Project E:
Port–Hamiltonian Systems on Graphs

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Outline

Linear port–Hamiltonian systems

Constant Dirac structures

Dynamics on general Dirac structures

Open graphs

2 Special Dirac structures

Mass-Spring-Damper system

Composition of graph Dirac structures

Motivation

What are port–Hamiltonian systems?

$\rightarrow$ Generalization of Hamiltonian systems

- additional ports for interconnection
- possible dissipation

$\rightarrow$ Developed by Bernhard Maschke and Arjan van der Schaft in 1992

$\rightarrow$ Covers many (multi-body) physical systems

$\rightarrow$ Energy based modelling

$\rightarrow$ Structured/modular

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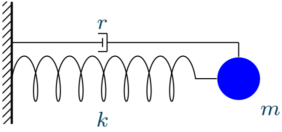
A simple example

Mass–spring–damper system:

$$m\ddot{z} = -k(z - z_0) - r\dot{z},$$



where m is a mass, k the spring constant, z_0 the rest length of the spring and $r \geq 0$ is the damping coefficient.

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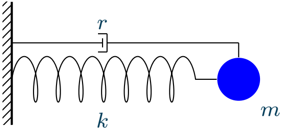

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Hamiltonian(-ish) formulation:

$$H(q,p) = \tfrac{1}{2}kq^2 + \tfrac{1}{2m}p^2 \qquad \text{(Hamiltonian)}$$

$$\tfrac{d}{dt}\begin{bmatrix}(z - z_0)\\ m\dot{z}\end{bmatrix} = \begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -r\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q}(q,p)\\ \frac{\partial H}{\partial p}(q,p)\end{bmatrix}$$

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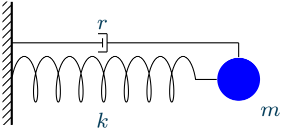

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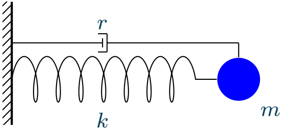

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$$\frac{d}{dt}\begin{bmatrix}(z - z_0)\\ m\dot{z}\end{bmatrix} = \begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = [J - R]\begin{bmatrix}\frac{\partial H}{\partial q}(q,p)\\ \frac{\partial H}{\partial p}(q,p)\end{bmatrix}$$

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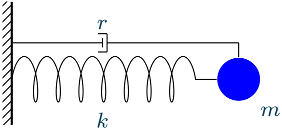

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Hamiltonian(-ish) formulation:

$$H(x) = \frac{1}{2}x^T Qx \qquad \text{(Hamiltonian)}$$

$$\dot{x} = [J - R]\nabla H(x)$$

Linear port–Hamiltonian Systems

Consider the linear system of equations

$$\dot{x} = [J - R]\, Qx + [B - P]\, u\,,$$
$$y = [B + P]^T\, Qx + [D + S]\, u\,,$$

where $x : \mathbb{R} \to \mathbb{R}^n$ and $u, y : \mathbb{R} \to \mathbb{R}^m$.

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where $x : \mathbb{R} \to \mathbb{R}^n$ and $u, y : \mathbb{R} \to \mathbb{R}^m$.

Take the matrices to be as follows :

- $J = -J^T \in \mathbb{R}^{n\times n}$,
- $R = R^T \in \mathbb{R}^{n\times n}$,
- $Q = Q^T \in \mathbb{R}^{n\times n}$ with $Q \geq 0$,
- $B \in \mathbb{R}^{n\times m}$, $P \in \mathbb{R}^{n\times m}$,
- $D = -D^T \in \mathbb{R}^{m\times m}$,
- $S = S^T \in \mathbb{R}^{m\times m}$,

such that $\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0.$

Energy interpretation

Introducing $H(x) = \frac{1}{2}x^T Q x$ results in

$$\dot{x} = [J - R]\,\nabla H(x) + [B - P]u\,,$$
$$y = [B + P]^T \nabla H(x) + [D + S]u\,.$$

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Interpretation:

- $x : \mathbb{R} \to \mathbb{R}^n$ is the state function.
- H is the energy/storage function, depending on x.
- $H(x)$ is the total stored energy in the system.
- u is the input of the system.
- y is the output of the system.
- $y^T \cdot u$ is the power supply.

Dissipative property

Consider the change in total energy $\frac{\mathrm{d}}{\mathrm{d}t}H(x(t))$.

