ISEM 26, Project J

Surjectivity of formal graph Laplacian

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Setup

Let X be at most countable set and b be graph on X, we define the formal Laplacian

$$\mathcal{L} : \mathcal{F} \to C(X), \quad \mathcal{L}f(x) = \sum_{y \in X} b(x,y)\,(f(x) - f(y))$$

with

$$\mathcal{F} = \{f \in C(X) : \sum_{y \in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\},$$

where

$$C(X) = \{f : X \to \mathbb{K}\}\,.$$

Remark

If X is finite, the non-injectivity of $\mathcal{L}$ implies its non-surjectivity.

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Objective: **The surjectivity of the formal Laplacian?**

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The prodiscrete topology

The approach is based on the paper by Ceccherini-Silberstein, Coornaert, and Dodziuk 2012.

The prodiscrete topology on $C(X)$ is the product topology obtained by taking the discrete topology on each factor $\mathbb{K}$ of

$$C(X) = \prod_{x \in X} \mathbb{K}.$$

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The prodiscrete topology on $C(X)$ is the product topology obtained by taking the discrete topology on each factor $\mathbb{K}$ of

$$C(X) = \prod_{x \in X} \mathbb{K}.$$

$$\boxed{\text{The prodiscrete topology on } C(X) \text{ is metrizable.}}$$

Proof

Consider $(\Omega_n)_{n\in\mathbb{N}}$ a non-decreasing sequence of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} \Omega_n$, then the metric d on $C(X)$ defined by

$$d(f,g) = \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} \delta_n(f,g) \qquad f,g \in C(X),$$

where

$$\delta_n(f,g) := \begin{cases} 0 & \text{if } f = g \text{ on } \Omega_n \\ 1 & \text{otherwise.} \end{cases}$$

induces the prodiscrete topology on $C(X)$.

Surjectivity of the Laplacian

Theorem

Assume that (X, b) is infinite, connected and locally finite. Then the formal Laplacian $\mathcal{L}$ is surjective.

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Assume that (X, b) is infinite, connected and locally finite. Then the formal Laplacian $\mathcal{L}$ is surjective.

Plan of the proof:

We equip $C(X)$ with its prodiscrete topology, we need to prove

$\rightarrow$ The image of $\mathcal{L}$ is closed.

$\rightarrow$ The image of $\mathcal{L}$ is dense.

Therefore, $\operatorname{Im} \mathcal{L} = C(X)$.

Projective sequence of sets

Projective sequence of sets (X_n, u_{nm}) consists of a sequence $(X_n)_{n\in\mathbb{N}}$ of sets and maps $u_{nm} : X_m \to X_n$ for all $n \leq m$ with

1. u_{nn} is the identity map on X_n for all $n \in \mathbb{N}$.
2. $u_{nk} = u_{nm} \circ u_{mk}$ for all $n, m, k \in \mathbb{N}$ such that $n \leq m \leq k$.

$$X_0 \xleftarrow{u_0} X_1 \xleftarrow{u_1} X_2 \xleftarrow{u_2} X_3 \xleftarrow{u_3} \ldots$$

Mittag-Leffler Condition

The projective sequence (X_n, u_{nm}) satisfies the Mittag-Leffler condition if, for each $n \in \mathbb{N}$, the sequence $(u_{nm}(X_m))_{m \geq n}$ stabilizes, that is, there exists $m_0 \geq n$ such that

$$u_{nm}(X_m) = u_{nm_0}(X_{m_0}), \qquad \text{for} \quad \text{all } m \geq m_0.$$

Projective limit

The projective limit of the projective sequence (X_n, u_n) is defined as

$$\varprojlim X_n = \{(x_n)_{n\in\mathbb{N}} \mid x_n = u_{nm}(x_m) \text{ for all } n, m \in \mathbb{N} \text{ with } n \leq m\}.$$

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A Mittag-Leffler-type lemma

If (X_n, u_{nm}) is a projective sequence of nonempty sets which satisfies the Mittag-Leffler condition, then $\varprojlim X_n$ is not empty.

