Laplacians on infinite graphs: from continuous to discrete and back

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Workshop Internet Seminar 26
Project I (Supervision: Noema Nicolussi)

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Motivation

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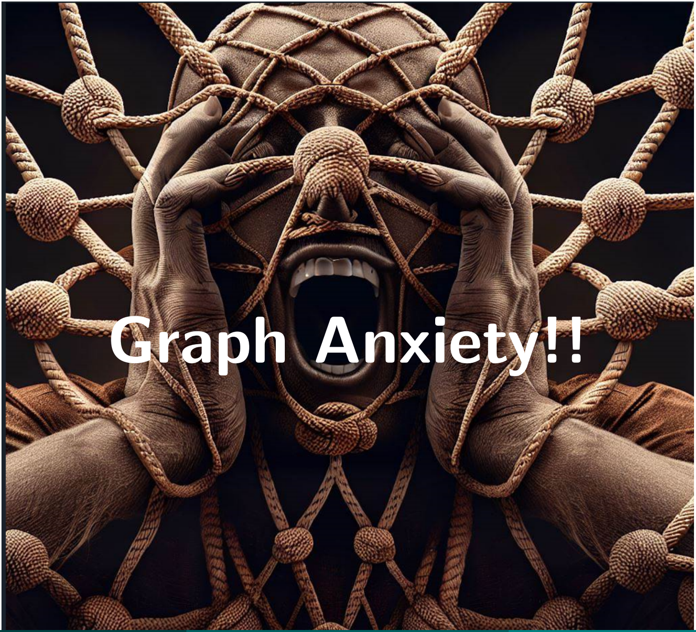
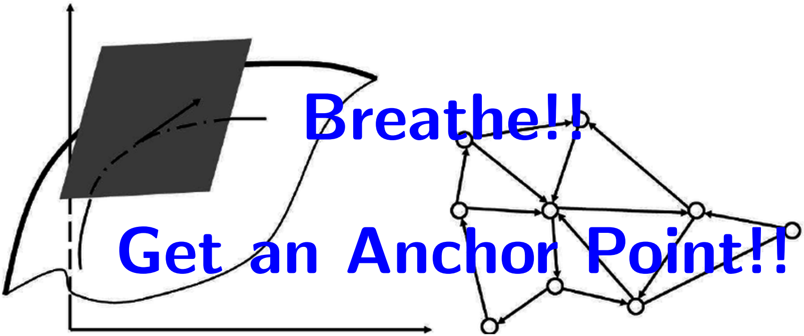
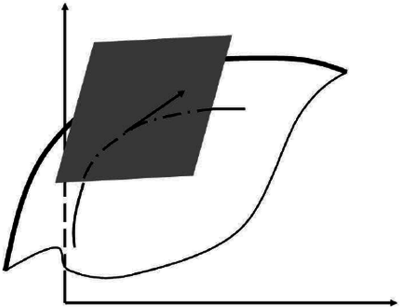


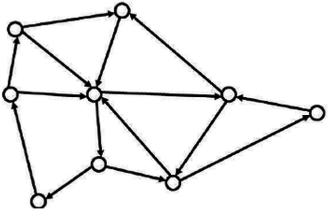


Breathe!!

Get an Anchor Point!!

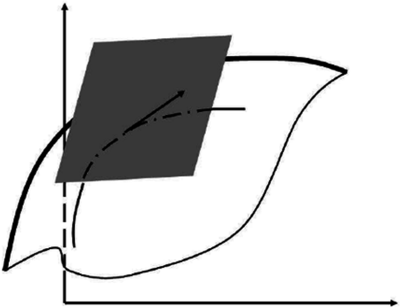
Motivation



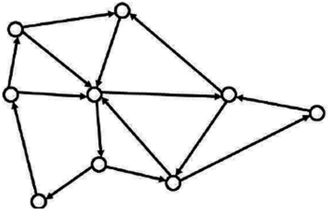


Motivation

$$\Delta f$$



$$-\frac{1}{\mu(x)}\frac{d}{dx}\nu(x)\frac{d}{dx}f$$



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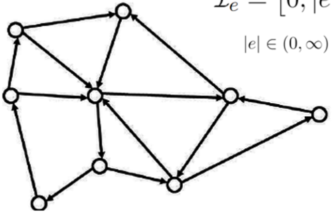


