TECHNISCHE UNIVERSITÄT DRESDEN

Jan Hausmann, Robert Wilke

26th Internet Seminar
Project K - General Dirichlet form theory

Workshop // Wuppertal, 18 July 2023

Contents

Recap Internet Seminar

Examples

Theorem of Beurling–Deny

Technical Tools

Proof of Beurling–Deny's Theorem

Recap Internet Seminar

Recap Internet Seminar

Definition (Form associated to a graph (b, c) over finite set X)

$\mathcal{Q}\colon C(X) \times C(X) \to \mathbb{R}$ **form**, if

$$\mathcal{Q}(f, g) = \mathcal{Q}_{b,c}(f, g) \coloneqq \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x)f(x)g(x).$$

Remark

Any symmetric form on a finite set X has this representation.

Basic Definitions

General Setting

- X locally compact separable metric space
- m positive Radon measure on X with $\operatorname{supp}(m) = X$
- $L^2(X; m)$ with inner product

$$\langle u \, , \, v \rangle = \int_X u(x)v(x)\,\mathrm{d}m(x)$$

- $C_c(X) = \{ f \colon X \to \mathbb{R} \mid f \text{ continuous}, \operatorname{supp} f \text{ compact} \}$

Basic Definitions

Definition (Symmetric Form)

Let H be H-space. $\mathcal{E}\colon \operatorname{dom}\mathcal{E} \times \operatorname{dom}\mathcal{E} \to \mathbb{R}$ **symmetric form** on H if densely defined and

(i) $\mathcal{E}(u,v) = \mathcal{E}(v,u)$, $\mathcal{E}(u,u) \geqslant 0$ $(u,v \in \operatorname{dom}\mathcal{E})$

(ii) $\mathcal{E}(u+v,w) = \mathcal{E}(u,v) + \mathcal{E}(v,w)$, $\mathcal{E}(\lambda u,v) = \lambda\mathcal{E}(u,v)$ $(u,v \in \operatorname{dom}\mathcal{E}, \lambda \in \mathbb{R})$

Definition

$\mathcal{E}$ on H **closed**, if $\operatorname{dom}\mathcal{E}$ is H-space w.r.t.

$$\langle u\,,\,v\rangle_{\operatorname{dom}\mathcal{E}} \coloneqq \mathcal{E}(u,v) + \langle u\,,\,v\rangle \qquad (u,v \in \operatorname{dom}\mathcal{E}).$$

General Dirichlet Forms

Definition (Dirichlet Form)

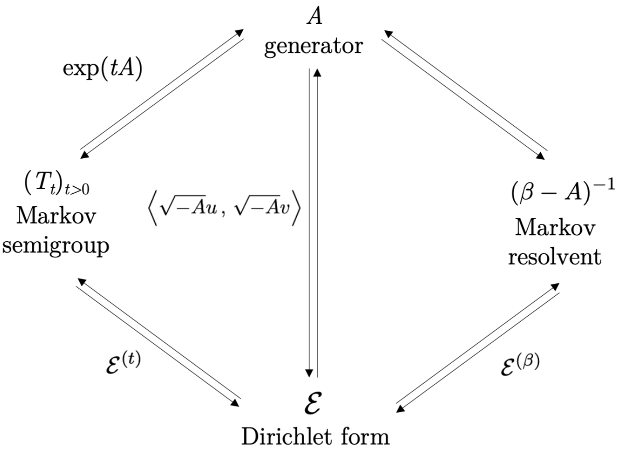
Closed symmetric form $\mathcal{E}$ on $L^2(X;m)$ is **Dirichlet form** if $0 \vee (u \wedge 1) \in \operatorname{dom}\mathcal{E}$ ($u \in \operatorname{dom}\mathcal{E}$) and

$$\mathcal{E}(0 \vee (u \wedge 1), 0 \vee (u \wedge 1)) \leqslant \mathcal{E}(u,u) \qquad (u \in \operatorname{dom}\mathcal{E}).$$

Definition

Symmetric form $\mathcal{E}$ on $L^2(X;m)$ **regular**, if $\operatorname{dom}\mathcal{E} \cap C_c(X)$ is dense in $\operatorname{dom}\mathcal{E}$ and $C_c(X)$.

