Energy Forms on Fractals

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Reference

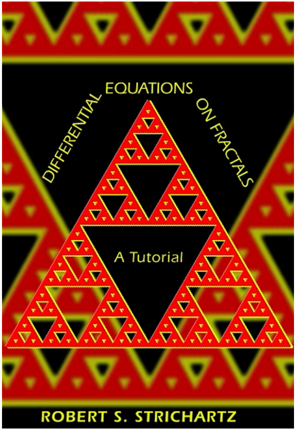


Figure: Book cover

Robert S. Strichartz
Differential Equations on Fractals
Princeton University Press (2006).

What is a fractal?

You know it, when you see it!

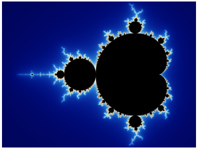


Figure: Mandelbrot set

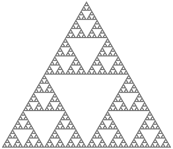


Figure: Sierpinksi gasket

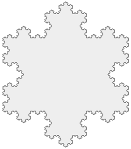


Figure: Koch snowflake

Common features:

- self-similarity
- roughness
- infinitely detailed
- generate complex structure from very simple procedure (e.g. from iterated function systems)

Iterated function systems

Definition 1.1

An *iterated function system (IFS)* is a family $F = \{F_i : \mathbb{R}^d \to \mathbb{R}^d : 1 \le i \le N\}$ of contractions.

Theorem 1.2 (Hutchinson, 1981)

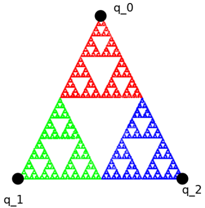
There is a unique non-empty, compact set $K \subset \mathbb{R}^d$ so that

$$K = F(K) := \bigcup_i F_i(K).$$

Example 1.3

(a) **Sierpinksi gasket:** Let $V_0 := \{q_0, q_1, q_2\}$ be the verticies of a proper triangle. Then these contractions characterize the Sierpinksi gasket $SG \subset \mathbb{R}^2$:

$$F_i(x) := \frac{1}{2}(x - q_i) + q_i$$



Addressing points in IFS fractals

Example 1.3 (Cont.)

(b) **Unit interval (not a fractal but self-similar):** Choose $V_0 := \{q_0 = 0, q_1 = 1\}$. Then

$$F_i(x) := \frac{1}{2}(x - q_i) + q_i$$

gives us the unit interval $I \subset \mathbb{R}$.

Iteratively, define $V_m := \bigcup_i F_i(V_{m-1})$ and note that one can write

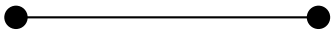
$$V_m = \bigcup_{|\omega|=m} F_\omega(V_0), \quad \text{where } F_\omega := F_{\omega_1} \circ \cdots \circ F_{\omega_m}$$

and $\omega \in \{1, \ldots, N\}^m$ is a *word* of length $|\omega| = m$.
So, $V_m \subset V_{m+1}$ and every point $x \in V_* := \bigcup_{m \geq 0} V_m$ can be written as $x = F_\omega q_i$. Every $x \in V_m \setminus V_0$ has exactly two addresses since $F_0 q_1 = F_1 q_0$, $F_1 q_2 = F_2 q_1$ and $F_2 q_0 = F_0 q_2$. Finally: Convince yourself, that $K = \overline{V_*}$.

Graph approximations

Example 1.4 (Graph approximations)

1. Unit interval:

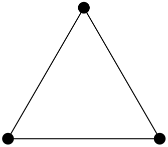


Γ_0

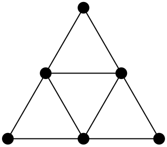
Γ_1

Γ_2

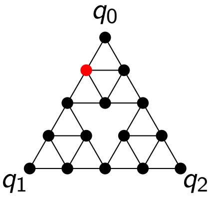
2. Sierpinksi gasket:



Γ_0



Γ_1



Γ_2

The red vertex is in $V_2 \setminus V_1$ and is addressed via $x = F_{(0,1)} q_0 = F_{(0,0)} q_1$.

Finite ramification and graph approximations

Definition 1.5

1. If ω is a word of with length $|\omega| = m$, we call $F_\omega(K)$ a *cell* of level m.
2. If two distinct cells of level m intersect in at most finitely many points, we say that K is *finitely ramified*.