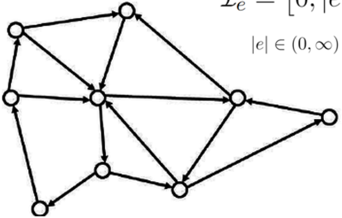
$$\mathcal{I}_e = \big[0, |e|\big],$$

$$|e| \in (0, \infty)$$

Quantum Graphs

Metric Laplacian

$$-\frac{1}{\mu(x)}\frac{d}{dx}\nu(x)\frac{d}{dx}f \qquad -\frac{1}{\mu(e)}\frac{d}{dx}\nu(e)\frac{d}{dx}f$$



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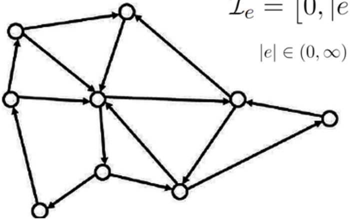

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Quantum Graphs **Metric Graphs**

Motivation

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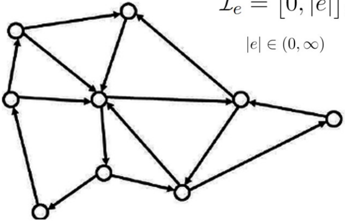
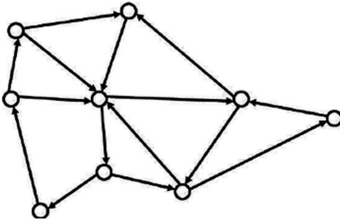


Metric Graphs

Discrete Laplacian

$$\frac{1}{m(v)}\sum_{u\in\mathcal{V}} b(v,u)\big(f(v)-f(u)\big)$$



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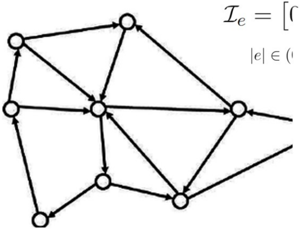
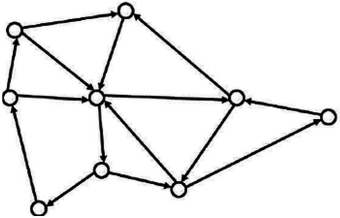


Project I



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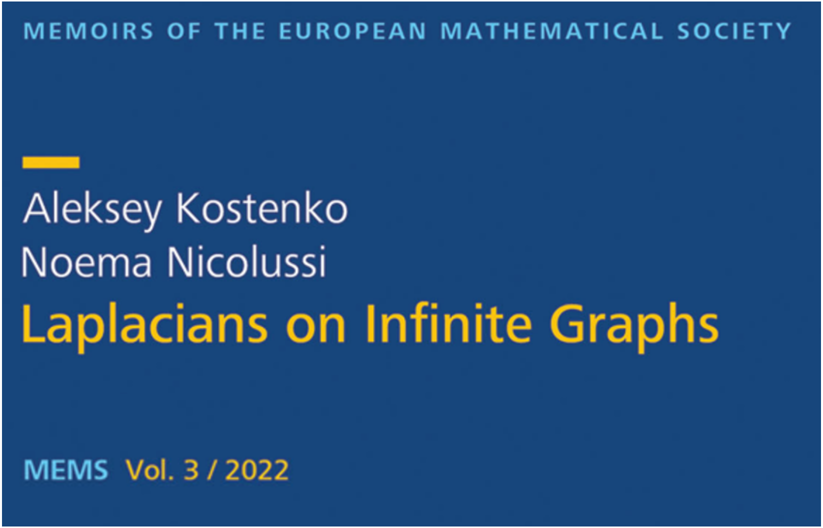
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Laplacians on Infinite Graphs
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Basic graph definitions