One-to-one correspondence



Approximating Form

Lemma

Let $A \sim \mathcal{E}$. *For* $\beta > 0$ *define* $R_\beta := (\beta - A)^{-1}$ *and*

$$\mathcal{E}^{(\beta)}(u,v) := \beta \left\langle u - \beta R_\beta u \, , \, v \right\rangle \qquad (u, v \in H).$$

Then

$$\mathcal{E}(u,v) = \lim_{\beta \to \infty} \mathcal{E}^{(\beta)}(u,v) \qquad (u, v \in \operatorname{dom} \mathcal{E}).$$

Examples

Example 1

- For $(a,b) \subseteq \mathbb{R}$ consider

$$\mathcal{E}_1(u,v) = \int_{(a,b)} u(x)v(x)\,\mathrm{d}\lambda(x) + \int_{(a,b)} u'(x)v'(x)\,\mathrm{d}\lambda(x)$$

  on

$$\mathcal{D}_1 = W^{1,2}(a,b) \qquad \text{and} \qquad \mathcal{D}_2 = W_0^{1,2}(a,b).$$

- corresponding operator: $Au = -u + u''$
- Then:
  - $\big(\mathcal{E}_1, \mathcal{D}_1\big)$ is Dirichlet form
  - $\big(\mathcal{E}_1, \mathcal{D}_2\big)$ is *regular* Dirichlet form

Example 2

- For $D \subseteq \mathbb{R}^d$ open, consider

$$\mathcal{E}_2(u,v) = \sum_{i=1}^{d} \int_D \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} \, \mathrm{d}\lambda(x)$$

  on

$$\mathcal{D}_1 = W^{1,2}(D) \qquad \text{and} \qquad \mathcal{D}_2 = W_0^{1,2}(D).$$

- corresponding operator: $Au = \Delta u$
- Then:
  - $\left(\mathcal{E}_2, \mathcal{D}_1\right)$ is Dirichlet form
  - $\left(\mathcal{E}_2, \mathcal{D}_2\right)$ is *regular* Dirichlet form

Examples at a glance

$$\mathcal{E}_1(u,v) = \int_{(a,b)} u(x)v(x)\,\mathrm{d}\lambda(x) + \frac{1}{2}\int_{(a,b)} u'(x)v'(x)\,\mathrm{d}\lambda(x)$$

$$\mathcal{E}_2(u,v) = \frac{1}{2}\sum_{i=1}^{d}\int_D \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}\,\mathrm{d}\lambda(x)$$

$$\mathcal{Q}_{b,c}(f,g) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c(x)f(x)g(x)$$

Theorem of Beurling–Deny

Representation of Dirichlet forms

Theorem (Beurling–Deny)

$\mathcal{E}$ regular Dirichlet form on $L^2(X;m)$. Then

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} (u(x)-u(y))(v(x)-v(y))\,\mathrm{d}J(x,y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

for $u,v \in \operatorname{dom}\mathcal{E}\cap C_c(X)$.

The local part $\mathcal{E}^{(c)}$

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} (u(x)-u(y))(v(x)-v(y))\,\mathrm{d}J(x,y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

- $\mathcal{E}^{(c)}$ symmetric form with $\operatorname{dom}\mathcal{E}^{(c)} = \operatorname{dom}\mathcal{E} \cap C_c(X)$
- $\mathcal{E}^{(c)}(u,v) = 0$ for $u,v \in \operatorname{dom}\mathcal{E}^{(c)}$ and $v \equiv$ const. on nbhd of $\operatorname{supp} u$

The measures J and k

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,\mathrm{d}J(x,y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

- J unique positive symmetric Radon measure on $X \times X \setminus \Delta_X$
- k unique positive Radon measure on X

Technical Tools

Riesz–Markov

Theorem

Let $\varphi\colon C_c(X) \to \mathbb{R}$ be linear and positive. Then, there is unique Radon measure μ on $\mathcal{B}(X)$ with

$$\varphi(f) = \int_X f \,\mathrm{d}\mu \qquad \big(f \in C_c(X)\big).$$

In particular, $\mu(K) < \infty$ for $K \subseteq X$ compact.

