Harnack Estimates For The Discrete Porous Medium Equation Via Curvature Dimension Condition

Project H

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Continuous Porous Medium equation

Let $m \in (1, \infty)$ and M denote a complete d-dimensional Riemannian manifold with $Ricc(M) \geq 0$. (For the rest of the talk, M is fixed. In particular, $M = \mathbb{R}^d$.)

Let $u : (0, \infty) \times M \to (0, \infty)$ be C^1 in time. The continuous porous medium equation (PME) is given by

$$\partial_t u(t,x) - \Delta u^m(t,x) = 0, \qquad \forall \quad t > 0\,.$$

Observe that in the limit as $m \to 1$ one recovers the continuous heat equation, i.e.,

$$\partial_t u(t,x) - \Delta u(t,x) = 0, \qquad \forall \quad t > 0\,.$$

Continuous Aronson-Bénilan estimate

Let $v = \frac{m}{m-1} u^{m-1}$, known as the pressure function. We have the following:

Continuous Aronson-Bénilan estimate (*Vázquez*)

Let u be a positive solution to the PME on $(0, \infty) \times \mathbb{R}^d$. Then, the pressure function v satisfies

$$-\Delta v(t,x) = -\Delta(\frac{m}{m-1} u^{m-1}) \leq \frac{d}{2 + d(m-1)} \frac{1}{t} .$$

The above inequality implies the existence of a lower bound for the Laplacian of the pressure function v.

Remark: The Aronson-Bénilan estimate gives rise to the Harnack estimate (G. Auchmuty, D. Bao: *Harnack type inequalities for evolution equations*)

Verification

Recall

$$-\Delta v(t,x) = -\Delta\left(\frac{m}{m-1}u^{m-1}\right) \leq \frac{d}{2+d(m-1)}\frac{1}{t},$$

Observe that in the limit as $m \to 1$ in the above estimate yields the continuous **Li-Yau inequality**, i.e.,

$$\begin{aligned}\lim_{m\to 1}(-\Delta v(t,x)) &= \lim_{m\to 1}\left(-\Delta\left(\frac{m}{m-1}u^{m-1}\right)\right)(t,x)\\ &= -\lim_{m\to 1}\Big(\nabla(mu^{m-2}\nabla u)(t,x)\Big) = -\Big(\nabla\left(\frac{\nabla u}{u}\right)(t,x)\Big).\\ &\qquad\qquad = -\Delta \log u(t,x).\end{aligned}$$

On the other hand,

$$\lim_{m\to 1}\frac{d}{2+d(m-1)}\frac{1}{t} \leq \frac{d}{2\,t}.$$

The continuous evolution equation

Recall $v = \frac{m}{m-1} u^{m-1}$. The evolution equation for the function v is given by

$$\partial_t v = (m-1) v \Delta v + |\nabla v|^2 .$$

Observe that in the limit as $m \to 1$ we have

1) $\partial_t v = m u^{m-2} \partial_t u \to \frac{\partial_t u}{u} = \partial_t \log(u)$,
2) $(m-1) v \Delta v \to \Delta \log u$,
3) $|\nabla v|^2 \to |\frac{\nabla u}{u}|^2 = |\nabla \log u|^2$.

Equating both sides leads to

$$\partial_t \log(u) = \Delta \log u + |\nabla \log u|^2 ,$$

which is the evolution equation for $v = \log u$. (We already know from the heat equation.)

Towards Discrete setting

Recall the continuous PME on M with $Ricc(M) \geq 0$ was given by

$$\partial_t u(t,x) - \Delta u^m(t,x) = 0, \qquad \forall \quad t > 0\,.$$

We aim to deduce the discrete version of the PME, a discrete evolution equation for v and a discrete Aronson-Bénilan estimate. To this end, we need

A) to replace M with X a non-empty finite or countably infinite set.

B) to replace Δ with a discrete Laplacian, L encapsulating the replacement for the non-negative curvature of the underlying state space X (known as CD-condition).

C) to come up with a kind of chain rule for the discrete Laplacian L.

Assumptions

Let

A) X be a non-empty finite set.

B) $k : X \times X \to [0, \infty)$ be a non-trivial kernel such that

  B-1) it is symmetric,

  B-2) it induces a graph with vertex set X, which is connected,

  B-3) for every $x \in X$ there are finitely many $y \in X$ with $k(x, y) > 0$.

C) L be the discrete Laplacian generated by the kernel k.