In the case of $K \in \{I, SG\}$, two distinct m level cells intersect in at most one point and the set of those *junction points* is $V_m \setminus V_0$. We can thus define a relation on the set V_m through

$$x \sim_m y :\Leftrightarrow \text{there is a } \omega \text{ of length } |\omega| = m \text{ so that } x, y \in F_\omega K.$$

Now, we can consider graphs $\Gamma_m := (V_m, \sim_m)$.

Example where finite ramification fails

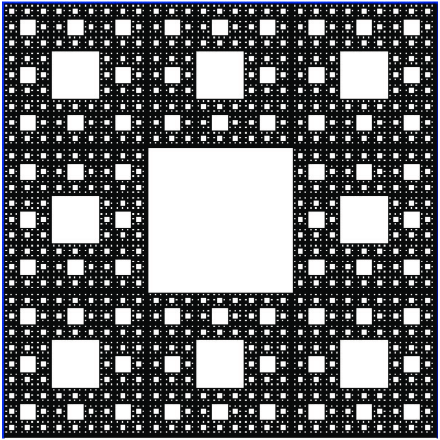
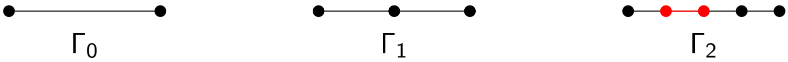


Figure: Sierpinksi carpet

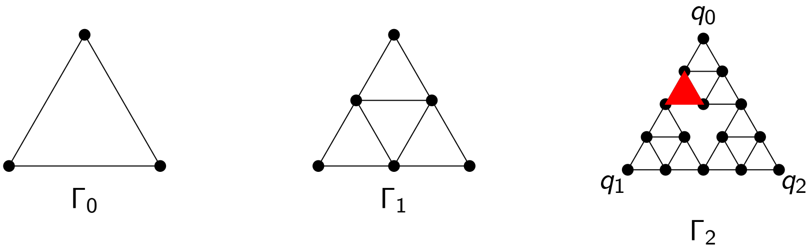
Graph approximations

Example 1.6 (Graph approximations)

1. Unit interval:



2. Sierpinksi gasket:



The red area is a level 2 cell and is given by $F_{(0,1)}K = F_0 F_1 K$.

Self-similar measures

Want: A measure μ on (K, Σ) which plays nicely with the self-similar structure of K.
Therefore, let Σ be the σ-field generated by all the cells of K, set

$$\mu(F_\omega SG) := \left(\frac{1}{3}\right)^{|\omega|}$$

and extend it to Σ by Carathéodory.
In the case of the unit interval, it is easy to see that $\mu(F_\omega I) := \left(\frac{1}{2}\right)^{|\omega|}$ extends to the Lebesgue measure.

Definition 1.7

We call the measure μ *self-similar* since

$$\mu(A) = \sum_i \mu_i \mu(F_i^{-1} A) \text{ for all } A \in \Sigma$$

for some positive weights with $\sum_i \mu_i = 1$.

Integration

We have canoncial measures on $K \in \{I, SG\}$, so we can integrate measurable functions $f : (K, \Sigma) \to (\mathbb{R}, \mathcal{B})$.

Lemma 1.8

For $f : K \to \mathbb{R}$ continuous, we obtain the usual integral via

$$\int_K f d\mu := \lim_{m\to\infty} \sum_{|\omega|=m} f(x_\omega)\mu(F_\omega K)$$

for some $x_\omega \in F_\omega K$.
This can also be written with a measure $\nu_m : V_m \to \mathbb{R}_+$ as

$$\int_K f d\mu = \lim_{m\to\infty} \frac{1}{3}\sum_{i=0}^{2} \sum_{|\omega|=m} f(F_\omega q_i)\mu(F_\omega K) = \lim_{m\to\infty} \int_{V_m} f(x) d\nu_m(x)$$

$$\lim_{m\to\infty} \overset{SG}{=} 3^{-m} \left(\frac{2}{3} \sum_{x\in V_m \setminus V_0} f(x) + \frac{1}{3} \sum_{x\in V_0} f(x) \right).$$

Dirichlet principle

Motivation: The Dirichlet principle states: Solving the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U \end{cases}$$

is equivalent to minimization of the energy functional

$$v \mapsto \frac{1}{2} \int_U \|\nabla v(x)\|^2 dx - \int_U f(x) v(x) dx$$

on a suitable function space.
Aim: Define something like the red energy via graph approx. for (continuous / measurable) functions $f : K \to \mathbb{R}$.
Use the unit interval $I = [0, 1]$ as a model case.