Proof

Assume that $X_n \neq \emptyset$ and consider $X'_n = \cap_{m \geq n} u_{nm}(X_m)$.

- The map u_{nm} induces by restriction $u'_{nm} : X'_m \to X'_n$, $\forall m \geq n$.

  Then (X'_n, u'_{nm}) is projective sequence

  and

  $$\varprojlim X'_n = \varprojlim X_n.$$

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- (X_n, u_{nm}) satisfies the Mittag-Leffler condition

$$\Longrightarrow \exists m_0 \geq n \text{ s.t. } u_{nm}(X_m) = u_{nm_0}(X_{m_0}) \text{ for all } m \geq m_0$$
$$\Longrightarrow X'_n = u_{nm_0}(X_{m_0}).$$

- Consider $u'_{n,n+1} : X'_{n+1} \to X'_n$ for each $n \in \mathbb{N}$.

Plan of the proof:

$$\begin{aligned} \varprojlim X_n \neq \emptyset &\iff \varprojlim X'_n \neq \emptyset \\ &\iff u'_{n,n+1} \text{ is surjective} \\ &\iff \forall x'_n \in X'_n,\ \exists x'_{n+1} \in X'_{n+1} \text{ s.t. } x'_n = u'_{n,n+1}(x'_{n+1}). \end{aligned}$$

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$$\iff \forall x'_n \in X'_n,\ \exists x'_{n+1} \in X'_{n+1} \text{ s.t. } x'_n = u'_{n,n+1}(x'_{n+1}).$$

Why is $u'_{n,n+1} : X'_{n+1} \to X'_n$ surjective?

Fix $n \in \mathbb{N}$ and consider $x'_n \in X'_n$. Then,

$$\exists p \geq n+1, \begin{cases} u_{nk}(X_k) = u_{np}(X_p) \\ u_{n+1,k}(X_k) = u_{n+1,p}(X_p), \end{cases} \qquad \forall k \geq p$$
$$\implies X'_n = u_{np}(X_p) \text{ and } X'_{n+1} = u_{n+1,p}(X_p)$$
$$\implies \exists x_p \in X_p \text{ s.t. } x'_n = u_{np}(x_p).$$

- Consider $x'_{n+1} = u_{n+1,p}(x_p) \in X'_{n+1}$.
  Then

$$\begin{aligned} u'_{n,n+1}(x'_{n+1}) &= u_{n,n+1}(x'_{n+1}) \\ &= u_{n,n+1} \circ u_{n+1,p}(x_p) \\ &= u_{np}(x_p) \\ &= x'_n. \end{aligned}$$

  Therefore,

$$\varprojlim X_n \neq \emptyset.$$

The image of $\mathcal{L}$ is closed

Theorem

The image of the formal Laplacian $\mathcal{L}$ is closed in the prodiscrete topology.

Let us fix $x_0 \in X$. For each $n \in \mathbb{N}$, consider

$$B_n = \{x \in X \mid d_G(x_0, x) \leq n\}.$$

Moreover, $\mathcal{L}$ induces by restriction a linear map

$$\mathcal{L}^{(n)} : C(B_{n+1}) \to C(B_n), \qquad \forall n \in \mathbb{N},$$

with

$$\mathcal{L}^{(n)} f_n(x) = \sum_{y \in B_n} b(x, y)\,(f_n(x) - f_n(y)), \text{ for all } f_n \in C(B_{n+1}) \text{ and } x \in B_n.$$

Let $g \in C(X)$ in the closure of $\mathcal{L}(C(X))$. Then

$$\forall n \in \mathbb{N}, \exists f_n \in C(X) \text{ s.t. } \mathcal{L} f_n = g \qquad \text{on } B_n \text{ for every } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, we define the affine subspace $X_n \subset C(B_{n+1})$ s.t.

$$X_n = (\mathcal{L}^{(n)})^{-1}\left(g_{|B_n}\right).$$

Observe

- $X_n \neq \emptyset$ since $f_n|B_{n+1} \in X_n$.
- X_n is a finite-dimensional affine subspace.