Recall,

$$\dot{x} = [J - R]\,\nabla H(x) + [B - P]\,u\,,$$
$$y = [B + P]^T\,\nabla H(x) + [D + S]\,u\,,$$

where

- $J = -J^T$,
- $R = R^T$,
- $D = -D^T$,
- $S = S^T$,

such that $\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0.$

From LPHS to Dirac structures

Consider the general case

$$\dot{x} = [J - R]\,\nabla H(x) + [B - P]u\,,$$
$$y = [B + P]^T \nabla H(x) + [D + S]u\,.$$

Define the sets

$$\mathcal{D} := \{(f_S, f_{R,1}, f_{R,2}, f_P, e_S, e_{R,1}, e_{R,2}, e_P) \mid \begin{pmatrix} f_S \\ e_{R,1} \\ e_{R,2} \\ e_P \end{pmatrix} = \begin{pmatrix} -J & -I & 0 & -B \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ B^T & 0 & -I & D \end{pmatrix} \begin{pmatrix} e_S \\ f_{R,1} \\ f_{R,2} \\ f_P \end{pmatrix} \Big\},$$

$$\mathcal{R} := \left\{ (f_{R,1}, f_{R,2}, e_{R,1}, e_{R,2}) \mid \begin{pmatrix} f_{R,1} \\ f_{R,2} \end{pmatrix} = -\begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \begin{pmatrix} e_{R,1} \\ e_{R,2} \end{pmatrix} \right\}.$$

From LPHS to Dirac structures

$$\mathcal{D} := \left\{ (f_S, f_{R,1}, f_{R,2}, f_P, e_S, e_{R,1}, e_{R,2}, e_P) \mid \begin{pmatrix} f_S \\ e_{R,1} \\ e_{R,2} \\ e_P \end{pmatrix} = \begin{pmatrix} -J & -I & 0 & -B \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ B^T & 0 & -I & D \end{pmatrix} \begin{pmatrix} e_S \\ f_{R,1} \\ f_{R,2} \\ f_P \end{pmatrix} \right\},$$

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We reobtain the LPHS by the constitutive equations

$$f_S = -\dot{x}\,, \quad e_S = \nabla H(x)$$

together with the requirements

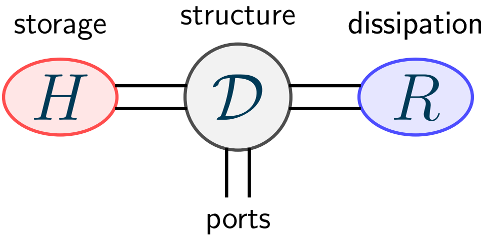
$$(f_S, f_R, f_P, e_S, e_R, e_P) \in \mathcal{D}, \quad (f_R, e_R) \in \mathcal{R}.$$

Some remarks

- The LPHS is fully described by the constitutive equations together with the pair $(\mathcal{D}, \mathcal{R})$.
- The splitting into storage, resisitve and external port variables has been made explicit.
- The matrix in the definition of $\mathcal{D}$ is skew symmetric, hence if $(f_S, f_R, f_P, e_S, e_R, e_P) \in \mathcal{D}$, then

$$\langle e_S, f_S \rangle + \langle e_R, f_R \rangle + \langle e_P, f_P \rangle = 0.$$

From LPHS to Dirac structures



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Abstract Dirac structures

We consider a finite dimensional linear space $\mathcal{F}$ together with its dual space $\mathcal{E} := \mathcal{F}^*$ and dual action

$$\langle \cdot \,|\, \cdot \rangle \, : \mathcal{E} \times \mathcal{F} \to \mathbb{R} \, .$$

Abstract Dirac structures

We consider a finite dimensional linear space $\mathcal{F}$ together with its dual space $\mathcal{E} := \mathcal{F}^*$ and dual action

$$\langle \cdot \,|\, \cdot \rangle \; : \mathcal{E} \times \mathcal{F} \to \mathbb{R} \,.$$

We call the elements of $\mathcal{F}$ flow variables and the elements of $\mathcal{E}$ effort variables.

Definition

A linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is called a (constant) Dirac structure if it satisfies the following two properties

(i) $\langle e \,|\, f \rangle = 0$ for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{F}$.

Alternative characterisation

Theorem

A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure if and only if it satisfies

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp} \, ,$$

where $\perp\!\!\!\perp$ denotes the orthogonal complement with respect to the indefinite inner product on $\mathcal{F} \times \mathcal{E}$ defined by

$$\langle\langle (f_1, e_1) \mid (f_2, e_2) \rangle\rangle = \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle \ .$$

Proof for the alternative characterisation

Assume : $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ satisfies $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}$.

Then for all $(f, e) \in \mathcal{D}$ we have

$$0 = \langle\langle (f, e) \,|\, (f, e) \rangle\rangle = 2 \langle e \,|\, f \rangle .$$

Furthermore, ($\perp\!\!\!\perp$ is non degenerate)

$$\dim \mathcal{D}^{\perp\!\!\!\perp} = \dim \mathcal{F} \times \mathcal{E} - \dim \mathcal{D} = 2 \dim \mathcal{F} - \dim \mathcal{D}^{\perp\!\!\!\perp}$$

implies that $\dim \mathcal{D} = \dim \mathcal{D}^{\perp\!\!\!\perp} = \dim \mathcal{F}$.

It follows : $\mathcal{D}$ is a Dirac structure.

Proof for the alternative characterisation

Assume : $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.