Graph energies for unit interval

Note that:

$$\int_U \|u'(x)\|^2 dx = \lim_{m\to\infty} \sum_{k=1}^{2^m} 2^{-m}[u'(x_k)]^2 \overset{\text{MVT}}{=} \lim_{m\to\infty} \sum_{k=1}^{2^m} 2^{-m} \left(\frac{u(k2^{-m}) - u((k-1)2^{-m})}{2^{-m}}\right)^2$$

$$= \lim_{m\to\infty} 2^m \sum_{x\sim_m y} [u(x) - u(y)]^2 =: \lim_{m\to\infty} r^{-m} E_m(u) =: \lim_{m\to\infty} \mathcal{E}_m(u)$$

with appropriate choices of $x_k \in ((k-1)2^{-m}, k2^m)$.

Systematic approach: One can show via harmonic extensions that

$$E_{m+1}(u) \geq E_{m+1}(\tilde{u}) = r \cdot E_m(u)$$

and thus

$$\mathcal{E}_{m+1}(u) = r^{-(m+1)} E_{m+1}(u) \geq r^{-m} E_m(u) = \mathcal{E}_m(u)$$

allows definition of

$$\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}_m(u).$$

Harmonic extension approach to graph energies

For $u : \mathbb{R} \to \mathbb{R}$, we define the m-th (graph) energies

$$E_m(u) := \sum_{x \sim_m y} (u(x) - u(y))^2 .$$

Question: What is the harmonic extension of u from Γ_m to Γ_{m+1}? I.e. find $\tilde{u}$ with $\tilde{u}(x) = u(x)$ for $x \in V_m$ and $\tilde{u}$ is a minimizer of E_{m+1}.

Now:

$$\begin{aligned} E_{m+1}(\tilde{u}) &= \sum_{k=1}^{2^{m+1}} [\tilde{u}(k2^{-(m+1)}) - \tilde{u}((k-1)2^{-(m+1)})]^2 \\ &= \sum_{k=1}^{2^m} [\tilde{u}(k2^{-m}) - \tilde{u}((k-1/2)2^{-m})]^2 + [\tilde{u}((k-1/2)2^{-m}) - \tilde{u}((k-1)2^{-m})]^2 \end{aligned}$$

Optimization yields $\tilde{u}((k-1/2)2^{-m}) = \frac{u(k2^{-m}) + u((k-1)2^{-m})}{2}$ and $E_{m+1}(\tilde{u}) = \frac{1}{2} \cdot E_m(u)$. Thus, $r = 1/2$.

Graph Energies

- The graph energy, on a connected finite graph G, is defined as

$$E_G(u) := \sum_{x \sim y} (u(x) - u(y))^2 \tag{1}$$

- It is a quadratic form in u associated to the following bilinear form

$$E_G(u, v) := \sum_{x \sim y} (u(x) - u(y))\,(v(x) - v(y)) \tag{2}$$

Let's denote by $G := (V, E)$, where

$$\begin{cases} V = & \text{set of vertices} \\ E = & \text{set of edges.} \end{cases}$$

Markov Property (Compatibility with with normal contraction)

- If $u : V \to \mathbb{R}_+$, we denote by $[u] = \min\{1, u \vee 0\}$. Then the energy $E_G(\cdot)$ satisfies the so called **Markov Property**, which is
$$E_G([u]) \leq E_G(u). \tag{3}$$
- Let $G' := (V', E')$ such that $G \subseteq G'$ is a sub-graph. For $u : V \to \mathbb{R}_+$ given, $\exists\ \tilde{u} : V' \to \mathbb{R}$ such that $\tilde{u}|_V = u$ and that
$$\inf_{u'|_V = u} E_{G'}(u') = E_{G'}(\tilde{u})$$
- $\tilde{u}$ is called the **harmonic extension**
- For all examples of interest to us (I and SG included) we have
$$E_{G'}(\tilde{u}) = r\, E_G(u) \quad \text{for all } u : V \to \mathbb{R}. \tag{4}$$

  for some $r \in (0.1)$. This called the **renormalization equation**.