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- X_n is a finite-dimensional affine subspace.

$C(B_{m+1}) \to C(B_{n+1})$ induces by restriction an affine map

$$u_{nm} : X_m \to X_n,\ f \mapsto f|_{B_{n+1}},\ \text{for all}\ \ m \geq n$$

$$\Downarrow$$

(X_n, u_{nm}) is a projective sequence.

Then, $u_{nm}(X_m)$ is a non-increasing sequence of affine subspaces

$$\Downarrow$$

(X_n, u_{nm}) satisfies Mittag-Leffler condition

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$$\varprojlim X_n \neq \emptyset.$$

Choose $(h_n)_{n\in\mathbb{N}} \in \varprojlim X_n$. We have $h_n \in C(B_{n+1})$. Furthermore, one has

$$h_{n+1} = h_n \qquad \text{on } \ B_{n+1} \ \text{ for every } \ n \in \mathbb{N}.$$

As $X = \bigcup_{n\in\mathbb{N}} B_{n+1}$, then

there exists a unique $f \in C(X)$ s.t. $f_{|B_{n+1}} = h_n, \forall n \in \mathbb{N}$

Since $h_n \in X_n$, we have

$$(\mathcal{L}f)_{|B_n} = \mathcal{L}^{(n)}(h_n) = g_{|B_n}, \qquad \text{for all } n.$$

As $X = \bigcup_{n \in \mathbb{N}} B_n$, it follows that

$$\mathcal{L}f = g.$$

The image of $\mathcal{L}$ is dense

Let $F_n = \{f \in C(X) \mid \operatorname{supp} f \subset B_n\}$ and the linear map $\Delta_n : F_n \to F_n$

$$\Delta_n(f)(x) = \begin{cases} \mathcal{L}f(x) & \text{if } x \in B_n \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } f \in F_n \text{ and } x \in X.$$

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Why Δ_n is injective?

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Why Δ_n is injective?

$$\begin{aligned} f \in \operatorname{Ker} \Delta_n &\Longrightarrow \Delta_n(f) = 0 \\ &\Longrightarrow |f(x)| \leq \frac{1}{deg(x)} \sum_{y \in X} b(x,y)|f(y)|, \ \forall x \in B_n. \end{aligned}$$

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$$\begin{aligned} \text{if } x \in B_n \text{ s.t. } |f(x)| = \max|f| = M &\implies |f(y)| = M, \quad \forall y \in X \text{ with } x \sim y \\ &\implies |f| = cste \qquad \text{on } \ B_{n+1} \\ &\implies f = 0. \end{aligned}$$

Since F_n is finite-dimensional, it follows that

$$\Delta_n \text{ is injective}$$
$$\Downarrow$$
$$\Delta_n \text{ is surjective}$$

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$$\text{for all } g \in C(X)\ \exists f \in F_n \text{ s.t. } \ \mathcal{L}f = g \text{ on } \ B_n$$
$$\Downarrow$$
$$\operatorname{Im}\mathcal{L} \text{ is dense in } \ C(X) \text{ in the prodiscrete topology.}$$

Setup

We equip the space

$$C(X) = \{f : X \to \mathbb{K}\}$$

with the following family of seminorms p_K, $K \subseteq X$ finite

$$p_K : C(X) \to [0, \infty)\,, \qquad p_K(f) = \sum_{x \in K} |f(x)|.$$

Continuity of the Laplacian

Lemma

The formal Laplacian $\mathcal{L}$ is continuous if, and only if, (X, b) is locally finite. In particular $\mathcal{F} = C(X)$ holds.

