The p-Laplacian on discrete graphs and the torsion function

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Final Workshop of ISem 26 - Group B

supervised by
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Agenda

1. Motivation
2. p-Laplacian on Graphs
3. p-Torsion
4. p-Torsional Rigidity
5. Torsional Rigidity, Ground State Energy and p-Cheeger Constant
6. Conclusions and Open Problems

Outline

Motivation - p-Laplacian

The classical Laplacian is

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What if I want an non-homogeneous diffusion rate?

One possibility is to define the p-Laplacian

$$\Delta^p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$$

Motivation - p-Laplacian

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$p > 2$

- If $|\nabla u| \ll 1 \to$ slow diffusion
- If $|\nabla u| \gg 1 \to$ fast diffusion
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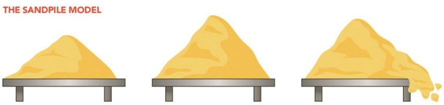
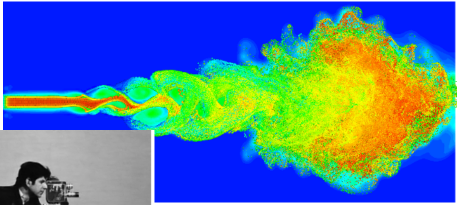
$p < 2$

- If $|\nabla u| \ll 1 \to$ fast diffusion
- If $|\nabla u| \gg 1 \to$ slow diffusion
- If $|\nabla u| \sim 1 \to$ intermediate situation

Applications

- For $p \sim 1$, the p-Laplacian is used for *image denoising*
- For $p \gg 1$, it can be used to model *growing/collapsing sandpiles*
- Fluid dynamics in turbulent regime



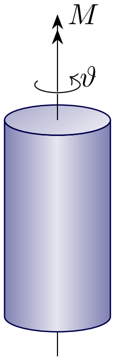


Motivation - Torsional Rigidity

- Twist isotropic cylinder around axis of symmetry
- Cross section is a bounded region $D \subset \mathbb{R}^2$
- The resulting stress function u fulfills

$$u_{xx} + u_{yy} + 1 = \Delta u + 1 = 0, \; u|_{\partial D} = 0$$

- Suppose twist by momentum M achieves a twist angle of ϑ, then $P\vartheta = M$ with *torsional rigidity* $P = 4 \iint_D u \mathrm{d}x \mathrm{d}y$
- P only depends on the cross-section geometry



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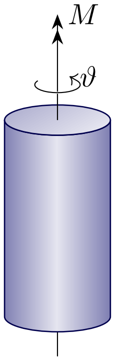
- Which shape has the maximal P?



Motivation - Circular Cross Section has Maximal Torsional Rigidity

- $\Delta u + 1 = 0$, $u|_{\partial D} = 0$, $P = 4 \iint_D u \mathrm{d}x\mathrm{d}y$, u stress
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Symmetrization wrt. plane of symmetrization

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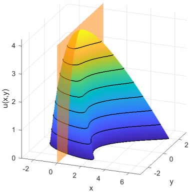

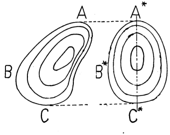

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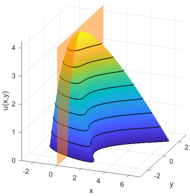

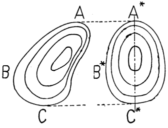

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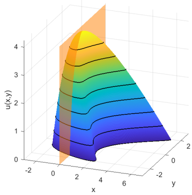

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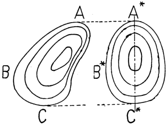

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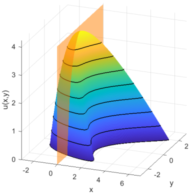

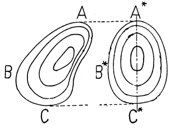

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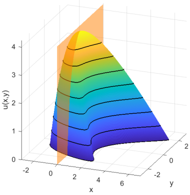

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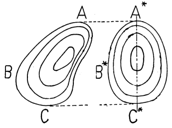
- increases torsional rigidity



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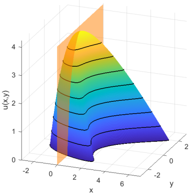

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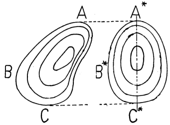

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Pólya concluded: *of all plane domains with a given area, the circle has the maximum torsional rigidity*

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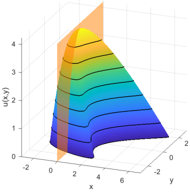

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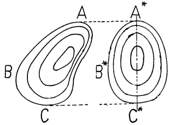

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What are analogies for graphs?