We then have for all $(f_1, e_1), (f_2, e_2) \in \mathcal{D}$ that $(f_1 + f_2, e_1 + e_2) \in \mathcal{D}$ and that

$$\begin{aligned} 0 &= \langle e_1 + e_2 \,|\, f_1 + f_2 \rangle \\ &= \langle e_1 \,|\, f_1 \rangle + \langle e_1 \,|\, f_2 \rangle + \langle e_2 \,|\, f_1 \rangle + \langle e_2 \,|\, f_2 \rangle \\ &= \langle e_1 \,|\, f_2 \rangle + \langle e_2 \,|\, f_1 \rangle \\ &= \langle\langle (f_1, e_1) \mid (f_2, e_2) \rangle\rangle \end{aligned}$$

and hence $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\!\perp}$. Again we have

$$\dim \mathcal{D}^{\perp\!\!\!\perp} = \dim \mathcal{F} \times \mathcal{E} - \dim \mathcal{D} = 2 \dim \mathcal{F} - \dim \mathcal{D} = \dim \mathcal{D}.$$

It follows : $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}$.

Separable Dirac structure

Definition

A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is called a separable Dirac structure if there is a subspace $\mathcal{K} \subset \mathcal{F}$ such that

$$\mathcal{D} = \mathcal{K} \times \mathcal{K}^{\perp},$$

where $\mathcal{K}^{\perp} := \{e \in \mathcal{E} \mid \forall f \in \mathcal{K} : \langle e \mid f \rangle = 0\}$.

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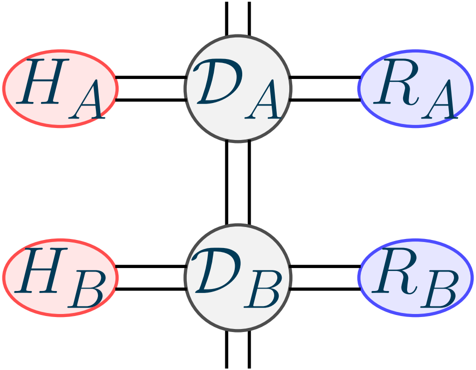
where $\mathcal{K}^{\perp} := \{e \in \mathcal{E} \mid \forall f \in \mathcal{K} : \langle e \mid f \rangle = 0\}$.

A Dirac structure $\mathcal{D}$ is separable if and only if we have

$$\langle e_1 \mid f_2 \rangle = 0$$

for all (f_1, e_1) and (f_2, e_2) in $\mathcal{D}$.

Composition of Dirac structures



Composition of Dirac structures

Suppose that we have two linear spaces

$$\mathcal{F}_A = \mathcal{F}_a \times \mathcal{F}_c\,, \quad \mathcal{F}_A^* = \mathcal{F}_a^* \times \mathcal{F}_c^*\,,$$
$$\mathcal{F}_B = \mathcal{F}_b \times \mathcal{F}_c\,, \quad \mathcal{F}_B^* = \mathcal{F}_b^* \times \mathcal{F}_c^*\,.$$

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$$\mathcal{F}_B = \mathcal{F}_b \times \mathcal{F}_c\,, \quad \mathcal{F}_B^* = \mathcal{F}_b^* \times \mathcal{F}_c^*\,.$$

Definition

The composition of two Dirac structures $\mathcal{D}_A \subset \mathcal{F}_A \times \mathcal{F}_A^*$ and $\mathcal{D}_B \subset \mathcal{F}_B \times \mathcal{F}_B^*$ (along $\mathcal{F}_c$) is defined as

$$\mathcal{D}_A \circ \mathcal{D}_B \subset (\mathcal{F}_a \times \mathcal{F}_b) \times (\mathcal{F}_a^* \times \mathcal{F}_a^*)$$

with

$$\mathcal{D}_A \circ \mathcal{D}_B \coloneqq$$
$$\{\,(f_A, f_B, e_A, e_B) \in (\mathcal{F}_a \times \mathcal{F}_b) \times (\mathcal{F}_a^* \times \mathcal{F}_a^*) \;\mid$$
$$\exists\,(f, e) \in \mathcal{F}_c \times \mathcal{F}_c^* : (f_A, f, e_A, e) \in \mathcal{D}_A,\ (f_B, -f, e_B, e) \in \mathcal{D}_B\}.$$

Composition of separable Dirac structures

For two separable Dirac structures

$$\mathcal{D}_A = \mathcal{K}_A \times \mathcal{K}_A^\perp \,,$$
$$\mathcal{D}_B = \mathcal{K}_B \times \mathcal{K}_B^\perp$$

the composition $\mathcal{D}_A \circ \mathcal{D}_B$ takes the simpler form

$$\mathcal{D}_A \circ \mathcal{D}_B = (\mathcal{K}_A \circ \mathcal{K}_B) \times (\mathcal{K}_A \circ \mathcal{K}_B)^\perp$$

with

$$\mathcal{K}_A \circ \mathcal{K}_B := \left\{ (f_A, f_B) \in (\mathcal{F}_a \times \mathcal{F}_b) \;\middle|\; \exists f \in \mathcal{F}_c : \begin{array}{c} (f_A, f) \in \mathcal{K}_A \,, \\ (f_B, -f,) \in \mathcal{K}_B \end{array} \right\} .$$