Assume that (X, b) is locally finite, one has

$$\mathcal{L}f(x) = \sum_{y \in X} b(x, y)(f(x) - f(y)), \text{ for all } f \in C(X) \text{ and } x \in X.$$

Clearly, if $f_n \to f$ pointwise then one can get

$$\mathcal{L}f_n \to \mathcal{L}f \qquad \text{pointwise.}$$

Proof

Assume that $\mathcal{L}$ continuous, i.e. for all $y \in X$ there exists $\{x_1, ..., x_n\} \subseteq X$ and $C > 0$ such that

$$|\mathcal{L}f(y)| \leq C \sum_{k=1}^{n} |f(x_k)|, \qquad \text{for all } f \in \mathcal{F}.$$

Proof

Assume that $\mathcal{L}$ continuous, i.e. for all $y \in X$ there exists $\{x_1, ..., x_n\} \subseteq X$ and $C > 0$ such that

$$|\mathcal{L}f(y)| \leq C \sum_{k=1}^{n} |f(x_k)|, \qquad \text{for all } f \in \mathcal{F}.$$

Let us prove that(X, b) is locally finite, that is,

$$\{z \in X : b(y, z) \neq 0\} \subseteq \{x_1, ..., x_n\}.$$

Assume there exists $z_0 \notin \{x_1, .., x_n\}$ such that $b(y, z_0) > 0$. We define

$$f_{z_0}(x) := \begin{cases} 0, & x \neq z_0 \\ 1, & x = z_0. \end{cases}$$

Then

$$0 < |b(y, z_0)| = |\mathcal{L}f_{z_0}(y)| \leq C \sum_{k=1}^{n} |f_{z_0}(x_k)| = 0.$$

Hence (X, b) locally finite.

Dual Operator

The dual space of $C(X)$ can be identified with $C_c(X)$ by the isomorphism

$$C_c(X) \to C(X)', \qquad \varphi \mapsto L_\varphi$$

with
$$L_\varphi(f) = \sum_{x \in X} f(x)\varphi(x).$$

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For $f \in C(X)$ and $\varphi \in C_c(X)$, Green's formula implies

$$\begin{aligned}(\mathcal{L}f, \varphi) &= \sum_{x\in X} \mathcal{L}f(x)\varphi(x) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi(x)-\varphi(y))(f(x)-f(y)) \\ &= \sum_{x\in X} f(x)\mathcal{L}'\varphi(x) \\ &= (f, \mathcal{L}'\varphi).\end{aligned}$$

Hence
$$\mathcal{L}' = \mathcal{L}|_{C_c(X)}.$$

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This approach can be found in Koberstein and Schmidt 2020.
Moreover see Kalmes 2016.

Theorem

Let $A : C(X) \to C(X)$ be a continuous operator. The following assertions are equivalent.

(i) *A is surjective.*

(ii) *The adjoint operator $A' : C_c(X) \to C_c(X)$ is injective.*

- $(i) \Rightarrow (ii)$: Let $\varphi \in C_c(X)$ with $A'\varphi = 0$. Then

$$0 = (f, A'\varphi) = (Af, \varphi)$$

for all $f \in C(X)$. Since A is surjective

$$(g, \varphi) = 0 \text{ for all } g \in C(X).$$

- Given two Fréchet spaces E, F. Hahn-Banach theorem: If $A : E \to F$ is a continuous linear map, then

$$A \text{ has dense image } \Leftrightarrow A' \text{ is injective.}$$

We will show

$$A' \text{ injective } \Rightarrow A \text{ open.}$$

For $K \subseteq X$ finite we define

$$U_K := \{f \in C(X) : p_K(f) \leq 1\}.$$

By scaling and linearity openness of A means: For each $\emptyset \neq K \subseteq X$ finite there exist $\varepsilon > 0$ and $\emptyset \neq K' \subseteq X$ finite such that

$$A(U_K) \supseteq \varepsilon U_{K'}.$$

This implies surjectivity of A with

$$AC(X) = \bigcup_{n \in \mathbb{N}} nAU_K \supseteq \bigcup_{n \in \mathbb{N}} n\varepsilon U_{K'} = C(X).$$