Representation Lemma

Lemma

Let S be positive, symmetric operator on $L^2(X;m)$. Then, there is unique positive Radon measure σ on $\mathcal{B}(X \times X)$ with

$$\langle u\,,\,Sv\rangle = \int_{X\times X} u(x)v(y)\,\mathrm{d}\sigma(x,y) \qquad \big(u,v \in L^2(X;m)\big).$$

Proof of Representation Lemma

- $u \otimes v \colon X \times X \to \mathbb{R}$, $(x, y) \mapsto u(x)v(y)$
- Consider $u_i, v_i \in C_c(X)$ with $f \coloneqq \sum_{i=1}^{l} u_i \otimes v_i \geqslant 0$ ($i \in \{1, \ldots, l\}$, $l \in \mathbb{N}$)
- Idea: Show

$$\sum_{i=1}^{l} \langle u_i \, , \, S v_i \rangle \geqslant 0$$

- Define $K \coloneqq \bigcup_{i=1}^{l} \operatorname{supp} u_i$

Proof of Representation Lemma ctd

- Let $\varepsilon > 0$ and choose $\delta > 0$ according to uniform continuity of u_i

$$K = \bigcup_{x \in K} B(x, \delta)$$

- Choose finite subcover and make pw disjoint

$$K = \bigcup_{k=1}^{p} E_k, \qquad \big(E_k \in \mathcal{B}(X)\big),$$

  and points $\xi_k \in E_k$ $(1 \leqslant k \leqslant p)$
- Note: $\operatorname{diam} E_k \leqslant 2\delta$ for all $k \in \mathbb{N}$
- For $x \in K$ set $\widetilde{u}_i(x) \coloneqq \sum_{k=1}^{p} u_i(\xi_k) \mathbb{1}_{E_k}(x)$ $(1 \leqslant i \leqslant l)$

$$\implies \sup_{x \in K} \left| u_i(x) - \widetilde{u}_i(x) \right| \leqslant \varepsilon$$

Proof of Representation Lemma - Strategy

- Consider

$$f := \sum_{i=1}^{l} u_i \otimes v_i \geqslant 0$$

- Show that

$$\sum_{i=1}^{l} \langle u_i \,, Sv_i \rangle \geqslant 0$$

- Strategy: Approximate u_i by $\widetilde{u}_i$, show $\sum_{i=1}^{l} \langle \widetilde{u}_i \,, Sv_i \rangle \geqslant 0$ and

$$\left| \sum_{i=1}^{l} \langle u_i \,, Sv_i \rangle - \sum_{i=1}^{l} \langle \widetilde{u}_i \,, Sv_i \rangle \right| \xrightarrow{\varepsilon \to 0} 0$$

Proof of Representation Lemma ctd

$$
\begin{aligned}
\left| \sum_{i=1}^{l} \langle u_i \, , \, S v_i \rangle - \sum_{i=1}^{l} \langle \tilde{u}_i \, , \, S v_i \rangle \right| &\leqslant \sum_{i=1}^{l} \left| \langle u_i - \tilde{u}_i \, , \, S v_i) \rangle \right| \\
&= \sum_{i=1}^{l} \int_X \left| u_i - \tilde{u}_i \right| \cdot |S v_i| \, \mathrm{d}m \\
&\leqslant \varepsilon \sum_{i=1}^{l} \int_X \mathbb{1}_K \, |S v_i| \, \mathrm{d}m \\
&\leqslant \varepsilon \sum_{i=1}^{l} \langle \mathbb{1}_K \, , \, |S v_i| \rangle
\end{aligned}
$$