Outline

Definitions (infinite case)

(X, m) discrete measure space, (b, c) graph on (X, m)

$$\mathcal{F} := \left\{ u \in C(X) : \sum_{y \in X} b(x,y)|u(y)| < \infty (x \in X) \right\}$$

$$\mathcal{D}_{b,c} := \left\{ u \in C(X) : \frac{1}{2} \sum_{x,y \in X} b(x,y)\big(u(x) - u(y)\big)^2 + \sum_{x \in X} c(x) u^2(x) < \infty \right\}$$

$$\mathcal{Q}_{b,c}(u) := \begin{cases} \frac{1}{2} \sum_{x,y} b(x,y)\big(u(x) - u(y)\big)^2 + \sum_{x \in X} c(x) u^2(x), & u \in \mathcal{D}_{b,c} \\ \infty, & \text{else.} \end{cases}$$

- $\left(\mathcal{D}_{b,c} \cap \ell^2(X,m), u \mapsto \|u\|_{\ell^2(X,m)} + \mathcal{Q}_{b,c}(u)^{\frac{1}{2}} \right)$ is a Banach space
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Laplacian as Fréchet-Derivative of $\mathcal{Q}_p$

Lemma

$\mathcal{E}_p : \mathcal{D}_p \cap \ell^2(X,m) \to \mathbb{R}, u \mapsto \frac{1}{p}\mathcal{Q}_p(u)$ *is F-differentiable (e.g. Isem 13, Thm 4.2).*

Computing the G-derivative at u yields for $v \in \mathcal{D}_p \cap \ell^2(X,m)$

$$\begin{aligned}\mathcal{E}_p'(u)v &= \lim_{t\to 0} \frac{\mathcal{E}_p(u+tv) - \mathcal{E}_p(u)}{t} \\ &= \sum_x v(x) \sum_y b(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y)) + c(x)|u(x)|^{p-2}u(x) \\ &= \sum_x v(x)\partial\mathcal{E}_p(u)m(x) = \langle v, \partial\mathcal{E}_p(u)\rangle_{\ell^2(X,m)}\end{aligned}$$

p-Laplacian $L_{b,c}^{(p)}$ is defined as the F-gradient of $u \mapsto \frac{1}{p}\mathcal{Q}_p(u)$ on $\mathcal{D}_p \cap \ell^2(X,m)$

Laplacian as Fréchet-Derivative of $\mathcal{Q}_p$

Definition

Define the p-Laplacian $L_{b,c}^{(p)}$ as F-gradient of $u \mapsto \frac{1}{p}\mathcal{Q}_p(u)$ on $\mathcal{D}_p \cap \ell^2(X,m)$, i.e.

$$L_{b,c}^{(p)}u(x) = \frac{1}{m(x)}\sum_y b(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y)) + \frac{c(x)}{m(x)}|u(x)|^{p-2}u(x)$$

Using $\nabla_{x,y}u := u(x)-u(y)$ we get

$$L_{b,0}^{(p)}u(x) = \frac{1}{m(x)}\sum_y b(x,y)|\nabla_{x,y}u|^{p-2}\nabla_{x,y}u$$

from which we can recover

$$L_{b,0}^{(p)}u = \operatorname{div}\big(|\nabla_{x,y}u|^{p-2}\nabla_{x,y}u\big).$$

Ground state energy λ_0

$\lambda_0(L^{(p)}_{b,c}) := \inf_{u\in\mathcal{D}_p\cap\ell^2(X,m)} \frac{\mathcal{Q}^{(p)}_{b,c}(u)}{\|u\|^p_{\ell^p(X)}}$ is the *ground state energy*

Lemma

Suppose there is a $v \in \mathcal{D}_p \cap \ell^2(X,m)$ such that

$$L^{(p)}_{b,c} v = \lambda_0(L^{(p)}_{b,c})|v|^{p-2}v.$$

If $c \neq 0$ then the ground state energy is strictly positive.