Combinatorial Graph

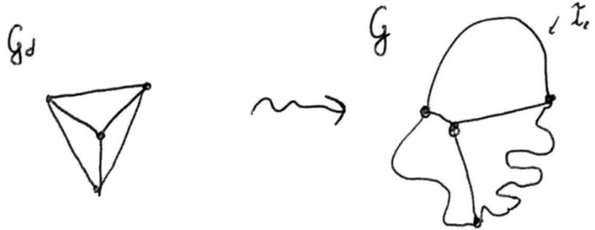
We consider *combinatorial graphs* $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$:

- $\mathcal{V}$ is countable and called the vertex set
- $\mathcal{E}$ is called the edge set
- $\deg(v) < \infty \quad \forall v \in \mathcal{V}$, i.e., locally finite

Metric graphs

Let $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ be a combinatorial graph.
If we assign to each edge $e \in \mathcal{E}$ a length $|e| \in (0, +\infty)$, then $\mathcal{G} = (\mathcal{G}_d, |\cdot|) = (\mathcal{V}, \mathcal{E}, |\cdot|)$ is called a metric graph

One can view $\mathcal{G}$ as a topological space by "glueing together intervals"



Replace each edge $e \in \mathcal{E}$ by $\mathcal{I}_e = [0, |e|]$

Basic graph notation and definitions

Discrete graph

Let $\mathcal{V}$ be a finite or countable set and $m \colon \mathcal{V} \to (0, +\infty)$ a measure of full support on $\mathcal{V}$.
A discrete graph on $(\mathcal{V}, m)$ is a pair (b, c) consisting of a function $b \colon \mathcal{V} \times \mathcal{V} \to [0, +\infty)$ satisfying:

i) Symmetry
ii) Vanishing diagonal : $b(v, v) = 0$ for all $v \in \mathcal{V}$
iii) Local summability : $\sum_{v \in \mathcal{V}} b(u, v) < \infty$ for all $u \in \mathcal{V}$

and $c \colon \mathcal{V} \to [0, +\infty)$

Discrete Laplacian

Energy form $\mathfrak{q}$

With each graph $(b, 0)$ one can associate the energy form $\mathfrak{q} := C(\mathcal{V}) \to [0, \infty]$ defined by

$$\mathfrak{q}[f] := \frac{1}{2} \sum_{u,v \in \mathcal{V}} b(v, u)|f(v) - f(u)|^2.$$

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Let $\mathcal{D}(\mathfrak{q}) :=$ set of finite energy functions.
Introduce the graph norm

$$\|f\|_{\mathfrak{q}}^2 := \mathfrak{q}(f) + \|f\|_{\ell^2(\mathcal{V},m)}^2$$

for all $f \in \mathcal{D} \cap \ell^2(\mathcal{V}, m) =: \mathrm{dom}(\mathfrak{q})$.

Discrete Laplacian

Dirichlet Forms

$$\mathfrak{q}_N := \mathfrak{q} \upharpoonright \mathrm{dom}(\mathfrak{q})$$
$$\mathfrak{q}_D := \mathfrak{q} \upharpoonright \mathrm{dom}(\mathfrak{q}_D)$$

with $\mathrm{dom}(\mathfrak{q}_D) := \overline{C_c(\mathcal{V})}^{\|\cdot\|_{\mathfrak{q}}}$

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Using the representation theorems for quadratic forms one can associate in $\ell^2(\mathcal{V}, m)$ the self-adjoint operators h_D, h_N

Formal Laplacian L

$$(Lf)(v) := \frac{1}{m(v)} \left(\sum_{u \in \mathcal{V}} b(v,u)(f(v) - f(u)) \right),$$

where $v \in \mathcal{V}, f \in \mathcal{F}_b(\mathcal{V})$

$$\mathcal{F}_b(\mathcal{V}) = \left\{ f \in C(\mathcal{V}) \;\middle|\; \sum_{u \in \mathcal{V}} b(v,u)|f(u)| < \infty, \forall v \in \mathcal{V} \right\}$$

$$\begin{aligned} h &:= L \restriction \mathrm{dom}(h), \\ \mathrm{dom}(h) &:= \{ f \in \mathcal{F}_b(\mathcal{V}) \cap \ell^2(\mathcal{V}; m) \mid Lf \in \ell^2(\mathcal{V}; m) \} \end{aligned}$$

Lemma (Sy&Sunada 1992; Davies 1993; Keller&Lenz 2010.)