Discrete Harnack inequality for Porous Medium Equation

Let $m > 1$. Suppose $u : [0, \infty) \times X \to (0, \infty)$ is C^1 in time. Let $\lambda \in [0, 1)$ and $\mu > 0$. Suppose the function $v = \frac{m}{m-1} u^{m-1}$ satisfies the differential Harnack inequality, i.e.,

$$\partial_t v \geq (1 - \lambda)\tilde{\Psi}^m_\Upsilon(v) - \frac{\mu}{t} v.$$

The function $v(t, x)$ enjoys the following pointwise bound:

Discrete Harnack inequality

Let $N \in \mathbb{N}$, $i = \{0, \cdots, N\}$, $0 < t_1 < t_2$, and $x_1, x_2 \in X$. Let $\tau_i = t_1 + i\frac{t_2 - t_1}{N}$.Then, for every sequence of pairwise distinct points $(y_i)_{i=0,\cdots,N}$ such that $y_0 = x_1$, $y_N = x_2$ and $k(y_{i-1}, y_i) > 0$, it holds

$$t_1^\mu v(t_1, x_1) \leq t_2^\mu v(t_2, x_2) + \frac{2N^2}{(1-\alpha)(\mu+1)(t_2-t_1)^2} \sum_{j=1}^{N} \frac{\tau_j^{\mu+1} - \tau_{j-1}^{\mu+1}}{k(y_{j-1}, y_j)} .$$

Discrete Laplacian L

Let X be a finite or countably infinite set and $k : X \times X \to [0, \infty)$ denote a kernel such that for any $x \in X$ there are at most finitely many y with $k(x, y) > 0$.

Consider function $u : X \to \mathbb{R}$. The discrete Laplacian L generated by the kernel k is given by

$$L\,u(x) = \sum_{y \in X} k(x, y)\Big(u(y) - u(x)\Big),\ x \in X\,.$$

Towards Discrete setting

Guiding principle: In the limit as $m \to 1$ we should recover

1) The discrete heat equation, i.e.,

$$\partial_t u(t,x) = L\, u(t,x)\,.$$

2) The discrete evolution equation for the heat equation, i.e.,

$$\partial_t \log u = L \log u + \Psi_\Upsilon \log u\,,$$

3) The discrete Li-Yau inequality, i.e.,

$$-L\, \log(u(t,x)) \leq \frac{d}{t}\,.$$

Discrete PME

Let $m > 1$. The discrete PME is given by

$$\partial_t u(t,x) - L\,(u^m)(t,x) = 0\,.$$

Notice that in the limit as $m \to 1$ we recover the discrete heat equation, i.e.,

$$\partial_t u(t,x) = L\,u(t,x)\,.$$

Discrete Evolution Equation via adapted Chain rule

Recall $v = \frac{m}{m-1} u^{m-1}$. Let u be a positive solution to the PME. It holds

$$\partial_t v = m u^{m-2} \partial_t u = m u^{m-2} L\, u^m \,.$$

Equivalently,

$$\partial_t v = m u^{m-2} \partial_t u = (m-1) v\, L\, v + \Big(m u^{m-2} L u^m - (m-1) v\, L\, v \Big)$$

Using the preceding result, one infers chain rule

$$\partial_t v = (m-1) v \;\, L \;\, v + \tilde{\Psi}^{(m)}_{\Upsilon}(v)\,,$$

where

$$\tilde{\Psi}^{(m)}_{\Upsilon}(u) = u^2 \Psi_{\tilde{\Upsilon}}(\log u)(x)$$

and the function $\tilde{\Upsilon} : \mathbb{R} \to \mathbb{R}$ is given by

$$\tilde{\Upsilon}(z) = \frac{(m-1)^2}{m} \Upsilon\Big(\frac{m}{m-1} z\Big) - (m-1)\Upsilon(z)\,.$$

Implications

Recall

$$\tilde{\Psi}_{\Upsilon}^{(m)}(u)(x) = u^2(x)\,\Psi_{\tilde{\Upsilon}}(\log u)(x)\,.$$

Observe that

A) $\tilde{\Psi}_{\Upsilon}^{(m)}(u) \geq 0$ (Due to the convexity of $\tilde{\Upsilon}$) .

B) $\lim_{m\to 1} \tilde{\Psi}_{\Upsilon}^{(m)}(v) = \Psi_{\Upsilon}(\log u)(x)$. (An application of the l'H$\hat{o}$spital's rule).