Proof of Representation Lemma ctd

- S is positive and $f = \sum_{i=1}^{l} u_i \otimes v_i \geqslant 0$:

$$\sum_{i=1}^{l} \langle \widetilde{u}_i \, , \, S v_i \rangle$$

Proof of Representation Lemma ctd

- S is positive and $f = \sum_{i=1}^{l} u_i \otimes v_i \geqslant 0$:

$$
\begin{aligned}
\sum_{i=1}^{l} \langle \widetilde{u}_i \, , \, S v_i \rangle &= \sum_{i=1}^{l} \sum_{k=1}^{p} \langle u_i(\xi_k) \mathbb{1}_{E_k} \, , \, S v_i \rangle \\
&= \sum_{k=1}^{p} \left\langle \mathbb{1}_{E_k} \, , \, \sum_{i=1}^{l} u_i(\xi_k) S v_i(\cdot) \right\rangle \\
&= \sum_{k=1}^{p} \left(\mathbb{1}_{E_k} , S \left[\sum_{i=1}^{l} u_i(\xi_k) v_i(\cdot) \right] \right) \\
&= \sum_{k=1}^{p} \langle \mathbb{1}_{E_k} \, , \, S f(\xi_k, \cdot) \rangle \geqslant 0
\end{aligned}
$$

Proof of Representation Lemma ctd

- Conclusion:

$$\sum_{i=1}^{l} \langle u_i \,,\, Sv_i \rangle \geqslant 0$$

- Define

$$\widetilde{C}_c(X \times X) \coloneqq \operatorname{span}\left\{u \otimes v \,\middle|\, u, v \in C_c(X)\right\} \subseteq C_c(X \times X)$$

and

$$I \colon \widetilde{C}_c(X \times X) \to \mathbb{R}, \qquad f \mapsto I(f) \coloneqq \sum_{i=1}^{l} \langle u_i \,,\, Sv_i \rangle$$

Proof of Representation Lemma ctd

- I positive functional on $\widetilde{C}_c(X \times X)$
- $I(f)$ independent of representation of $f \in \widetilde{C}_c(X \times X)$
- Extend I by Stone–Weierstraß on $C_c(X \times X)$
- By Riesz–Markov, there is measure $\sigma \colon \mathcal{B}(X \times X) \to [0, \infty]$ with

$$I(f) = \int_{X \times X} f(x, y) \, \mathrm{d}\sigma(x, y)$$

- Plugging in $f = u \otimes v$ $\big(u, v \in L^2(X; m)\big)$ yields

$$\langle u \, , \, Sv \rangle = I(f) = \int_{X \times X} u(x) v(y) \, \mathrm{d}\sigma(x, y)$$

$\square$

Vague convergence

Definition

Radon measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathcal{B}(X)$ **vaguely** converges to Radon measure μ on $\mathcal{B}(X)$ if

$$\int_X f \,\mathrm{d}\mu_n \xrightarrow{n\to\infty} \int_X f \,\mathrm{d}\mu \qquad \big(f \in C_c(X)\big).$$

Notation:

$$\mu_n \xrightarrow{v} \mu$$

Vague convergence

Theorem

Let $(\mu_n)_{n\in\mathbb{N}}$ *Radon measures on* $\mathcal{B}(X)$ *with*

$$\sup_{n\in\mathbb{N}} \mu_n(K) < \infty \qquad\qquad (K \subseteq X \text{ compact}).$$