The Laplacian $L = L_{0,b,m}$ is bounded on $\ell^2(\mathcal{V}, m)$ if and only if the weighted degree function $\mathrm{Deg}\colon \mathcal{V} \to [0, \infty)$ given by

$$\mathrm{Deg}(v) := \frac{1}{m(v)} \sum_{u \in \mathcal{V}} b(u, v)$$

is bounded on $\mathcal{V}$. In this case, $h_D = h_N = h$.

Laplacians on metric graphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$ be a metric graph.
Let $\mu(e), \nu(e)$ be weight functions assigning positive weights $\forall\ e \in \mathcal{E}$.
Goal: Define Laplacian H_D by quadratic form approach.

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Hilbert Space

Given the Lebesgue measure $\mu(dx) := \mu(e)dx_e,\ e \in \mathcal{E}$, we define the Hilbert space $L^2(\mathcal{G}; \mu)$ of measurable functions $f : \mathcal{G} \to \mathbb{C}$ which are square integrable w.r.t. μ.

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Quadratic Form $\mathcal{Q}$

Energy of a "sufficiently smooth" $f : \mathcal{G} \to \mathbb{C}$

$$\mathcal{Q}[f] := \sum_e \int_{\mathcal{I}_e} |f'(x)|^2 \nu(e) \, \mathrm{d}x_e$$

What is smooth enough?

Sobolev space $H^1(\mathcal{G})$

$$H^1(\mathcal{G}) := \{f : \mathcal{G} \to \mathbb{C} \,|\, f \text{ is continous, } f \in L^2(\mathcal{G}, \mu), \\ f|_{\mathcal{I}_e} \in H^1(\mathcal{I}_e) \text{ and } \mathcal{Q}[f] < \infty\}.$$

Note that $\mathcal{Q}|_{H_1(\mathcal{G})}$ is a closed, non-negative form.

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Sobolev space $H^1_0(\mathcal{G})$

$H^1_0(\mathcal{G}) :=$ closure of $H^1(\mathcal{G}) \cap C_c(\mathcal{G})$ w.r.t. $\|\cdot\|^2 := \|\cdot\|^2_{L^2(\mathcal{G})} + \mathcal{Q}[\cdot]$

Note that $\mathcal{Q}|_{H^1_0(\mathcal{G})}$ is a closed, non-negative form.

Neumann Laplacian

The Neumann Laplacian $H_N : \mathrm{dom}(H_N) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G})$ is the non-negative self-adjoint operator associated to $\mathcal{Q}$ on $H^1(\mathcal{G})$

Laplacians on metric graphs

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Dirichlet Laplacian

The Dirichlet Laplacian $H_D : \mathrm{dom}(H_D) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G})$ is the non-negative self-adjoint operator associated to $\mathcal{Q}$ on $H_0^1(\mathcal{G})$

Laplacians on metric graphs

How does H_D act?
Let $\mu(e), \nu(e)$ be two weight functions assigning a positive weight.

Formal Laplacian

For $f : \mathcal{G} \to \mathbb{C}$ "sufficiently smooth", define $\Delta f : \mathcal{G} \to \mathbb{C}$ by

$$\Delta f|_{\mathcal{I}_e} := -\frac{1}{\mu(e)} \frac{d}{dx} \nu(e) \frac{df}{dx} \tag{2.1}$$

"edgewise second derivative"

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"edgewise second derivative"

Kirchhoff boundary conditions

$$\begin{cases} f \text{ is continuous at } v, \\ \sum_{e \in \mathcal{E}} \nu(e) \partial_e f(v) = 0, \end{cases} \qquad v \in \mathcal{V} \tag{2.2}$$

Maximal Laplacian in $L^2(G)$

Let

$$\begin{aligned}\operatorname{dom}(H) := \{f \in L^2(\mathcal{G}) | f|_{\mathcal{I}_e} \in H^2(\mathcal{I}_e), \forall\ e \in \mathcal{E},\\ \Delta f \in L^2(\mathcal{G}),\ \text{and } f \text{ satisfies the Kirchhoff conditions.}\}\end{aligned}$$

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We then have $\mathrm{dom}(H_D) \subset \mathrm{dom}(H)$

and $H_D = H \restriction \mathrm{dom}(H_D)$.