Therefore we have

$$\frac{1}{r}\, E_{G'}(u') \geq E_G(u). \tag{5}$$

This means that energy increase with **general extension**, except in the case of harmonic extension where the energy is **constant**.

Example 2.1

Let's consider $K = SG$ and suppose $u(\cdot)$ is defined on V_0 by

$$\begin{cases} u(q_0) = a \\ u(q_1) = b \\ u(q_0) = c \end{cases} .$$

Then

$$\begin{aligned} E_1(\tilde{u}) = (a-z)^2 + (b-z)^2 + (b-x)^2 + (c-x)^2 + \\ + (a-y)^2 + (c-y)^2 + (x-y)^2 + (z-y)^2 + (x-z)^2. \end{aligned}$$

is to be minimized.

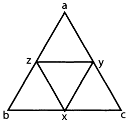


Figure: harmonic extension on V_1

- By calculating partial derivatives on x,y and z, we obtain

$$\begin{cases} u(F_0 q_1) = u(F_1 q_0) = x := & \frac{1}{5}a + \frac{2}{5}b + \frac{2}{5}c \\ u(F_0 q_2) = u(F_2 q_0) = y := & \frac{2}{5}a + \frac{1}{5}b + \frac{2}{5}c \\ u(F_1 q_2) = u(F_2 q_1) = z := & \frac{2}{5}a + \frac{2}{5}b + \frac{1}{5}c \end{cases} \tag{6}$$

- In the particular **symmetric** case, i.e. when $a = 1$ and $b = c = 0$, we obtain $x = 1/5$ and $y = z = 2/5$.

- Using (6), we can show easily that

$$E_1(\tilde{u}) = rE_0(u), \tag{7}$$

where $r = 3/5$, so the **renormalized energy** is given by:

$$\mathcal{E}_1(\tilde{u}) := r^{-1}E_1(\tilde{u}) = E_0(u).$$

- $\tilde{u}$ is the **harmonic extension** from V_1 to V_2.
- The same idea could be done, for $u : V_m \to \mathbb{R}$ given, to obtain the **harmonic extension** form V_m to V_{m+1} such that:

$$\begin{cases} E_{m+1}(\tilde{u}) = rE_m(u) \\ \mathcal{E}_{m+1}(\tilde{u}) = r^{-1}E_m(u) \end{cases} \tag{8}$$

- If $u : V_m \to \mathbb{R}$ is given and $u' : V_{m+1} \to \mathbb{R}$ to be any arbitrary extension, then

$$E_{m+1}(u') = \sum_{|\omega|=m} E_1(u' \circ F_\omega) \tag{9}$$

and

$$E_{m+1}(u') = \sum_{i=0,1,2} E_m(u' \circ F_i) \tag{10}$$

- The renormalized energy

$$\mathcal{E}_m(u) = (3/5)^{-m} E_m(u).,$$

is constant under the harmonic extension. Moreover, it's a nondecreasing sequence for any extension. i.e.

$$\mathcal{E}_0(u) \leq \mathcal{E}_1(u) \leq \ldots . \tag{11}$$

Summary

TO summarize: Let u be a function on V_m, then the harmonic extension $\tilde{u}$ to V_{m+1} may be characterized in three ways:

(i) it minimizes $\mathcal{E}_{m+1}(\tilde{u})$ at the value $\mathcal{E}_m(u)$;

(ii) at each new point $x \in V_{m+1} \backslash V_m$, $\tilde{u}(x)$ is **the average of the values at the four neighboring points in** V_{m+1}; i.e.

$$\tilde{u}(x) = \frac{1}{4} \sum_{\substack{y \in V_{m+1} \\ y \underset{m+1}{\sim} x}} \tilde{u}(y)$$

(iii) it satisfies the " $\frac{1}{5} - \frac{2}{5}$ rule" at the new points in $V_{m+1} \backslash V_m$.