Then, there is subsequence $(\mu_{n_i})_{i\in\mathbb{N}}$ *and Radon measure* μ *on* $\mathcal{B}(X)$ *s.t.*

$$\mu_{n_i} \xrightarrow{v} \mu.$$

Proof of the Theorem

- Construct increasing seq. $(U_k)_{k\in\mathbb{N}}$ consisting of rel. compact and open sets $U_k \subseteq X$ with $\overline{U_k} \subseteq U_{k+1}$ and
$$\bigcup_{k=1}^{\infty} U_k = X,$$
- μ_n induces positive functional on $C\left(\overline{U_k}\right)$ for $k, n \in \mathbb{N}$

Proof of the Theorem ctd

- $(\mu_n)_{n\in\mathbb{N}}$ uniformly bounded on $\overline{U_1}$
- Banach–Alaoglu: weak*-convergent subsequence $(\mu_{n,1})_{n\in\mathbb{N}}$ in $\left(C(\overline{U_1})\right)'$
- $(\mu_{n,1})_{n\in\mathbb{N}}$ uniformly bounded on $\overline{U_2}$
- Banach–Alaoglu: weak*-convergent subsequence $(\mu_{n,2})_{n\in\mathbb{N}}$ in $\left(C(\overline{U_2})\right)'$

$$\vdots$$

- Construct diagonal sequence $(\nu_n)_{n\in\mathbb{N}} \coloneqq (\mu_{n,n})_{n\in\mathbb{N}}$
- ν_n weak*-convergent in $\left(C(\overline{U_k})\right)'$ for all $k \in \mathbb{N}$

Proof of the Theorem ctd

- ν_n weak*-convergent in $\left(C(\overline{U}_k)\right)'$ for all $k \in \mathbb{N}$
- For $f \in C_c(X)$, there is $k \in \mathbb{N}$: $\operatorname{supp} f \subseteq \overline{U}_k$ and

$$\lim_{n\to\infty} \int_X f \,\mathrm{d}\nu_n = \lim_{n\to\infty} \int_{\overline{U}_k} f \,\mathrm{d}\nu_n =: \varphi(f).$$

- By Riesz–Markov, there is Radon measure μ, i.e.

$$\nu_n \xrightarrow{v} \mu.$$

$\square$

Proof of Beurling–Deny's Theorem

Uniqueness of J

- Recall

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} (u(x)-u(y))\,(v(x)-v(y))\,\mathrm{d}J(x,y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

  and $\mathcal{E}^{(c)}(u,v) = 0$ for $v \equiv$ const. on nbhd of $\operatorname{supp} u$.
- If $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$:

$$\mathcal{E}(u,v) = -2\int_{X\times X} u(x)v(y)\,\mathrm{d}J(x,y)$$

- $\operatorname{span}\{u \otimes v \mid u, v \in \operatorname{dom}\mathcal{E} \cap C_c(X), \operatorname{supp} u \cap \operatorname{supp} v = \emptyset\}$ dense subspace of $C_c(X \times X \setminus \Delta_X) \implies J$ unique

Uniqueness of k and $\mathcal{E}^{(c)}$

- Recall

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,\mathrm{d}J(x,y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

and $\mathcal{E}^{(c)}(u,v) = 0$ for $v \equiv$ const. on nbhd of $\operatorname{supp} u$.

- Thus,

$$\mathcal{E}(u,v) = \int_{X\times X\setminus\Delta_X} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,\mathrm{d}J(x,y) + \int_X u(x)\,\mathrm{d}k(x)$$

for $u, v \in \operatorname{dom}\mathcal{E} \cap C_c(X)$ with $v \equiv 1$ on nbhd of $\operatorname{supp} u$

$$\Longrightarrow k \text{ unique}$$

$$\Longrightarrow \mathcal{E}^{(c)} \text{ unique}$$

Construction of J

- Apply representation lemma to βR_β:

$$\langle u\,,\ \beta R_\beta v\rangle = \int_{X\times X} u(x)v(y)\,\mathrm{d}\sigma_\beta(x,y)$$