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Designating Variables

- Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ be a Dirac structure
- We first designate the storage, resistive and external port variables, i.e. we set

$$\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$$

and therefore

$$\mathcal{E} = \mathcal{F}_S^* \times \mathcal{F}_R^* \times \mathcal{F}_P^* =: \mathcal{E}_S \times \mathcal{E}_R \times \mathcal{E}_P \,.$$

Energy storage

- Define an (quadratic) energy function $H \in \mathcal{C}^1\left(\mathcal{F}_S, \mathbb{R}\right)$ denoting the total energy in the system.
- For $(f_S, e_S) \in (\mathcal{F}_S, \mathcal{E}_S)$ consider the constitutive equations given by

$$\dot{x} = -f_S, \qquad e_S = \nabla H\left(x\right),$$

or dually

$$\dot{x} = e_S, \qquad f_S = -\nabla H(x).$$

Resistive relation

- Define a *energy-dissipating (resistive) relation*

$$\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$$

such that $\langle e_R \mid f_R \rangle \leq 0$ for all $(f_R, e_R) \in \mathcal{R}$.

Dynamics on general Dirac structures

We can now define a dynamic on $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ by

$$\left(-\dot{x}\left(t\right), f_R\left(t\right), f_P\left(t\right), \nabla H\left(x\left(t\right)\right), e_R\left(t\right), e_P\left(t\right)\right) \in \mathcal{D},$$
$$\left(f_R\left(t\right), e_R\left(t\right)\right) \in \mathcal{R}\,.$$

- There is no unique dynamic on $\mathcal{D}$.
- We have freedom of choice in H, the splitting of $\mathcal{F}$ and also the constitutive equations.
- The derived systems are not always LPHS of the form introduced before $\rightsquigarrow$ DAE

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Outline

- $\rightarrow$ Starting from a Directed Graph;
- $\rightarrow$ Assign particular vertices (boundary vertices);
- $\rightarrow$ Introduce the incidence matrix; Functional spaces;
- $\rightarrow$ Define different Dirac structures;
- $\rightarrow$ Mass-spring-damper system: PH System;
- $\rightarrow$ Composition of graph Dirac structures.

Directed Graph

Definition

A **directed graph** $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ where:

- $\mathcal{V}$ is a set of vertices.
- $\mathcal{E}$ is a set of directed edges.

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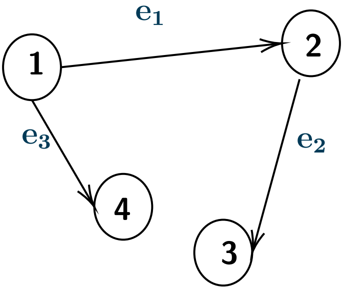
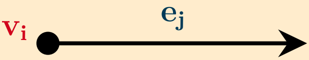
Example:

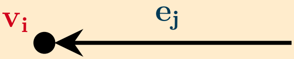


Incidence matrix

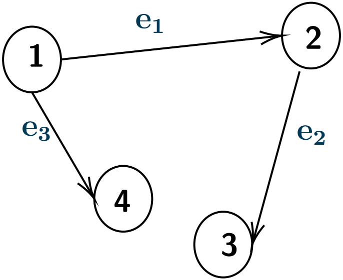
Definition

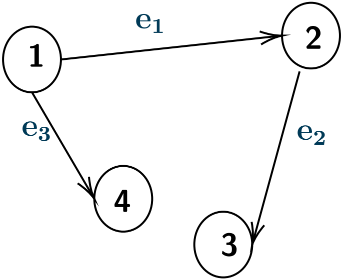
The **incidence matrix** B of $\mathcal{G}$ is an $n \times m$ matrix, where $n = |\mathcal{V}|$ and $m = |\mathcal{E}|$. Each row of B represents a vertex of $\mathcal{G}$, and each column represents an edge of $\mathcal{G}$, where

- $b_{i,j} = 1$, if 

- $b_{i,j} = -1$, if 

- $b_{i,j} = 0$, otherwise.


1
2
3
4
$\mathbf{e_1}$
$\mathbf{e_2}$
$\mathbf{e_3}$



$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we set

Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we set

$$\mathbb{R}^n \cong \Lambda_0 := \{f : \mathcal{V} \to \mathbb{R}\}, \quad \Lambda^0 := \Lambda_0^*,$$
$$\mathbb{R}^m \cong \Lambda_1 := \{f : \mathcal{E} \to \mathbb{R}\}, \quad \Lambda^1 := \Lambda_1^*,$$

flows efforts

Open Graphs

Definition

Let $\mathcal{V} = \mathcal{V}_i \cup \mathcal{V}_b$, where $\mathcal{V}_i \cap \mathcal{V}_b = \emptyset$.
Then, such a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called open graph with internal vertices $\mathcal{V}_i$ and boundary vertices $\mathcal{V}_b$.