- For $\inf_{x\in X} c(x) > 0$ directly from

$$\lambda_0(L^{(p)}_{b,c}) \geq \inf_{u\in\mathcal{D}_p\cap\ell^2(X,m)} \frac{\mathcal{Q}^{(p)}_{b}(u)+\inf_{x\in X} c(x)\|u\|^p_{\ell^p(X)}}{\|u\|^p_{\ell^p(X)}} \geq \inf_{x\in X} c(x)$$

Outline

p-Torsion

Definition

Let (b, c) be a graph over the set X. We call *p-torsion* the function $\tau_{b,c}$ such that

$$L_{b,c}^{(p)} \tau_{b,c} = \mathbf{1},$$

where $\mathbf{1} \in \ell^\infty(X)$ is the function equal to 1 on X.

Weak formulation for the Poisson problem

Consider the equation

$$L^{(p)}_{b,c} u = \mathbf{1} \quad \text{in } X$$

Multiplying both sides by v in $\ell^2(X)$

$$\langle L^{(p)}_{b,c} u, v \rangle = \langle \mathbf{1}, v \rangle$$

By Green's formula

$$\mathcal{Q}^{(p)}_{b,c}(u, v) - \sum_{x \in X} v(x) m(x) = 0 \quad \forall v \in \ell^2(X, m)$$

The functional $\mathfrak{F}_{b,c}^{(p)}$

We define the functional $\mathfrak{F}_{b,c}^{(p)}$, acting on $\mathcal{D}_p \cap \ell^1(X,m)$ as

$$\mathfrak{F}_{b,c}^{(p)} u := \mathcal{E}_p(u) - \sum_{x\in X} u(x)m(x) = \frac{1}{p}\mathcal{Q}_{b,c}^{(p)}(u) - \sum_{x\in X} u(x)m(x)$$

Then, the Frèchet derivative of $\mathfrak{F}_{b,c}^{(p)}$ $(=: \mathfrak{F})$ is

$$\mathfrak{F}'(u)v = \mathcal{E}_p'(u)v - \sum_{x\in X} v(x)m(x), \quad v \in \mathcal{D}_p \cap \ell^1(X,m)$$

Hence

$$\mathcal{Q}_{b,c}^{(p)}(u,v) - \sum_{x\in X} v(x)m(x) = 0 \iff \mathfrak{F}'(u)v = 0$$

$\rightsquigarrow$ **A weak solution is a stationary point of $\mathfrak{F}$!**

Hunting for a solution

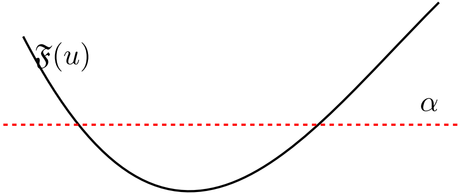
To find a weak solution, we can thus look for a minimum of $\mathfrak{F}$. This certainly exists and it's unique if

(i) $\mathfrak{F}$ is strictly convex, namely

$$\mathfrak{F}(\lambda u + (1-\lambda)v) < \lambda\mathfrak{F}(u) + (1-\lambda)\mathfrak{F}(v), \quad \forall \lambda \in (0,1)$$

(ii) $\mathfrak{F}$ is coercive, that is

$$\{u \in \mathcal{D}_p \cap \ell^1(X,m) : \mathfrak{F}(u) \leq \alpha\} \text{ is bounded in } \mathcal{D}_p \cap \ell^1(X,m), \quad (\alpha \in \mathbb{R})$$



Sufficient condition for existence of a solution

Lemma

Let $c \not\equiv 0$. The functional $\mathfrak{F}_{b,c}^{(p)}$ is differentiable and strictly convex. Furthermore, if there exists $M > 0$ such that

$$\|u\|_{\ell^1(X,m)} \leq M[\mathcal{Q}_{b,c}^{(p)}(u)]^{\frac{1}{p}}, \quad \forall u \in \mathcal{D}_p \cap \ell^1(X,m),$$

then it is coercive.