Towards the discrete Aronson-Bénilan

To infer the discrete Li-Yau inequality, i.e., $-L\log(u(t,x)) \leq \frac{d}{t}$, we utilized the condition $CD(F,0)$.

Similarly, in order to induce the discrete Aronson-Bénilan estimate we need to come up with a correct counterpart of $CD(F,0)$ in the case of dealing with the PME.

We shall denote such objects by $CD_m(0,d)$ and $CD_{m,\alpha}(0,d)$ each of which is suitable for different kind of graph structures.

$CD_m(0,d)$ condition

Let $m > 1$, $d > 0$, and $u : X \to [0, \infty)$. For $x \in X$, we set

$$\mathcal{D}_m(u)(x) := m \sum_{y \in X} k(x,y)\Big(u^{m-2}(y)Lu^m(y) - u^{m-2}(x)Lu^m(x)\Big) .$$

Observe that if u is a positive solution to the PME, then we have

$$\mathcal{D}_m(u)(x) = \partial_t \, L \, v(t,x) \, ,$$

where L is the discrete Laplacian and $v = \frac{m}{m-1} \, u^{m-1}$ denotes the pressure function.

$CD_m(0, d)$ condition

Definition

The operator L satisfies $CD_m(0, d)$ condition at $x \in X$, if for every function $u : X \to [0, \infty)$ such that the function $v = \frac{m}{m-1} u^{m-1}$ satisfies

A) $-Lv(x) > 0$

B) $-Lv(x) \geq -Lv(y)$ for all $y \in X$ with $k(x, y) > 0$

it holds

$$\mathcal{D}_m(u)(x) \geq \frac{1}{d}\Big(- Lv(x)\Big)^2 .$$

The discrete Laplacian L satisfies $CD_m(0, d)$ if it satisfies $CD_m(0, d)$ at any $x \in X$.

Remark:

Observe that in the limit as $m \to 1$ in $CD_m(0, d)$ we recover $CD(F; 0)$ with CD-function $F(x) = \frac{1}{d}x^2$.

The discrete Aronson-Bénilan

Let L denote the discrete Laplacian generated by k and assume L satisfies $CD_m(0, d)$ with some $d > 0$. Let u be a positive solution to the PME, i.e., $\partial_t u(t,x) - L(u^m)(t,x) = 0$ on $(0,\infty) \times X$. Then the function $v = \frac{m}{m-1} u^{m-1}$ satisfies

$$-Lv(t,x) \leq \frac{d}{t}\,.$$

Idea of the proof:

Recall the set X is finite. Let

$$G(t,x) = -\frac{t}{d}\, Lv(t,x)\,.$$

Now, demand $G(t,x) \leq 1$ by appealing to the maximum principle. (Similar to what we did in the case of the discrete heat equation.)

Let $X = \{x_1, x_2\}$, $m > 2$, and $u : X \to [0, \infty)$. Then

$$Lu(x_1) = u(x_2) - u(x_1)\,.$$

Now, suppose

$$-\,L\,v(x_1) > 0 \qquad \text{and} \qquad -\,L\,v(x_1) \geq -\,L\,v(x_2)\,,$$

which implies $u(x_1) > u(x_2)$. By definition,

$$\mathcal{D}_m u(x_1) = m\Big(u^{m-2}(x_2)L\,u(x_2)^m - u^{m-2}(x_1)L\,u(x_1)^m\Big) =: H$$

Moreover,

$$\Big(-\,Lv(x)\Big)^2 = \Big(v(x_2) - v(x_1)\Big)^2 =: G$$

Let $z = \frac{u(x_2)}{u(x_1)} \in [0, 1)$, and set

$$f_{\nu,m}(z) = \nu\,H - G\,.$$

Fact: Observe that the discrete Laplacian L satisfies $CD_m(0, \nu)$ if and only if $f_{\nu,m}(z) \geq 0$.

Explicitly, we have

$$f_{\nu,m}(z) = \nu m(z^{m-2} - z^{2m-2} - z^m + 1) - \frac{m^2}{(m-1)^2}(z^{2m-2} - 2z^{m-1} + 1)\,.$$

Recall $z \in [0,1)$. Hence, $z^m \leq z$ for all $m > 2$. Now, setting $z = 0$ implies that $f_{\nu,m}(z) \geq 0$ if $\nu_* \geq \frac{m}{(m-1)^2}$.