Question: How to obtain a discrete graph from a metric graph?

Connecting the settings

To do this, let us consider a metric graph $\mathcal{G}$ and two edge weights $\nu, \mu \colon \mathcal{E} \to (0, \infty)$.
With this metric graph we associate two functions:

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$$m(v) = \sum_{e \in \mathcal{E}_v} |e|\mu(e), \tag{3.1}$$

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  $\mathcal{E}_v$: set of edges at v

- *The edge weight* $b\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$,

$$b(u, v) = \begin{cases} \sum_{e \in \mathcal{E}} \frac{\nu(e)}{|e|} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases} \tag{3.2}$$

Connecting the settings

It is straightforward to verify that m is strictly positive and defines a measure of full support on $\mathcal{V}$ and b is a graph over $(\mathcal{V}, m)$, that is, b satisfies the following properties:

i) Symmetry

ii) Vanishing diagonal : $b(v, v) = 0$ for all $v \in \mathcal{V}$

iii) Local summability : $\sum_{v \in \mathcal{V}} b(u, v) < \infty$ for all $u \in \mathcal{V}$

Equilateral graphs as special case

Equilateral Graphs

A metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$ is called *equilateral graph* if $|e| = \mu(e) = \nu(e) = 1$ for all $e \in \mathcal{E}$ and $\mathcal{G}$ has no loops or multiple edges.

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Definition of Normalized Laplacian

Let $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$; *the normalized Laplacian* h_{norm} is the discrete Laplacian associated to $m \colon \mathcal{V} \to (0, \infty)$ and $b \colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ given by

$$m(v) = \deg(v)$$

and

$$b(u, v) = \begin{cases} 1 & \text{if u and v are neighbors} \\ 0 & \text{else} \end{cases}$$

Equilateral graphs as special case

Theorem

Let $\sigma(h_{norm})$ and $\sigma(H_D)$ be the spectrum of h_{norm} and H_D, respectively. If $\lambda \geq 0$ and $\lambda \notin \{(n\pi)^2 | n \in \mathbb{N}\}$ then

$$\lambda \in \sigma(H_D) \Longleftrightarrow (1 - \cos(\sqrt{\lambda})) \in \sigma(h_{norm}). \tag{3.3}$$

see as reference (*G. Berkolaiko and P. Kuchment, Introduction to Quantum Graphs, Amer. Maths. Soc., Providence, RI, 2013*)

Proof of Theorem for finite graphs

$\Longrightarrow$ Let $\lambda \in \sigma(H_D)$. Then for $f \in dom(H_D)$, we have

$$H_D f = \lambda f \iff -\Delta f = \lambda f \tag{3.4}$$

Thus any eigenfunction f must satisfy the equation $-f_e^{''} = \lambda f_e$ on every edge $e = \mathcal{I}_e$. Hence, we have by an easy computation on an edge $e = \mathcal{I}_e$ with left and right endpoints $v, w \in \mathcal{V}$,

$$f(w) - \cos(\sqrt{\lambda})f(v) = f_e^{'}(v)\frac{\sin(\sqrt{\lambda})}{\sqrt{\lambda}} \tag{3.5}$$

since $\mathcal{G}$ is equilateral. Summing over all neighbors w of v, the Kirchhoff conditions imply that

$$h_{norm}\mathbf{f} = (1 - \cos(\sqrt{\lambda}))\mathbf{f} \tag{3.6}$$

for $\mathbf{f} = f|_{\mathcal{V}}$.

$\Longleftarrow$ For the converse direction, simply carry out the above steps in the reverse direction. More precisely, we show that every function f satisfying (3.6) is equal to the restriction $f = f|_{\mathcal{V}}$ of a function $f \in \mathrm{dom}(H_D)$ satisfying (3.4).