Proof of the Lemma: Strict Convexity

Split $\mathfrak{F}u$ as

$$\begin{aligned}\mathfrak{F}u &= \mathfrak{F}^{(1)}u + \mathfrak{F}^{(2)}u \\ &= \left(\sum_{x\in X} c(x)|u(x)|^p\right) + \left(\frac{1}{2}\sum_{x,y\in X} b(x,y)|u(x)-u(y)|^p - \sum_{x\in X} u(x)m(x)\right)\end{aligned}$$

Then

- $\mathfrak{F}^{(1)}$ is strictly convex
- $\mathfrak{F}^{(2)}$ is convex

and the sum of a convex function with a strictly convex function is strictly convex.

Proof of the Lemma: Coercivity

Fix $\alpha \in \mathbb{R}$ and let $u \in \mathcal{D}_p \cap \ell^1(X,m)$ be such that $\mathfrak{F}(u) \leq \alpha$. Then

$$\frac{1}{p}\mathcal{Q}_{b,c}^{(p)}(u) - \sum_{x\in X} u(x)m(x) \leq \alpha$$
$$\Rightarrow \quad [\mathcal{Q}_{b,c}^{(p)}(u)]^{\frac{1}{p}} \leq (\alpha p + p\|u\|_{\ell^1(X,m)})^{\frac{1}{p}}$$

Therefore

$$\|u\|_{\ell^1(X,m)} + \mathcal{Q}_{b,c}^{(p)}(u)^{\frac{1}{p}} \leq (1+M)\mathcal{Q}_{b,c}^{(p)}(u)^{\frac{1}{p}} \leq (1+M)(\alpha p + p\|u\|_{\ell^1(X,m)})^{\frac{1}{p}}$$

So that $\|u\|_{\ell^1(X,m)}$ is bounded by a uniform constant. But then, $\mathcal{Q}_{b,c}^{(p)}(u)^{\frac{1}{p}}$ is too.

Graphs with p-Torsion

Graphs with p-Torsion

Well...

Graphs with p-Torsion

Well... It's complicated.

Graphs with p-Torsion

Graphs with a **finite** number of vertices admits a p-Torsion function provided that

$$\inf_{x\in X} c(x) > 0$$

Indeed,

$$\begin{aligned}
\|u\|_{\ell^1(X,m)} &= \sum_{x\in X} |u(x)|\, m(x) \\
&\le m(X)\Big(\sum_{x\in X} \frac{1}{c(x)^{\frac{p}{q}}}\Big)^{\frac{1}{q}} \Big(\sum_{x\in X} c(x)|u(x)|^p\Big)^{\frac{1}{p}} \\
&\le M[\mathcal{Q}^{(p)}_{b,c}(u)]^{\frac{1}{p}}
\end{aligned}$$

which is the sufficient condition in the Lemma!

The case of infinite graphs

Definition

A graph (b, c) over some (possibly infinite) discrete measure space (X, m) is called (*uniquely*) formally p-torsion admissible with respect to some domain $\mathcal{D}(L_{b,c}^{(p)})$ if there exists a (unique) solution $\tau \in C(X)$ to the p-Poisson equation

$$L_{b,c}^{(p)} \tau = \mathbf{1}$$

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$\rightsquigarrow$ $\mathbf{1}$ should belong to $\ell^2(X, m)$
$\rightsquigarrow$ We ask that $m(X) < \infty$

A positivity property

Lemma

The operator $L_{b,c}^{(p)}$ is order-preserving, that is

$$L_{b,c}^{(p)}u \geq L_{b,c}^{(p)}v \Rightarrow u \geq v.$$

In particular, the p-Torsion function $\tau_{b,c}^{(p)}$ is strictly positive

Outline

p-Torsional Rigidity

Definition

Let (b, c) be a formally p-torsion admissible graph over some (finite) discrete measure space (X, m). The p-Torsional rigidity $T_p(b, c)$ of (b, c) with respect to the potential c is

$$T_p(b, c) := \left\| \tau_{b,c}^{(p)} \right\|_{\ell^1(X,m)}^{p-1}$$

Observe that

$$\left\| \tau_{b,c}^{(p)} \right\|_{\ell^1(X,m)} = \langle \tau_{b,c}^{(p)}, \mathbf{1} \rangle = \langle \tau_{b,c}^{(p)}, L_{b,c}^{(p)} \tau_{b,c}^{(p)} \rangle = \mathcal{Q}_{b,c}^{(p)}(\tau_{b,c}^{(p)})$$