To show the choice $\nu_* \geq \frac{m}{(m-1)^2}$ is sufficient, let

$$f_{\nu_*,m}(z) = \frac{m^2}{(m-1)^2}(z^{m-2} - 2z^{2m-2} - z^m + 2z^{m-1})\,.$$

Observe that $f_{\nu_*,m}(z) \geq 0$ for all $z \in [0,1)$ if

$$\frac{1}{z} - 2z^{m-1} - z + 2 \geq 0\,,$$

which holds true for all $z \in [0,1)$ and $m > 2$.

Negative Example: Chain like graph with $m = 2$

Let $X = \{x, y, z\}$ and set $k(x, z) = k(x, y) = 1$, $k(y, z) = 0$. Let L be the discrete Laplacian generated by k.

Suppose $u(x) = 1, u(y) = 1.5$ and $u(z) = 0$ and consider $m = 2$. Recall $v = \frac{m}{m-1} u^{m-1}$. It holds

1) $-Lv(x) = -2Lu(x) = 1$.
2) $-Lv(y) = 1$. Thus, $-Lv(y) \leq -Lv(x)$.
3) $-Lv(z) = -2$. Hence, $-Lv(z) \leq -Lv(x)$.

On the other hand,

$$\mathcal{D}_2 \, u(x) = 2L \, u^2(y) + 2L \, u^2(z) - 4L \, u^2(x) = -\frac{3}{2} < 0 \, .$$

Hence, there is no $d > 0$ such that $\mathcal{D}_2 \, u(x) \geq \frac{1}{d} \Big(- Lv(x) \Big)^2$ holds, i.e., $CD_2(0, d)$ cannot be satisfied for any $d > 0$.

Problem

As the previous examples illustrated the $CD_m(0, d)$ is useful for a couple of graph structures e.g. complete graphs.

However, there is a problem for chain like graphs with more than two vertices including the discrete Laplacian on $\mathbb{Z}$.

To get around this problem, we generalize the CD condition using the evolution equation for the function v.

$CD_{m,\alpha}(0,d)$ condition

Let $u : X \to [0,\infty)$ and $x \in X$. We aim to generalize the operator $\mathcal{D}_m(u)(x)$. To this end, let $\alpha \in [0,1]$ be some constant and define

$$\mathcal{D}_{m,\alpha}(u)(x) := \sum_{y\in X} k(x,y)\Big((1-\alpha+\alpha\frac{u(y)}{u(x)})mu^{m-2}(y)L\,u^m(y) \\ -(m-\alpha+\alpha(\frac{u(y)}{u(x)})^m)u^{m-2}(x)L\,u^m(x)\Big)\,.$$

Obviously, $\mathcal{D}_{m,0} = \mathcal{D}_m$. Furthermore, if u is a solution to the PME, then

$$\mathcal{D}_{m,\alpha}(u)(x) = \partial_t\left(Lv + \alpha\frac{\tilde{\Psi}_{\Upsilon}^{(m)}(v)}{(m-1)v}\right)(x)\,.$$

$CD_{m,\alpha}(0,d)$ condition

Denote $-(Lv + \alpha\frac{\tilde{\Psi}_{\Upsilon}^{(m)}(v)}{(m-1)v})(x) := J(v)(x)$. We have the following definition:

Definition

Let $m > 1$ and $\alpha \in [0,1]$. We say the discrete Laplacian L satisfies $CD_{m,\alpha}(0,d)$ with $d > 0$ at $x \in X$, if for every function $u : X \to (0,\infty)$ such that the function $v = \frac{m}{m-1}u^{m-1}$ satisfies

1) $J(v)(x) > 0$

2) $J(v)(x) \geq J(v)(y)$ for all $y \in X$ such that $k(x,y) > 0$

it holds

$$\mathcal{D}_m(u)(x) \geq \frac{1}{d}\Big(J(v)(x)\Big)^2.$$

We say that L satisfies $CD_{m,\alpha}(0,d)$ condition, if it satisfies $CD_{m,\alpha}(0,d)$ condition at any $x \in X$.

Recall the chain like graph $\mathbb{Z}$. We have the following:

Positive Example for $CD_{m,\alpha}(0, d)$ condition

For all $m > 1$, the discrete Laplacian on the unweighted lattice $\mathbb{Z}$ satisfies $CD_{m,1}(0, \frac{1}{m-1})$.