Energy

- Since, for any function u on V_*, the sequence of energies $\mathcal{E}_m(u)$ is nondecreasing. It make sense to define

$$\mathcal{E}(u) := \lim_{m \to +\infty} \mathcal{E}_m(u).$$

- Moreover

$$\mathcal{E}(u) = 0 \quad \Leftrightarrow \quad u = const, \tag{12}$$

and

$$\mathrm{dom}(\mathcal{E}) = \{u : V_* \to \mathbb{R} \, : \, \mathcal{E}(u) < \infty\}. \tag{13}$$

Consequences

- Let $u : V_* \to \mathbb{R}$ such that $\mathcal{E}(u) < \infty$. Therefore

$$|u(x) - u(y)| \leq r^{m/2}\, \mathcal{E}^{1/2}(u) \quad \text{for all } x \underset{m}{\sim} y \in V_m. \tag{14}$$

- From the geometry of K (either I or SG) it can be seen that if $x, y \in V_*$ belong to the **same** or **adjacent** m-cell, with some technical calculation, we obtain

$$|u(x) - u(y)| \leq \frac{r^{m/2}}{1 - r^{1/2}}\, \mathcal{E}^{1/2}(u). \tag{15}$$

- When $K = I$ is the **interval**, (15) ensures that u is 1/2-Hölder continuous. Precisely; since $r = 1/2$, if x, y are such that $|x - y| \leq 1/2^m$ then x and y belongs to the same or adjacent m-cell. Hence (15) ensures that

$$|u(x) - u(y)| \leq M|x - y|^{1/2}. \tag{16}$$

- When $K = SG$, the $1/2$-Hölder continuity can be obtained in **another metric**, called the **resistance metric**, which we denote by $R(\cdot,\cdot)$. Hence if $u \in \mathrm{dom}(\mathcal{E})$ then for all $m \geq 1$

$$|u(x) - u(y)| \leq M \cdot R(x,y)^{1/2}, \tag{17}$$

  for all $x, y \in K$ such that $R(x,y) \leq r^m$.
- Moreover, this resistance metric $R(\cdot,\cdot)$ is equivalent to the Euclidean metric with some exponent $0 < \beta < 1$, i.e.

$$R(x,y) \sim \|x - y\|^{\beta} \quad \text{for} \quad \beta = \log \frac{5}{3} / \log 2. \tag{18}$$

Hilbert space property

Theorem 3.1

dom$(\mathcal{E})$*/constants forms a Hilbert space with the inner product given by*

$$(u, v)_{\mathcal{E}} := \mathcal{E}(u, v) := \lim_{m \to \infty} \mathcal{E}_m(u, v) \qquad \textit{for all } u, v \in \mathrm{dom}(\mathcal{E}) \quad (19)$$

Self-Similarity of the energy

Theorem 3.2

If $u \in \mathrm{dom}(\mathcal{E})$ *then* $u \circ F_i \in \mathrm{dom}(\mathcal{E})$ *for all* i*, and*

$$\mathcal{E}(u) = \sum_i r^{-1} \mathcal{E}\left(u \circ F_i\right) \tag{20}$$

This self-similarity of the energy follows from the following fact

$$\mathcal{E}_{m+1}(u) = \sum_{i=0}^{2} r^{-1} \mathcal{E}_m\left(u \circ F_i\right).$$

Moreover, for any partition $\mathcal{P}$, using the subdivision:

$$K = \bigcup_{\omega \in \mathcal{P}} F_\omega K, \tag{21}$$

and replacing r^{-1} by $r^{-|\omega|}$, we obtain

$$\mathcal{E}(u) = \sum_{\omega \in \mathcal{P}} r^{-|\omega|} \mathcal{E}\left(u \circ F_\omega\right). \tag{22}$$

Markov property/Compatibility with normal contraction

- The Markov property/Compatibility with normal contraction holds for all $u \in \mathrm{dom}(\mathcal{E})$, i.e.

$$\mathcal{E}([u]) \leq \mathcal{E}(u),$$

where $[u] = min\{1, u \vee 0\}$.

Electric Network Interpretation

Network Nodes (vertices) V and conductors (edges) with conductance (edge weight) C

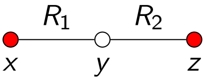
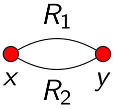
Resistance Reciprocal of conductance $R = 1/C$

Energy Voltage applied to nodes results in electric energy

$$E(u) = \sum_{x \sim y} C_{xy} \left| u(x) - u(y) \right|^2 \qquad (u : V \to \mathbb{R})$$