Equivalence of heat kernel decay

Theorem

Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. Let H_D be Dirichlet Laplacian on $(\mathcal{G}, \mu, \nu)$ and let h_D the Dirichlet Laplacian on the associated discrete graph $(\mathcal{V}, m, b)$. We have

$$\begin{array}{c} \exists C > 0,\ \forall t > 0 \\ \left\| e^{-tH_D} \right\|_{\mathcal{L}(L^1, L^\infty)} \leq C t^{-d/2} \end{array} \iff \begin{array}{c} \exists C > 0,\ \forall t > 0 \\ \left\| e^{-th_D} \right\|_{\mathcal{L}(\ell^1, \ell^\infty)} \leq C t^{-d/2} \end{array}$$

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where $\eta_d^* = \sup \left(|e| \, \mu(e) \right)^{1-d/2} \frac{|e|}{\nu(e)}$

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Proof: two key ingredients:

- Key ingredient 1: special space of test functions: edge-wise affine functions
- Key ingredient 2: Nash estimates

Our toolbox

Decomposition of $f \in C(\mathcal{G})$

affine linear part $f_{\text{lin}} \in CA(\mathcal{G} \backslash \mathcal{V})$
and rest f_0

Relation of $\mathcal{Q}$ and $\mathfrak{q}$

$\mathcal{Q}[f_{\text{lin}}] = \mathfrak{q}[i_{\mathcal{V}}(f_{\text{lin}})]$
for sufficiently smooth f_{lin}

Decomposition of $\mathcal{Q}$

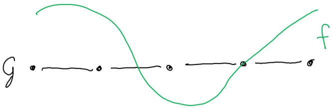
corresponding to affine linear part
$\mathcal{Q}[f_{\text{lin}}]$ and rest $\mathcal{Q}[f_0]$

L^p-ℓ^p-estimates

comparing functions on $\mathcal{G}$ and $\mathcal{V}$
and their p-norms

Decomposition of $f \in C(\mathcal{G})$

- $f \in C(\mathcal{G})$ on metric graph, $\mathfrak{f} \in C(\mathcal{V}) = \{\mathfrak{g} \colon \mathcal{V} \to \mathbb{C}\}$ on discrete one



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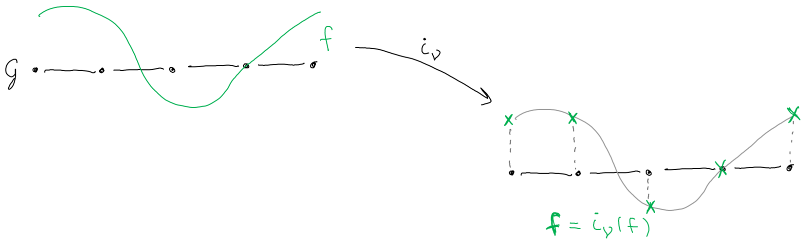
- $f \in C(\mathcal{G})$ on metric graph, $\mathfrak{f} \in C(\mathcal{V}) = \{\mathfrak{g} : \mathcal{V} \to \mathbb{C}\}$ on discrete one
- *Restriction* to vertices: $i_{\mathcal{V}} : C(\mathcal{G}) \to C(\mathcal{V}),\ f \mapsto (f(v))_{v \in \mathcal{V}}$



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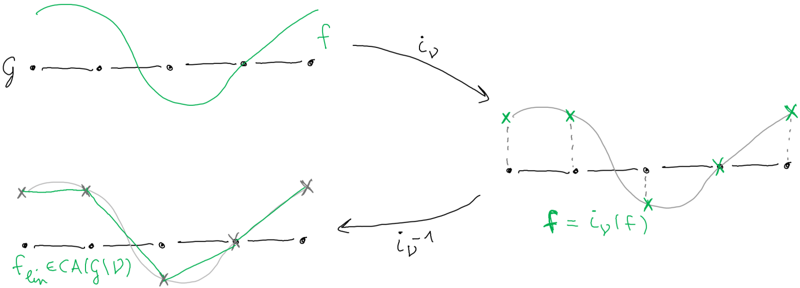