Variational characterization of p-Torsional rigidty

Given a graph (b, c), we define the p-Polya quotients as

$$\mathsf{P}^{(p)}_{b,c}(u) := \frac{\|u\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_{b,c}(u)}$$

Lemma

There holds

$$T_p(b,c) = \max_{u \in \mathcal{D}_p \cap \ell^2(X,m)} \mathsf{P}^{(p)}_{b,c}(u).$$

Moreover, if there exists a unique minimizer of $\mathfrak{F}^{(p)}_{b,c}(u)$*, then*

$$T_p(b,c) = \left(\frac{p}{1-p} \min_{u \in \mathcal{D}_p \cap \ell^1(X,m)} \mathfrak{F}^{(p)}_{b,c}(u) \right)^{p-1}$$

Proof

Since (b,c) is p-torsion admissible, then $\tau_{b,c}^{(p)} \in \mathcal{D}_p \cap \ell^1(X,m)$ minimizes the functional $\mathfrak{F}_{b,c}^{(p)}$. Therefore

$$\begin{aligned}
\min_{u \in \mathcal{D}_p \cap \ell^1(X,m)} \mathfrak{F}_{b,c}^{(p)}(u) &= -\sum_{x \in X} \tau_{b,c}^{(p)}(x) m(x) + \frac{1}{p} \mathcal{Q}_{b,c}^{(p)}(\tau_{b,c}^{(p)}) \\
&= -\left\| \tau_{b,c}^{(p)} \right\|_{\ell^1(X,m)} + \frac{1}{p} \left\| \tau_{b,c}^{(p)} \right\|_{\ell^1(X,m)} \\
&= \frac{1-p}{p} \left\| \tau_{b,c}^{(p)} \right\|_{\ell^1(X,m)} \\
&= \frac{1-p}{p} [T_p(b,c)]^{\frac{1}{p-1}}
\end{aligned}$$

Proof (continued)

Since the p-torsion is positive, making the change of variables $u \mapsto tu$ $(t > 0)$,

$$\min_{u \in \mathcal{D}_p \cap \ell^1(X,m)} \mathfrak{F}^{(p)}_{b,c}(u) = \min_{u \in \mathcal{D}_p \cap \ell^1(X,m)} \min_{t \in \mathbb{R}^+} \left(\frac{t^p}{p} \mathcal{Q}^{(p)}_{b,c}(u) - t \|u\|_{\ell^1(X,m)} \right)$$

But

$$\min_{t \in \mathbb{R}} \left(\frac{t^p}{p} \mathcal{Q}^{(p)}_{b,c}(u) - t \|u\|_{\ell^1(X,m)} \right) = \frac{1-p}{p} \left[\frac{\|u\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_{b,c}(u)} \right]^{\frac{1}{p-1}}$$

Hence

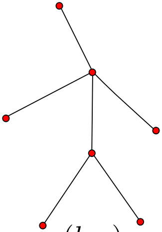
$$\min_{u \in \mathcal{D}_p \cap \ell^1(X,m)} \mathfrak{F}^{(p)}_{b,c}(u) = \frac{1-p}{p} \min \left[\mathsf{P}^{(p)}_{b,c} \right]^{\frac{1}{p-1}} = \frac{p-1}{p} \max \left[\mathsf{P}^{(p)}_{b,c} \right]^{\frac{1}{p-1}}$$

Properties of the torsional rigidity

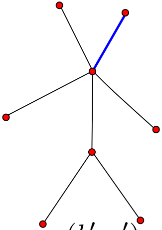
Lemma (Monotonicity wrt. edges)