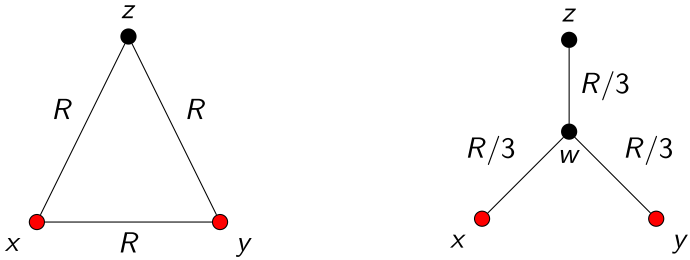
Restriction Only apply voltage to *some* nodes V' and let others settle into value

Familiar rules:

Resistors in series	Resistors in parallel
	
$R_{xz} = R_1 + R_2$	$\frac{1}{R_{xy}} = \frac{1}{R_1} + \frac{1}{R_2}$

Δ-Y-Transform

Claim. Resistance equivalent networks:



Proof. Combine rules for resistors in series and in parallel:

Left:

$$R_{xy} = \frac{1}{\frac{1}{R} + \frac{1}{2\,R}} = \frac{2\,R}{3}$$

Right:

$$R_{xy} = \frac{R}{3} + \frac{R}{3} = \frac{2\,R}{3}$$

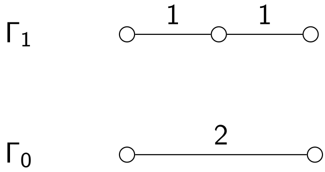
Remark

There exists a formula for the general case of unequal resistances.

Renormalization Problem

Set all resistances equal to 1 and compute resistance equivalent network!

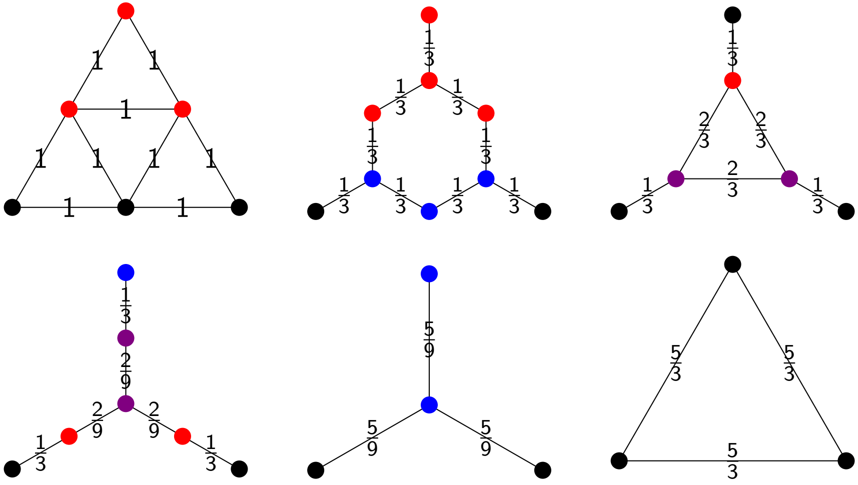
Unit interval: Apply rule for resistors in series (trivial):



Hence renormalization factor $r = \frac{1}{2}$.

Renormalization Problem II

SG:



Hence renormalization factor $r = \frac{3}{5}$.

Effective Resistance I

Given a nework, we define the effective resistance between any two points as the resistance between them when restricting to these two points:

Definition 5.1 (Effective Resistance)

Let $x, y \in V$.

$$R(x,y)^{-1} := \min\left\{\mathcal{E}(u) \mid u(x) = 0 \text{ and } u(y) = 1\right\}$$

Claim. Equivalent: Minimum value for R such that

$$|u(x) - u(y)|^2 \leq R\,\mathcal{E}(u) \qquad\qquad \text{for all } u \in \operatorname{dom}(\mathcal{E})\,.$$

Effective Resistance II

Proof. Let $u \in \mathrm{dom}\,(\mathcal{E})$ such that $u(x) = 0$ and $u(y) = 1$ and u minimizes $\mathcal{E}(u)$. Thus

$$1 = |u(x) - u(y)|^2 \leq R\,\mathcal{E}(u) = \frac{R}{R(x,y)}$$

yields $R(x,y) \leq R$.
For arbitrary $u \in \mathrm{dom}\,(\mathcal{E})$ with $u(x) \neq u(y)$ denote $v := \frac{u-u(x)}{u(y)-u(x)}$.
Then $v(x) = 0$ and $v(y) = 1$ and

$$R(x,y)^{-1} \leq \mathcal{E}(v) = \frac{\mathcal{E}(u)}{|u(x) - u(y)|^2}$$

yields $|u(x) - u(y)|^2 \leq R(x,y)\,\mathcal{E}(u)$ and therefore $R \leq R(x,y)$.