Lemma

Let $A : C(X) \to C(X)$ be continuous and assume for every $\emptyset \neq K \subseteq X$ finite there exist $\varepsilon > 0$ and $\emptyset \neq K' \subseteq X$ finite such that

$$\overline{A(U_K)} \supseteq \varepsilon U_{K'}.$$

Then the map A is open.

Theorem

Let A' be injective on $C_c(X)$. For every finite $\emptyset \neq K \subseteq X$ there exist $\varepsilon > 0$ and $\emptyset \neq K' \subseteq X$ finite such that

$$\overline{A(U_K)} \supseteq \varepsilon U_{K'}.$$

Proof of the theorem

The bipolar theorem implies

$$\overline{A(U_K)} = \{f \in C(X) : |(f,\varphi)| \leq 1 \text{ for all } \varphi \in A(U_K)^\circ\}$$
$$= \{f \in C(X) : |(f,\varphi)| \leq 1 \text{ for all } \varphi \text{ s.t. } |(g, A'\varphi)| \leq 1 \text{ for all } g \in U_K\}$$

By the definition of the dual pairing between $C(X)$ and $C_c(X)$ we have

$$|\sum_{x\in X} g(x)A'\varphi(x)| = |(g, A'\varphi)| \leq 1 \text{ for all } g \in U_K$$

if, and only if

$$\operatorname{supp} A'\varphi \subseteq K \quad \text{and} \quad \max_{x\in K} |A'\varphi(x)| \leq 1.$$

Proof of the theorem

We will show: There exists $K' \subseteq X$ finite and $C > 0$ such that for all $\varphi \in C_c(X)$ with

$$\operatorname{supp} A'\varphi \subseteq K \quad \text{and} \quad \max_{x\in K} |A'\varphi(x)| \leq 1$$

we have $\operatorname{supp}\varphi \subseteq K'$ and $\max_{x\in K'} |\varphi(x)| \leq C$.

For such φ and $f \in \varepsilon U_{K'}$ we have

$$|(f,\varphi)| = \left|\sum_{x\in X} \varphi(x) f(x)\right| \leq \sum_{x\in K'} |\varphi(x)||f(x)| \leq C p_{K'}(f) \leq C\varepsilon.$$

Proof of the theorem

We define

$$V := \{\varphi \in C_c(X) : \operatorname{supp} A'\varphi \subseteq K\}.$$

Since

$$A' \text{ injective} \Rightarrow V \text{ is finite-dimensional}$$

and

$$A'|_V : V \to A'(V)$$

is a vector space isomorphism. Hence we can choose a finite basis $(\varphi_i)_{i\in I}$ of V and set

$$K' = \cup_i \operatorname{supp} \varphi_i.$$

For $\varphi \in V$ this implies $\operatorname{supp} \varphi \subseteq K'$.

Proof of the theorem

Since linear operators on finite-dimensional normed spaces are always continuous, the inverse

$$(A'|_V)^{-1} : A'(V) \to V$$

is continuous with respect to $\|.\|_\infty$. Hence there is some constant $C > 0$ such that

$$\|\varphi\|_\infty = \|(A'|_V)^{-1}(A'\varphi)\|_\infty \leq C$$

for all $\varphi \in V$ with $\max_{x \in X} |A'\varphi(x)| \leq 1$. Alltogether:

$$\operatorname{supp} A'\varphi \subseteq K \quad \text{and} \quad \max_{x \in K} |A'\varphi(x)| \leq 1$$

implies

$$\operatorname{supp} \varphi \subseteq K' \quad \text{and} \quad \max_{x \in K'} |\varphi(x)| \leq C.$$

$\square$

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Theorem

If (X, b) is locally finite and all connected components are infinite, then $\mathcal{L}$ is surjective.