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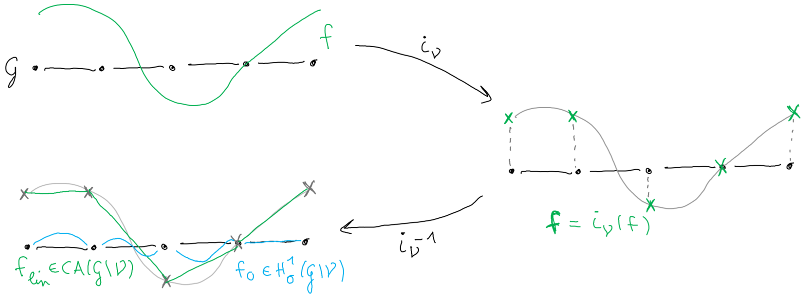
- If f is edgewise H^1, then $f_0 \in H_0^1(\mathcal{G} \backslash \mathcal{V})$.



Lemma (Relation of $\mathcal{Q}$ and $\mathfrak{q}$)

Let $f \in CA(\mathcal{G}\setminus\mathcal{V}) \cap H^1(\mathcal{G})$. Then $\mathfrak{f} := i_{\mathcal{V}}(f) \in H^1(\mathcal{V})$ and

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Relation of $\mathcal{Q}$ and $\mathfrak{q}$ & Decomposition of $\mathcal{Q}$

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Comparing L^p and ℓ^p-norms

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Let $1 \leq p < \infty$ and $f \in CA(\mathcal{G}\backslash\mathcal{V}) \cap L^p(\mathcal{G};\mu)$. Then $\mathfrak{f} := i_{\mathcal{V}}(f) \in \ell^p(\mathcal{V}; m)$.

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Proof. Let's start with second $\leq$. First consider $p = 1$.

1. affine f on edge $[0, \ell]$: case distinction w.r.t. sign of $|f(0)f(\ell)|$ yields

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$$\|f\|_{L^1(\mathcal{G};\mu)} \geq \frac{1}{4}\sum_{e\in\mathcal{E}} |e|\mu(e)(|f(e_\iota)| + |f(e_\tau)|)$$
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Comparing L^p and ℓ^p-norms

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3. Sum over all edges and proceed as for $p = 1$ to obtain

$$\|\mathfrak{f}\|_{\ell^p(\mathcal{V};m)} \leq 4^p\|f\|^p_{L^p(\mathcal{G};\mu)}.$$

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Proof of equivalence of Heat kernel decay

Recall

Theorem

We have

$$\begin{array}{ccc} \exists C > 0,\ \forall t > 0 & \overset{d\geq 1}{\Longrightarrow} & \exists C > 0,\ \forall t > 0 \\ \left\| e^{-tH_D} \right\|_{\mathcal{L}(L^1, L^\infty)} \leq Ct^{-d/2} & \Longleftrightarrow & \left\| e^{-th_D} \right\|_{\mathcal{L}(\ell^1, \ell^\infty)} \leq Ct^{-d/2} \\ & \underset{d>2,\ \eta_d^* < \infty}{\Longleftarrow} & \end{array}$$

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Key ingredient 2: Nash estimates: for an operator A, with associated Dirichlet form Q_A, we have

$$\forall t>0,\ \left\|e^{-tA}\right\|_{\mathcal{L}(L^1,L^\infty)} \le Ct^{-d/2} \Longleftrightarrow \forall f,\ \|f\|_{L^2}^{2+4/d} \le CQ_A(f)\,\|f\|_{L^1}^{4/d}$$

$$\underset{d>2}{\Longleftrightarrow} \forall f,\ \|f\|_{L^{2d/(d-2)}}^2 \le CQ_A(f)$$

From metric to discrete

$\Longrightarrow$ Assume $d/2$ decay rate for e^{-tH_D}, i.e.,

$$\exists C > 0,\ \forall f \in L^1 \cap H^1(\mathcal{G}), \quad \|f\|_{L^2}^{2+4/d} \leq C\,\mathcal{Q}(f)\,\|f\|_{L^1}^{4/d}.$$