Effective Resistance on Unit Interval

Same as Eucledian distance:

- Function u achieving minimum in definition of eff. resistance is harmonic in the complement of x, y. Thus u is the linear extension:

$$u(t) = \begin{cases} 0 & t \in [0, x), \\ \frac{t-x}{y-x} & t \in [x, y), \\ 1 & t \in [y, 1] \end{cases} \qquad \text{(assume } x < y\text{)}.$$

- Hence

$$\mathcal{E}(u) = \int_0^1 \left| u'(t) \right|^2 \mathrm{d}t = \int_x^y \frac{1}{(y-x)^2} \mathrm{d}t = \frac{1}{y-x}$$

- We obtain

$$R(x, y) = |y - x| .$$

Effective Resistance is Metric

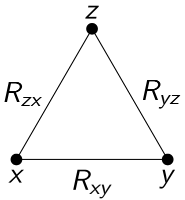
Theorem 5.2 (Tower Property)

Suppose $V'' \subseteq V' \subseteq V$. Given a network on V, the restriction to V'' is equal to the restriction V'' of the restrction to V'.

Claim. Effective resistance fullfills triangle inequality.

Proof. Only need to consider Δ-shaped networks. Combine rules for resistors in series and in parallel:

$$R(x,y) = \frac{1}{\frac{1}{R_{xy}} + \frac{1}{R_{yz}+R_{zx}}} = \frac{R_{xy}\,(R_{yz}+R_{zx})}{R_{xy}+R_{yz}+R_{zx}}$$



Short computation shows

$$R(x,y) + R(y,z) = R(z,x) + \frac{2\,R_{xy}\,R_{yz}}{R_{xy}+R_{yz}+R_{zx}} \geq R(z,x).$$

Effective Resistance on SG I

- Consider neighboring vertices $x, y \in V_m$.
- Chose $u = \psi_y^m$ as piecewise harmonic spline, i.e. $\psi_y^m(z) = \delta_{yz}$ for all $z \in V_m$. Then

$$\mathcal{E}(u) = \mathcal{E}_m(u) = r^{-m}\, E_m(u) = r^{-m} \sum_{x \sim_m y} |u(x) - u(y)|^2 \leq 4\, r^{-m}.$$

  Hence $R(x, y)^{-1} \leq \mathcal{E}(u) \leq 4\, r^{-m}$.
- $u(x) = 0$ and $u(y) = 1$ implies

$$\mathcal{E}(u) \geq \mathcal{E}_m(u) = r^{-m}\, E_m(u) \geq r^{-m}\, |u(x) - u(y)|^2 = r^{-m}.$$

- Conclude

$$r^{-m} \leq R(x, y)^{-1} \leq 4\, r^{-m}.$$

  Thus $R(x, y) \sim r^m$.

Effective Resistance on SG II

Expansion to other points is possible:

Theorem 5.3 (Estimates for R)

There exist $C_1, C_2 > 0$ *such that*

(a) *if* x, y *are in the same or adjecent m-cells*

$$R(x, y) \leq C_1 \, r^m.$$

(b) *if* x, y *are not in the same or adjecent m-cells*

$$R(x, y) \geq C_2 \, r^m.$$

Effective Resistance on SG III

Consider SG on equilateral triangle with edge length 1 and neighboring vertices $x, y \in V_m$:

- $|x-y| = 2^{-m} = \exp(m \log 1/2) \quad \Longrightarrow \quad m = \frac{\log|x-y|}{\log 1/2}$.
- Obtain

$$r^m = \exp(m \log r) = \exp\left(\frac{\log|x-y|}{\log 1/2} \log r\right) =: |x-y|^\beta$$

with

$$\beta = \frac{\log r}{\log 1/2} = \frac{-\log 1/r}{-\log 2} = \frac{\log 5/3}{\log 2}$$

- Thus

$$R(x,y) \sim |x-y|^\beta \qquad \text{with } \beta < 1.$$

- Effective resistance and Euclidian metric are topologically equivalent but they are not equivalent metrics.