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$$\Lambda_0 = \Lambda_{0i} \oplus \Lambda_{0b}$$
$$\Lambda^0 = \Lambda^{0i} \oplus \Lambda^{0b},$$
$$B = B_i \oplus B_b,$$

where

$\Lambda_{0i} \coloneqq \{f : \mathcal{V} \to \mathbb{R} | \operatorname{supp}(f) \subset \mathcal{V}_i\}, \quad \Lambda^{0i} \coloneqq \Lambda^*_{0i},$
$\Lambda_{0b} \coloneqq \{f : \mathcal{V} \to \mathbb{R} | \operatorname{supp}(f) \subset \mathcal{V}_b\}, \quad \Lambda^{0b} \coloneqq \Lambda^*_{0b},$

Open Graphs

Definition

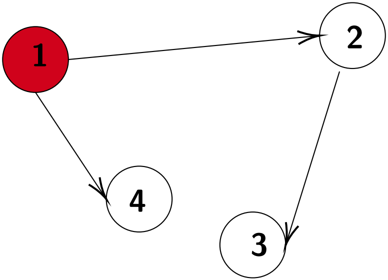
Let $\mathcal{V} = \mathcal{V}_i \cup \mathcal{V}_b$, where $\mathcal{V}_i \cap \mathcal{V}_b = \emptyset$.
Then, such a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called open graph with internal vertices $\mathcal{V}_i$ and boundary vertices $\mathcal{V}_b$.

$$\Lambda_0 = \Lambda_{0i} \oplus \Lambda_{0b}$$
$$\Lambda^0 = \Lambda^{0i} \oplus \Lambda^{0b},$$
$$B = B_i \oplus B_b,$$

where

$\Lambda_{0i} := \{f : \mathcal{V} \to \mathbb{R} | \operatorname{supp}(f) \subset \mathcal{V}_i\}, \quad \Lambda^{0i} := \Lambda_{0i}^*,$
$\Lambda_{0b} := \{f : \mathcal{V} \to \mathbb{R} | \operatorname{supp}(f) \subset \mathcal{V}_b\}, \quad \Lambda^{0b} := \Lambda_{0b}^*,$
$\Lambda_b := \{f : \mathcal{V}_b \to \mathbb{R}\}, \quad \Lambda^b := \Lambda_b^*.$

Example



Decomposition

$$B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{B_i} + \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{B_b}$$

$$\Lambda_{0i} \cong \mathbb{R}^3,\ \Lambda_{0b} \cong \mathbb{R}$$

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Dirac Structures

Flow-Continuous Dirac Structure

$$\mathcal{D}_f := \Big\{ \big(f_1, f_{0i}, f_b, e^1, e^{0i}, e^b\big) \in \Lambda_1 \times \Lambda_{0i} \times \Lambda_b \times \Lambda^1 \times \Lambda^{0i} \times \Lambda^b : \\ B_i f_1 = f_{0i},\ B_b f_1 = f_b,\ e^1 = -B_i^* e^{0i} - B_b^* e^b \Big\}$$

Effort-Continuous Dirac Structure

$$\mathcal{D}_e := \Big\{ \big(f_1, f_0, f_b, e^1, e^0, e^b\big) \in \Lambda_1 \times \Lambda_0 \times \Lambda_b \times \Lambda^1 \times \Lambda^0 \times \Lambda^b : \\ B_i f_1 = f_{0i},\ B_b f_1 = f_b + f_{0b},\ e^1 = -B^* e^0,\ e^b = e^{0b} \Big\}$$

Dirac Structures

Flow-Continuous Dirac Structure

$$\mathcal{D}_f := \Big\{ \big(f_1, f_{0i}, f_b, e^1, e^{0i}, e^b\big) \in \Lambda_1 \times \Lambda_{0i} \times \Lambda_b \times \Lambda^1 \times \Lambda^{0i} \times \Lambda^b : \\ B_i f_1 = f_{0i},\ B_b f_1 = f_b,\ e^1 = -B_i^* e^{0i} - B_b^* e^b \Big\}$$

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Theorem

$\mathcal{D}_f$ and $\mathcal{D}_e$ define two separable Dirac structures.

Theorem

$\mathcal{D}_f$ and $\mathcal{D}_e$ define two separable Dirac structures.

Recall (Separable Dirac structure)

$$\mathcal{D} = \mathcal{K} \times \mathcal{K}^{\perp}$$

Proof ($\mathcal{D}_f$).

We set

$$\mathcal{K} := \{(f_1, f_{0i}, f_b) \in \Lambda_1 \times \Lambda_{0i} \times \Lambda_b : B_i f_1 = f_{0i},\ B_b f_1 = f_b\}\,.$$

We will prove that

$$\mathcal{K}^\perp = \left\{\left(e^1, e^{0i}, e^b\right) \in \Lambda^1 \times \Lambda^{0i} \times \Lambda^b : e^1 = -B_i^* e^{0i} - B_b^* e^b\right\}.$$

Therefore, we obtain $\mathcal{D}_f = \mathcal{K} \times \mathcal{K}^\perp$ (separable Dirac structure).