- $u, v \in \operatorname{dom}\mathcal{E} \cap C_c(X)$ with $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$:

$$\begin{aligned}\mathcal{E}(u,v) \xleftarrow{\beta\to\infty} \mathcal{E}^{(\beta)}(u,v) &= \beta\,\langle u - \beta R_\beta u\,,\ v\rangle \\ &= \beta\int_X u(x)v(x)\,\mathrm{d}m(x) - \beta\int_{X\times X} u(x)v(y)\,\mathrm{d}\sigma_\beta(x,y) \\ &= -\beta\int_{X\times X} u(x)v(y)\,\mathrm{d}\sigma_\beta(x,y)\end{aligned}$$

Construction of J ctd

- For $u, v \in \operatorname{dom}\mathcal{E} \cap C_c(X)$ with $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$:

$$-\beta \int_{X\times X} u(x)v(y)\,\mathrm{d}\sigma_\beta(x,y) \xrightarrow{\beta\to\infty} \mathcal{E}(u,v).$$

- $\left(\beta\sigma_\beta\right)_{\beta>0}$ uniformly bdd on compact subsets of $X \times X \setminus \Delta_X$
- Find subsequence converging vaguely to pos. Radon measure J:

$$\frac{\beta_n}{2}\sigma_{\beta_n} \xrightarrow{v} J \quad \text{on } X \times X \setminus \Delta_X$$

Construction of k

- ρ the metric on X
- For $\delta_l \downarrow 0$ and $E_l \uparrow X$, E_l relatively compact and open, define

$$\Gamma_l := \{(x,y) \in E_l \times E_l \mid \rho(x,y) \geqslant \delta_l\}$$

- If $u \in \operatorname{dom} \mathcal{E} \cap C_c(X)$ with $\operatorname{supp} u \subseteq E_l$:

$$\begin{aligned}\mathcal{E}^{(\beta_n)}(u,u) &= \beta_n \langle u - \beta_n R_{\beta_n} u \,,\, u \rangle \\ &= \beta_n \int_{E_l} u(x)^2 \big(1 - \beta_n R_{\beta_n} \mathbb{1}_{E_l}(x)\big) \,\mathrm{d}m(x) \\ &\quad + \frac{1}{2} \beta_n \int_{E_l \times E_l} \big(u(x) - u(y)\big)^2 \,\mathrm{d}\sigma_{\beta_n}(x,y)\end{aligned}$$

Construction of k ctd

$$
\begin{aligned}
&\beta_n \int_{E_l} u(x)^2 \left(1 - \beta_n R_{\beta_n} \mathbb{1}_{E_l}(x)\right) \mathrm{d}m(x) + \frac{1}{2}\beta_n \int_{E_l \times E_l} \left(u(x) - u(y)\right)^2 \mathrm{d}\sigma_{\beta_n}(x,y) \\
&= \beta_n \langle u \,,\, u \rangle - \beta_n \left\langle u^2 \,,\, \beta_n R_{\beta_n} \mathbb{1}_{E_l} \right\rangle + \frac{1}{2}\beta_n \int_{E_l \times E_l} \left(u(x) - u(y)\right)^2 \mathrm{d}\sigma_{\beta_n}(x,y) \\
&= \beta_n \langle u \,,\, u \rangle - \beta_n \left\langle u^2 \,,\, \beta_n R_{\beta_n} \mathbb{1}_{E_l} \right\rangle \\
&\quad + \beta_n \int_{X \times X} u(x)^2 \mathbb{1}_{E_l}(y) \,\mathrm{d}\sigma_{\beta_n}(x,y) - \beta_n \int_{X \times X} u(x) u(y) \,\mathrm{d}\sigma_{\beta_n}(x,y)
\end{aligned}
$$