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We deduce

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$$\Longleftrightarrow$$

$d/2$ decay rate for e^{-th_D}

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For $\mathfrak{f} = i_{\mathcal{V}}(f_{\mathrm{lin}})$, we have

$$\|f_{\mathrm{lin}}\|_{L^q} \leq \|\mathfrak{f}\|_{\ell^q} \leq C\mathfrak{q}(\mathfrak{f})^{1/2} = C\mathcal{Q}(f_{\mathrm{lin}})^{1/2}$$

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$$\forall \mathfrak{f} \in H^1(\mathcal{V}), \quad \|\mathfrak{f}\|_{\ell^q}^2 \leq C\mathfrak{q}(\mathfrak{f}) \qquad \left(q = \frac{2d}{d-2}\right).$$

Then, for $f \in H^1(\mathcal{G})$, we have

$$f = f_{\text{lin}} + f_0 \quad \text{with} \quad \mathcal{Q}(f) = \mathcal{Q}(f_{\text{lin}}) + \|\nabla f_0\|_{L^2}^2$$

We have $f_0 \in H_0^1(\mathcal{G}) \subset L^q(\mathcal{G})$, thus

$$\|f\|_{L^q} \leq \|f_{\text{lin}}\|_{L^q} + \|f_0\|_{L^q} \leq \|f_{\text{lin}}\|_{L^q} + C\,\|\nabla f_0\|_{L^2}$$

For $\mathfrak{f} = i_{\mathcal{V}}(f_{\text{lin}})$, we have

$$\|f_{\text{lin}}\|_{L^q} \leq \|\mathfrak{f}\|_{\ell^q} \leq C\mathfrak{q}(\mathfrak{f})^{1/2} = C\mathcal{Q}(f_{\text{lin}})^{1/2}$$

We deduce

$$\|f\|_{L^q} \leq C\mathcal{Q}(f) \quad \Longleftrightarrow \quad d/2 \text{ decay rate for } e^{-tH_D}$$

Further connections

- Spectral estimate

$$\lambda_0(H_D) > 0 \quad \Longleftrightarrow \quad \lambda_0(h_D) > 0$$

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with $a = (\pi/\eta^*(\mathcal{E}))^2$ and $\eta^*(\mathcal{E}) = \sup_{e\in\mathcal{E}} |e| \sqrt{\frac{\mu(e)}{\nu(e)}}$.

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- Laplacians with δ-coupling

δ-Coupling Laplacian

$$\begin{cases} f \text{ is continous at } v, \\ \sum_{e \in \mathcal{E}} \nu(e) \partial_e f(v) = \alpha(v) f(v), \end{cases} \qquad \alpha(v) \in \mathbb{R} \bigcup \{\infty\}.$$

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$$\begin{cases} \alpha = 0, & \text{Kirchhoff-Neumann} \\ \alpha = \infty, & \text{Dirichlet} \\ \alpha \neq 0 \,\&\, \alpha \neq \infty, & \text{Robin} \end{cases}$$

δ-Coupling Laplacian

$$Q_\alpha(f) = \sum_{e\in\mathcal{E}} \nu(e) \int_{[0,|e|]} |f'(x)|^2\, dx + \sum_{v\in\mathcal{V}} \alpha(x)|f(x)|^2$$

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- We can either perturb the Kirchhoff condition or maintain it and perturb the operator.
- The killing term can be reinterpreted as the strength of a delta coupling potential.

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$$\lambda_0(h_D) > 0 \quad \Longleftrightarrow \quad \lambda_0(H_D) > 0$$

  More precisely

$$\min\left\{\lambda_0(h_D), \frac{a}{2}\right\} \leq \lambda_0(H_D) \leq \min\left\{\lambda_0(h_D), a\right\},$$

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- Laplacians with δ-coupling
- Recurrence/transience

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- Laplacians with δ-coupling
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- Stochastic completeness

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- Laplacians with δ-coupling
- Recurrence/transience
- Stochastic completeness
- Intrinsic metric
- (Noema's talk right after the end of this presentation)

Thank you for your attention!