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Therefore, we obtain $\mathcal{D}_f = \mathcal{K} \times \mathcal{K}^\perp$ (separable Dirac structure).

$$\mathcal{D}_f = \Big\{\left(f_1, f_{0i}, f_b, e^1, e^{0i}, e^b\right) \in \Lambda_1 \times \Lambda_{0i} \times \Lambda_b \times \Lambda^1 \times \Lambda^{0i} \times \Lambda^b : \\ B_i f_1 = f_{0i},\ B_b f_1 = f_b, \qquad e^1 = -B_i^* e^{0i} - B_b^* e^b\Big\}$$

Proof (Cont.).

Let $\left(e^1, e^{0i}, e^b\right) \in \mathcal{K}^\perp$ and let $f_1 \in \Lambda_1$. We set

$$f_{0i} = B_i f_1,\ f_b = B_b f_1,$$

Proof (Cont.).

Let $\left(e^1, e^{0i}, e^b\right) \in \mathcal{K}^\perp$ and let $f_1 \in \Lambda_1$. We set

$$f_{0i} = B_i f_1,\ f_b = B_b f_1,\ \text{i.e.,}\ (f_1, f_{0i}, f_b) \in \mathcal{K}.$$

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$$f_{0i} = B_i f_1,\ f_b = B_b f_1,\ \text{i.e.,}\ (f_1, f_{0i}, f_b) \in \mathcal{K}.$$

$$\left\langle e^1 \,|\, f_1 \right\rangle + \left\langle e^{0i} \,|\, f_{0i} \right\rangle + \left\langle e^b \,|\, f_b \right\rangle = 0,\ \text{for all}\ f_1,$$

Proof (Cont.).

Let $\left(e^1, e^{0i}, e^b\right) \in \mathcal{K}^\perp$ and let $f_1 \in \Lambda_1$. We set

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$\Longleftrightarrow$

$$\left\langle e^1 \,|\, f_1 \right\rangle + \left\langle e^{0i} \,|\, B_i f_1 \right\rangle + \left\langle e^b \,|\, B_b f_1 \right\rangle = 0,\ \text{for all } f_1,$$

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$\iff$

$$\left\langle e^1 \,|\, f_1 \right\rangle + \left\langle e^{0i} \,|\, B_i f_1 \right\rangle + \left\langle e^b \,|\, B_b f_1 \right\rangle = 0,\ \text{for all } f_1,$$

$\iff$

$$\left\langle e^1 + B_i^* e^{0i} + B_b^* e^b \,|\, f_1 \right\rangle = 0,\ \text{for all } f_1,$$

Proof (Cont.).

Let $\left(e^1, e^{0i}, e^b\right) \in \mathcal{K}^\perp$ and let $f_1 \in \Lambda_1$. We set

$$f_{0i} = B_i f_1, \; f_b = B_b f_1, \; \text{i.e., } \; (f_1, f_{0i}, f_b) \in \mathcal{K}.$$

$$\left\langle e^1 \,|\, f_1 \right\rangle + \left\langle e^{0i} \,|\, f_{0i} \right\rangle + \left\langle e^b \,|\, f_b \right\rangle = 0, \; \text{for all } f_1,$$

$\Longleftrightarrow$

$$\left\langle e^1 \,|\, f_1 \right\rangle + \left\langle e^{0i} \,|\, B_i f_1 \right\rangle + \left\langle e^b \,|\, B_b f_1 \right\rangle = 0, \; \text{for all } f_1,$$

$\Longleftrightarrow$

$$\left\langle e^1 + B_i^* e^{0i} + B_b^* e^b \,|\, f_1 \right\rangle = 0, \; \text{for all } f_1,$$

$\Longleftrightarrow$

$$e^1 = -B_i^* e^{0i} - B_b^* e^b.$$

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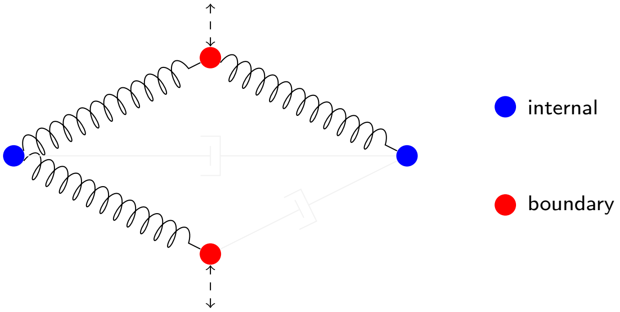
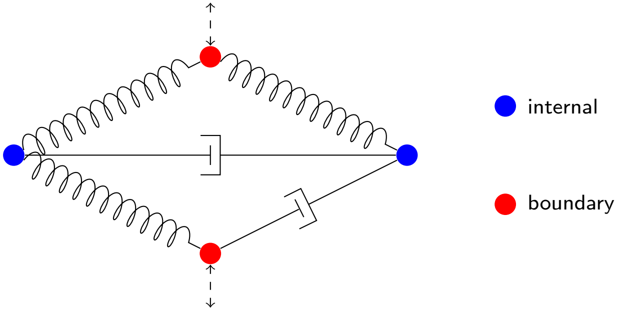


$$\mathcal{D}_f \; : \;\; B_{is} f_1 = f_{0i},\, B_{bs} f_1 = f_b,\, e^1 = -B_{is}^* e^{0i} - B_{bs}^* e^b$$