Proof.

We know

$$\mathcal{L} \text{ is surjective } \Leftrightarrow \mathcal{L}|_{C_c(X)} \text{ is injective.}$$

Let $\varphi \in C_c(X)$ with $\mathcal{L}\varphi = 0$. Green's formula:

$$0 = (\mathcal{L}\varphi, \varphi) = \frac{1}{2} \sum_{x,y \in X} b(x,y)(\varphi(x) - \varphi(y))^2.$$

Since all connected components are infinite and φ is finitely supported,

$$\varphi = 0$$

follows. □

Finite connected components

Let (X, b) be locally finite with one finite connected component K. We define

$$\delta_K(x) = \begin{cases} 1 & \text{if } x \in K; \\ 0 & \text{else.} \end{cases}$$

$\Rightarrow \delta_K \in C_c(X)$ and $\mathcal{L}\delta_K = 0$. Hence $\mathcal{L}$ can not be surjective.

A non locally finite graph

Let $X = \mathbb{N}_0$ and let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers with $\sum_n b_n < \infty$. We define $b : \mathbb{N}_0 \times \mathbb{N}_0 \to [0, \infty)$ by

$$b(n,0) = b(0,n) = b_n, \ \ n \in \mathbb{N} \quad \text{and} \quad b(n,m) = 0, \ \text{otherwise.}$$

(X, b) has one infinite connected component and is not locally finite.

Claim: The Laplacian on (X, b) is not surjective.

Example: Not locally finite

- (X, b) not locally finite we have

$$\mathcal{F} = \{f \in C(\mathbb{N}_0) : \sum_n b_n |f(n)| < \infty\}.$$

Let $f = \mathcal{L}g$ for $g \in \mathcal{F}$.

- For $n \in \mathbb{N}$:

$$f(n) = \mathcal{L}g(n) = b_n(g(n) - g(0)).$$

Since $g \in \mathcal{F} \Rightarrow f \in \ell^1(\mathbb{N}_0)$.

For $n = 0$:

$$f(0) = \mathcal{L}g(0) = \sum_{n \in \mathbb{N}} b_n(g(0) - g(n)).$$

Example: Not locally finite

- Substituting the second identity into the first:

$$f(0) = -\sum_{n\in\mathbb{N}} f(n).$$

We conclude

$$\mathcal{L}C(\mathbb{N}_0) \subseteq \{f \in \ell^1(\mathbb{N}_0) : f(0) = -\sum_{n\in\mathbb{N}} f(n)\} \subsetneq C(\mathbb{N}_0).$$

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Setup

We consider on

$$\mathcal{F} = \{f \in C(X) : \sum_{y \in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\}.$$

the locally convex topology generated by the family of seminorms q_K, $K \subseteq X$ finite, with

$$q_K(f) = \sum_{x \in K} \sum_{y \in X} b(x,y)|f(y)|.$$

If the graph has no isolated points, this topology is Hausdorff and $\mathcal{F}$ becomes a Fréchet space.
$\Rightarrow \mathcal{L} : \mathcal{F} \to C(X)$ is continuous with respect to this topology.

Setup

$$\deg_K : X \to [0, \infty), \quad \deg_K(x) = \sum_{y \in K} b(x, y).$$

Lemma

$$\mathcal{G} := \{\varphi \in C(X) : \exists M \geq 0,\ K \subseteq X \textit{ finite s.t. } |\varphi| \leq M \deg_K\}.$$

Then the map

$$\mathcal{G} \to \mathcal{F}', \quad \varphi \mapsto L_\varphi$$

with

$$L_\varphi(f) = \sum_{x \in X} f(x)\varphi(x)$$

is an isomorphism.

Remark

If (X, b) is locally finite, then $\mathcal{G} = C_c(X)$.

Setup

Proof.

