y(Y_{n-1})] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}_x\left[\frac{1}{\mathrm{Deg}(y)} \xi_n 1_y(Y_{n-1})\right] \\
&= \frac{1}{\mathrm{Deg}(y)} \sum_{n\in\mathbb{N}} \mathbb{E}_x[\xi_n]\, \mathbb{E}_x[1_y(Y_{n-1})] \\
&= \frac{1}{\mathrm{Deg}(y)} \sum_{n\in\mathbb{N}} \mathbb{P}_x(Y_{n-1} = y)
\end{aligned}$$

Stochastic Completeness

Theorem [Stochastic Completeness]

For a graph (b, c) the following assertions are equivalent;

(i) (b, c) is stochastic complete, i.e. for some/all $t \in (0, \infty)$:

$$e^{-tL}1 = 1$$

(ii) the lifetime ζ is almost surely infinite, i.e. for all $x \in X$:

$$\mathbb{P}_x(\zeta = \infty) = 1,$$

especially $c = 0$.

Let K_n be an exhausting increasing sequence of finite subsets of X.
For $x \in X$ and $t \in (0, \infty)$:

$$\begin{aligned} e^{-tL}1(x) &= \lim_{n\to\infty} e^{-tL}1_{K_n}(x) \\ &= \lim_{n\to\infty} \mathbb{E}_x[1_{\{t<\zeta\}}1_{K_n}(\mathbb{X}_t)] = \mathbb{E}_x[1_{\{t<\zeta\}}] \quad = \mathbb{P}_x(t < \zeta) \end{aligned}$$

Bibliography

- Keller, Lenz, Wojciechowsky - Graphs and Discrete Dirichlet Spaces
- Norris - Markov Chains
- Durett - Probability: Theory and Examples