Mass–spring–damper system



$$\mathcal{D}_f : \begin{cases} \begin{bmatrix} B_{is} & B_{id} \end{bmatrix} \begin{bmatrix} f_{1s} \\ f_{1d} \end{bmatrix} = f_{0i}, \\ \begin{bmatrix} B_{bs} & B_{bd} \end{bmatrix} \begin{bmatrix} f_{1s} \\ f_{1d} \end{bmatrix} = f_b, \\ \begin{bmatrix} e^{1s} \\ e^{1d} \end{bmatrix} = -\begin{bmatrix} B_{is} & B_{id} \end{bmatrix}^* e^{0i} - \begin{bmatrix} B_{bs} & B_{bd} \end{bmatrix}^* e^b \end{cases}$$

$$\mathcal{R} : \;\; f_{1d} = -Re^{1d}$$

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Given two open graphs

$$\mathcal{G}^\alpha = (\mathcal{V}^\alpha, \mathcal{E}^\alpha)$$

$$\mathcal{G}^\beta = \left(\mathcal{V}^\beta, \mathcal{E}^\beta\right)$$

where

$$\mathcal{V}^\alpha := \mathcal{V}_i^\alpha \cup \underbrace{(\mathcal{V}_b^\alpha \cup \mathcal{V}_{bc})}_{\text{boundary vertices of } \mathcal{G}^\alpha}$$

$$\mathcal{V}^\beta := \mathcal{V}_i^\beta \cup \underbrace{\left(\mathcal{V}_b^\beta \cup \mathcal{V}_{bc}\right)}_{\text{boundary vertices of } \mathcal{G}^\beta}$$

Given two open graphs

$$\mathcal{G}^\alpha = (\mathcal{V}^\alpha, \mathcal{E}^\alpha)$$

$$\mathcal{G}^\beta = \left(\mathcal{V}^\beta, \mathcal{E}^\beta\right)$$

where

$$\mathcal{V}^\alpha := \mathcal{V}_i^\alpha \cup \underbrace{(\mathcal{V}_b^\alpha \cup \mathcal{V}_{bc})}_{\text{boundary vertices of } \mathcal{G}^\alpha}$$

$$\mathcal{V}^\beta := \mathcal{V}_i^\beta \cup \underbrace{\left(\mathcal{V}_b^\beta \cup \mathcal{V}_{bc}\right)}_{\text{boundary vertices of } \mathcal{G}^\beta}$$

Definition

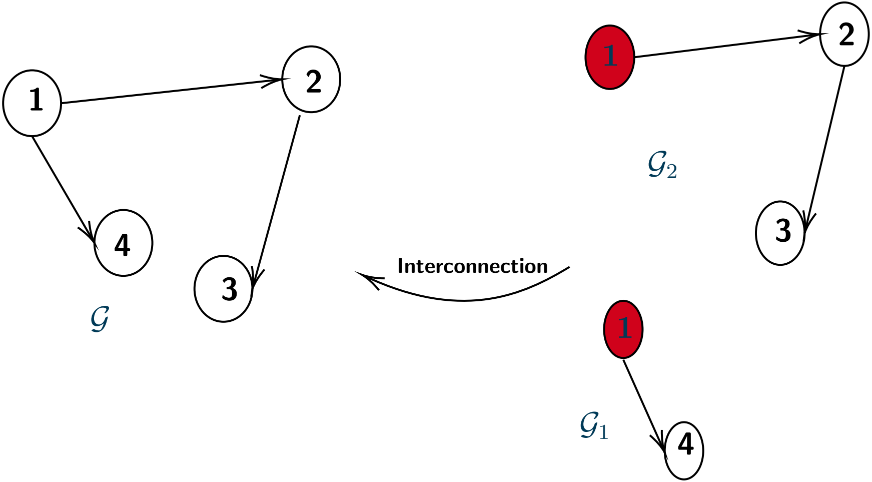
The interconnected graph from $\mathcal{G}^\alpha$ and $\mathcal{G}^\beta$ is an open graph $\mathcal{G}$ such that

$$\mathcal{V}_i := \mathcal{V}_i^\alpha \cup \mathcal{V}_i^\beta \cup \mathcal{V}_{bc}$$