$L_\varphi \in \mathcal{F}'$ for all $\varphi \in \mathcal{G}$: For $f \in \mathcal{F}$ we compute

$$\begin{aligned} |L_\varphi(f)| = |\sum_{x\in X} f(x)\varphi(x)| &\leq M \sum_{x\in X} |f(x)| \deg_K(x) \\ &\leq M \sum_{y\in K} \sum_{x\in X} b(x,y)|f(x)| = Mq_K(f) < \infty. \end{aligned}$$

Setup

Proof.

Surjectivity: Let $L \in \mathcal{F}'$ and $\varphi : X \to \mathbb{K}$ with $\varphi(x) = L(1_x)$.

$$L \text{ linear } \Rightarrow L(f) = \sum_{x \in X} f(x)\varphi(x) \quad \text{for } f \in C_c(X).$$

Since L is continuous there exists $K \subseteq X$ finite and $M \geq 0$ such that

$$|L(f)| \leq M q_K(f) = M \sum_{x \in K} \sum_{y \in X} b(x,y)|f(y)|$$

for $f \in \mathcal{F}$. Hence

$$|\varphi(y)| = |L(1_y)| \leq M \sum_{x \in K} b(x,y) = M \deg_K(y), \quad y \in X.$$

Setup

Proof.

We conclude
$$\varphi \in \mathcal{G}.$$
Since

- $C_c(X)$ is dense in $\mathcal{F}$;
- L and L_φ are continuous;
- $L = L_\varphi$ on $C_c(X)$, we get

$$L = L_\varphi \quad \text{on } \mathcal{F}.$$

□

Application to Laplacian

We define

$$N(K) := \{y \in X \,|\, \text{there exists } x \in K \text{ s.t. } y \sim x\}.$$

Theorem

The following assertions are equivalent:

(i) $\mathcal{L}$ *is surjective.*

(ii) (a) $\mathcal{L}|_{C_c(X)} = \mathcal{L}' : C_c(X) \to \mathcal{G}$ *is injective.*

(b) *For every finite* $K \subseteq X$ *there exists a finite* $H \subseteq X$ *such that for all* $\varphi \in C_c(X)$ *the following holds:* $\operatorname{supp} \mathcal{L}\varphi \subseteq N(K)$ *implies* $\operatorname{supp} \varphi \subseteq H$.

Remark

- If (X, b) is locally finite, then $N(K)$ is finite and the set

$$M := \{\varphi \in C_c(X) : \operatorname{supp} \mathcal{L}\varphi \subseteq N(K)\}$$

  is finite dimensional.
- If, additionally, $\mathcal{L}|_{C_c(X)}$ is injective (all connected components are infinite), the preimage of M under $\mathcal{L}$ is finite dimensional.

Hence, $M \subseteq C(H)$ for some finite $H \subseteq X$.

Proof.

$(ii) \Rightarrow (i)$: We want to show

$$\mathcal{L} : \mathcal{F} \to C(X)$$

is surjective. Recall

$$p_K(f) = \sum_{y \in K} |f(y)| \quad \text{and} \quad q_K(f) = \sum_{x \in K} \sum_{y \in X} b(x, y)|f(y)|.$$

We consider for $K \subseteq X$ finite:

$$V_K := \{f \in C(X) : q_K(f) \leq 1\}$$

and

$$U_K = \{f \in C(X) : p_K(f) \leq 1\}.$$

Proof.

As above it suffices to show: For every finite $K \subseteq X$ there exist $\varepsilon > 0$ and $K' \subseteq X$ finite such that

$$\overline{\mathcal{L}(V_K)} \supseteq \varepsilon U_{K'}.$$

With the bipolar theorem:

$$\overline{\mathcal{L}(U_K)} = \{f \in C(X) : |(f, \varphi)| \leq 1 \ \forall \varphi \in C_c(X) \text{ s.t. } |(g, \mathcal{L}\varphi)| \leq 1 \ \forall g \in V_K\}$$

Moreover

$$|(g, \mathcal{L}\varphi)| \leq 1 \text{ for all } g \in V_K$$

if and only if

$$\operatorname{supp} \mathcal{L}\varphi \subseteq N(K) \quad \text{and} \quad \sup_{x \in N(K)} \frac{|\mathcal{L}\varphi(x)|}{\deg_K(x)} \leq 1.$$

Proof.