Construction of k ctd

$$
\begin{aligned}
&= \beta_n \langle u\,,\, u\rangle - \beta_n \left\langle u^2\,,\, \beta_n R_{\beta_n} \mathbb{1}_{E_l} \right\rangle \\
&\quad + \beta_n \int_{X\times X} u(x)^2 \mathbb{1}_{E_l}(y)\,\mathrm{d}\sigma_{\beta_n}(x,y) - \beta_n \int_{X\times X} u(x)u(y)\,\mathrm{d}\sigma_{\beta_n}(x,y) \\
&= \beta_n \langle u\,,\, u\rangle - \beta_n \left\langle u^2\,,\, \beta_n R_{\beta_n} \mathbb{1}_{E_l} \right\rangle \\
&\quad + \beta_n \left\langle u^2\,,\, \beta_n R_{\beta_n} \mathbb{1}_{E_l} \right\rangle - \beta_n \int_{X\times X} u(x)u(y)\,\mathrm{d}\sigma_{\beta_n}(x,y) \\
&= \beta_n \langle u\,,\, u - \beta_n R_{\beta_n} u\rangle \\
&= \mathcal{E}^{(\beta_n)}(u,u)
\end{aligned}
$$

Construction of k ctd

- We have

$$\mathcal{E}^{(\beta_n)}(u,u) = \beta_n \int_{E_l} u(x)^2 \big(1-\beta_n R_{\beta_n} \mathbb{1}_{E_l}(x)\big)\, \mathrm{d}m(x) + \frac{1}{2}\beta_n \int_{E_l\times E_l} \big(u(x)-u(y)\big)^2 \,\mathrm{d}\sigma_{\beta_n}(x,y)$$

- Thus, if $\beta_n \to \infty$ there is pos. Radon measures k_l on E_l s.t.

$$\beta_n\big(1-\beta_n R_{\beta_n}\mathbb{1}_{E_l}\big)m \xrightarrow{v} k_l \quad \text{on } E_l$$

- Extend k_l to X by

$$k_l(A) := k_l(A\cap E_l) \qquad \big(A\in\mathcal{B}(X)\big)$$

- $(k_l)_l$ decreasing
- Hence, there is pos. Radon measure k on X s.t.

$$k_l \xrightarrow{v} k \quad \text{on } X$$

Construction of k and $\mathcal{E}^{(c)}$

- Combine everything! For every l and $u \in \operatorname{dom}\mathcal{E} \cap C_c(X)$ s.t. $\operatorname{supp} u \subseteq E_l$:

$$\mathcal{E}(u,u) = \lim_{n\to\infty} \frac{\beta_n}{2} \int_{X\times X\setminus\Gamma_l} (u(x)-u(y))^2 \,\mathrm{d}\sigma_{\beta_n}(x,y)$$
$$+ \int_{\Gamma_l} (u(x)-u(y))^2 \,\mathrm{d}J(x,y) + \int_{E_l} u(x)^2 \,\mathrm{d}k_l(x).$$

- $l \to \infty$ results in

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} (u(x)-u(y))(v(x)-v(y)) \,\mathrm{d}J(x,y) + \int_X u(x)v(x) \,\mathrm{d}k(x)$$

with

$$\mathcal{E}^{(c)}(u,v) := \lim_{l\to\infty} \lim_{n\to\infty} \frac{\beta_n}{2} \int_{\Gamma_l} (u(x)-u(y))(v(x)-v(y)) \,\mathrm{d}\sigma_{\beta_n}(x,y) \quad \square$$

Conclusion

$$\mathcal{E}_1(u,v) = \int_{[a,b]} u(x)v(x)\,\mathrm{d}\lambda(x) + \frac{1}{2}\int_{[a,b]} u'(x)v'(x)\,\mathrm{d}\lambda(x)$$

$$\mathcal{E}_2(u,v) = \frac{1}{2}\sum_{i=1}^{d}\int_D \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}\,\mathrm{d}\lambda(x)$$

$$\mathcal{Q}_{b,c}(f,g) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c(x)f(x)g(x)$$

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{X\times X\setminus\Delta_X} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,\mathrm{d}J(x,y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

Thank you for your attention!