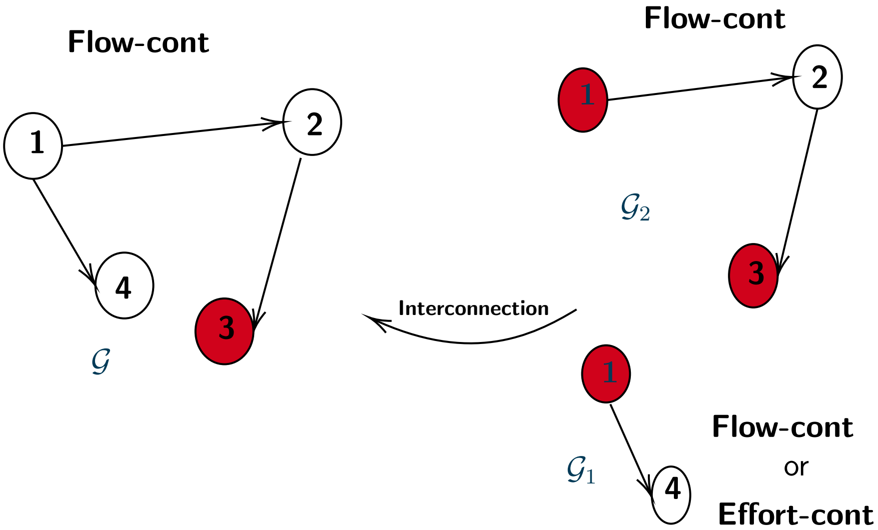
and

$$\mathcal{V}_b := \mathcal{V}_b^\alpha \cup \mathcal{V}_b^\beta$$

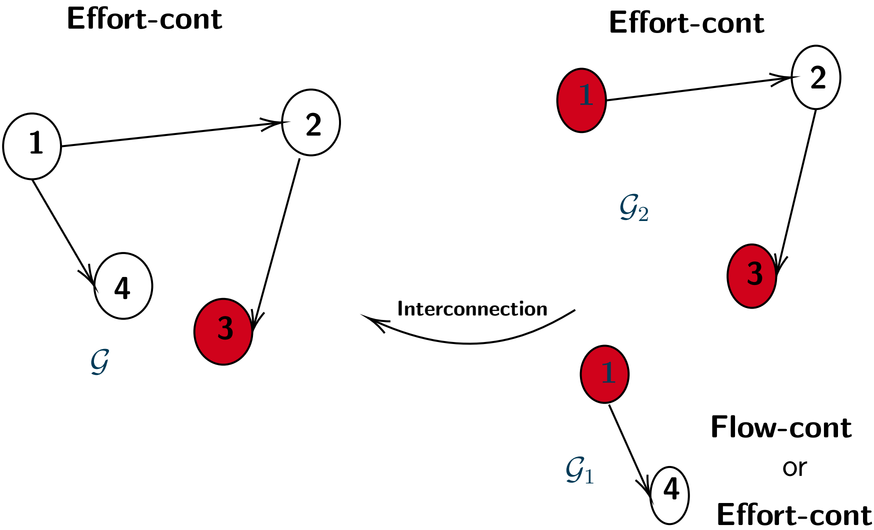
Derived Dirac structures



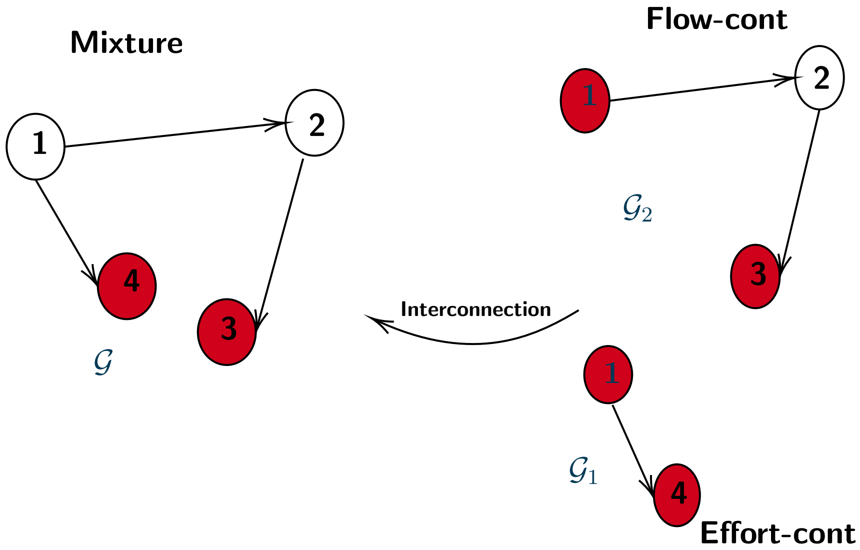
Derived Dirac structures



Derived Dirac structures



Derived Dirac structures



Conclusion

1. We introduced a geometric framework for treating physical dynamics on directed open graphs via Dirac structures.
2. The benefit of interconnections given intrinsically by the graph structure is that they allow to cover heterogeneous systems.
3. With the presented framework, we can also treat general mechanical systems, multibody systems, hydraulic networks, chemical reaction networks, etc.

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Thanks for your patience!