We define

$$V := \{\varphi \in C_c(X) : \operatorname{supp} \mathcal{L}\varphi \subseteq N(K)\}.$$

By assumption (ii)(b):

$$V \subseteq \{\varphi \in C_c(X) : \operatorname{supp} \varphi \subseteq H\}$$

for some finite $H \subseteq X$. Hence, V is finite-dimensional. Since $\mathcal{L}|_{C_c(X)}$ is injective,

$$\mathcal{L}|_V : V \to \mathcal{L}(V)$$

is a continuous bijection (with respect to any norm).

Proof.

We consider

$$\|\psi\|_{\mathcal{L}(V)} := \sup_{x \in N(K)} \frac{|\psi(x)|}{deg_K(x)} \text{ for } \psi \in \mathcal{L}(V)$$

and $\|\cdot\|_\infty$ on V. As before:

$$(\mathcal{L}|_V)^{-1} : \mathcal{L}(V) \to V$$

is continuous with respect to these norms. Altogether: $\exists C > 0$ such that

$$\|\varphi\|_\infty = \|(\mathcal{L}|_V)^{-1}(\mathcal{L}\varphi)\|_\infty \leq C$$

and $\operatorname{supp} \varphi \subseteq H$ for all $\varphi \in C_c(X)$ with

$$\operatorname{supp} \mathcal{L}\varphi \subseteq N(K) \quad \text{and} \quad \sup_{x \in N(K)} \frac{|\mathcal{L}\varphi(x)|}{\deg_K(x)} \leq 1.$$

Examples

Complete graph

Suppose (X, b) infinite complete graph, i.e.,

$$x \sim y \text{ for all } x \neq y.$$

If $x \neq y$, then

$$\operatorname{supp} \mathcal{L}\varphi \subseteq X = N(\{x, y\}) \text{ for all } \varphi \in C_c(X).$$

Infinite star graph

Suppose (X, b) is an infinite star graph (there exists $o \in X$ such that $N(\{o\}) = X \setminus \{o\}$). If $x \sim o$, then

$$\operatorname{supp} \mathcal{L}\varphi \subseteq X = N(\{o, x\}) \text{ for all } \varphi \in C_c(X).$$

Examples

Extended infinite star graph

Let $X = (\mathbb{N} \times \mathbb{N}) \cup \{0\}$ and consider a symmetric graph weight

$$b : X \times X \to [0, \infty)$$

such that

$$b(0, (i, 1)) > 0 \quad \text{for all } i \in \mathbb{N}$$

and

$$b((i, j), (k, l)) > 0 \Longleftrightarrow i = k \quad \text{and} \quad |j - l| = 1.$$

Thus

$$R_i := \{(i, n) : n \in \mathbb{N}\}$$

is the i-th ray.

References

Ceccherini-Silberstein, Tullio, Michel Coornaert, and Józef Dodziuk (2012). "The surjectivity of the combinatorial Laplacian on infinite graphs". In: L'Enseignement Math ́ematique 58, pp. 125–130.

Kalmes, Thomas (2016). "A short remark on the surjectivity of the combinatorial Laplacian on infinite graphs". In: Rev. R. Acad. Cienc. Exactas F ́ıs. Nat. Ser. A Math. RACSAM 110, pp. 695–698.

Koberstein, J. and M. Schmidt (2020). "A note on the surjectivity of operators on vector bundles over discrete spaces". In: Arch. Math. 114, pp. 313–329.