The discrete generalized Aronson-Bénilan

Let L denote the discrete Laplacian generated by the kernel k and assume L satisfies $CD_{m,\alpha}(0,d)$ with $d>0$. Let u be a positive solution to the PME, i.e., $\partial_t u(t,x) - L(u^m)(t,x) = 0$ on $(0,\infty)\times X$. Then the function $v = \frac{m}{m-1}u^{m-1}$ satisfies

$$-(Lv + \alpha\frac{\tilde{\Psi}_{\Upsilon}^{(m)}(v)}{(m-1)v}) \leq \frac{d}{t}\,.$$

Idea of the proof:

The idea of the proof is similar to the case $\alpha = 0$. To this end, let

$$G(t,x) = -\frac{t}{d}\left(L\,v + \alpha\frac{\tilde{\Psi}_{\Upsilon}^{(m)}(v)}{(m-1)v}\right)(t,x)\,.$$

and demand $G(t,x) \leq 1$.

Discrete Harnack inequality

Let m, X, k, L, v be as before. Suppose $u : [0,\infty) \times X \to (0,\infty)$ is C^1 in time. Let $\lambda \in [0,1)$ and $\mu > 0$. Suppose the function v satisfies the differential Harnack inequality, i.e.,

$$\partial_t v \geq (1-\lambda)\tilde{\Psi}^{(m)}_{\Upsilon}(v) - \frac{\mu}{t} v\,.$$

We have the following:

Discrete Harnack inequality

Let $N \in \mathbb{N}$, $i = \{0, \cdots, N\}$, $0 < t_1 < t_2$, and $x_1, x_2 \in X$. Then, for every sequence of pairwise distinct points $(y_i)_{i=0,\cdots,N}$ such that $y_0 = x_1$, $y_N = x_2$ and $k(y_{i-1}, y_i) > 0$, it holds

$$t_1^{\mu} v(t_1, x_1) \leq t_2^{\mu} v(t_2, x_2) + \frac{2N^2}{(1-\lambda)(\mu+1)(t_2-t_1)^2} \sum_{j=1}^{N} \frac{\tau_j^{\mu+1} - \tau_{j-1}^{\mu+1}}{k(y_{j-1}, y_j)},$$

where $\tau_i = t_1 + i\,\frac{t_2-t_1}{N}$.

Idea of the proof

By the assumption it holds

$$\partial_t v \geq (1-\lambda)\tilde{\Psi}_{\Upsilon}^{(m)}(v) - \frac{\mu}{t} v \,.$$

Multiplying both sides of the above inequality by t^{μ} yields

$$\partial_t (t^{\mu} v) \geq (1-\lambda) t^{\mu} \tilde{\Psi}_{\Upsilon}^{(m)}(v) \,.$$

Now, the statement follows by summation over a path in space and time.

More precisely, one first takes the two points which are directly connected by an edge, i.e., $N = 1$. Then, we apply the summation argument for a general case.

Idea of the proof

To proceed, recall

$$\tilde{\Psi}^{(m)}_{\Upsilon}(u)(x) = u^2(x)\,\Psi_{\tilde{\Upsilon}}(\log u)(x)\,.$$

and

$$\tilde{\Upsilon}(z) = \frac{(m-1)^2}{m}\Upsilon(\frac{m}{m-1}z) - (m-1)\Upsilon(z)\,.$$

Observe that depending on the value of m the function $\tilde{\Upsilon}(z)$ behaves differently, i.e.,

1) $\tilde{\Upsilon}(\log x) \geq \frac{1}{2}(x-1)^2$ for all $x \geq 1$, with $m \in (1,2]$
2) $\tilde{\Upsilon}(\log x) \geq \frac{1}{2}(x-1)^2$ for all $x \in (0,1]$, with $m \in [2,\infty)$

$\Rightarrow$ study two cases, i.e. $m \in (1,2]$ and $m \in [2,\infty)$ separately.

Conclusions

- We revisited the construction of continuous Aronson-Bénilan estimate for the PME on a complete d-dimensional Riemannian manifold with $Ricc(M) \geq 0$

- We introduced a discrete Aronson-Bénilan estimate for the discrete PME on a finite set X using the $CD_m(0, d)$ / $CD_{m,\alpha}(0, d)$ conditions

- We employed the discrete Aronson-Bénilan estimate to prove a discrete Harnack inequality for a solution to the discrete PME

Possible Directions for the Future Research

- Prove a discrete Aronson-Bénilan estimate and a discrete Harnack inequality on infinite graphs.