Let $X \subset X'$ be two sets and (b,c) a subgraph of (b',c') which are both formally p-torsion admissible. Assume that the graph b' is a tree (i.e. it does not contain a closed path) and that c' has support contained in X. Then

$$T_p(b,c) \leq T_p(b',c').$$

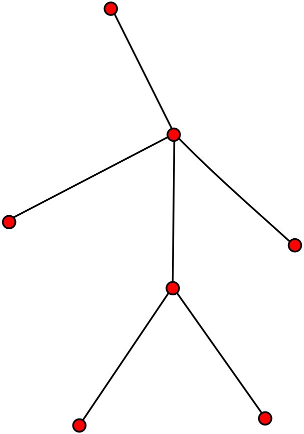


(b,c)



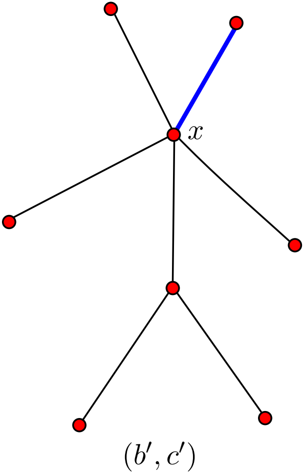
(b',c')

Sketch of Proof



(b, c)

Sketch of Proof



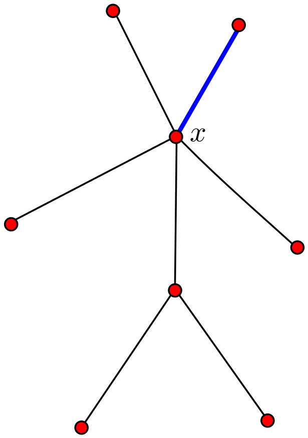
Sketch of Proof

Given the torsion function τ_p on (b, c), define the function:

$$\tau'_p(y) = \begin{cases} \tau_p(y), & y \in X \\ \tau_p(x), & y \in X' \setminus X \end{cases}$$

Then one can show that

$$\begin{aligned} T_p(b', c') &\geq \mathsf{P}^{(p)}_{b',c'}(\tau'_p) \\ &\geq \mathsf{P}^{(p)}_{b,c}(\tau_p) = T_p(b, c) \end{aligned}$$



(b', c')

Lemma (Monotonicity wrt. weights)

Let (b, c) and (b', c') be formally p-torsion admissible graphs over the same discrete measure space (X, m). If

$$b \leq b', \quad \text{and} \quad c \leq c'$$

pointwise, then

$$T_p(b', c') \leq T_p(b, c)$$

In particular,

$$\tau^{(p)}_{b',c'} \leq \tau^{(p)}_{b,c}$$

The proof is an immediate consequence of the fact that

$$\mathcal{Q}^{(p)}_{b,c}(u) \leq \mathcal{Q}^{(p)}_{b',c'}(u)$$

for every $u \in \mathcal{D}'_p$

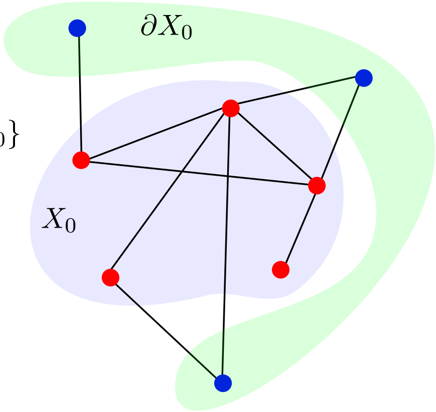
The Dirichlet case

Let $c \equiv 0$. Let $X_0 \subset X$ and consider the subgraph b_0 of b induced by X_0. We call

$$\partial X_0 = \{x \in X \setminus X_0 : x \sim y \text{ for some } y \in X_0\}$$

The p-Laplacian with Dirichlet boundary conditions on ∂X_0, i.e. $L^{(D)}_{b_0,p}$, is the nonlinear operator associated to the form

$$\mathcal{Q}^{(D)}_{b_0,p}(u) := \mathcal{Q}^{(p)}_{b_0}(\mathbf{1}_{X_0} u)$$



Dirichlet BC and Torsion

In the Dirichlet framework, we define the p-torsion function and p-torsional rigidity as

$$\tau_{b_0}^{(p)} := (L_{b_0,p}^{(D)})^{-1}\mathbf{1}, \qquad T_p(b, X_0) := \left\|\tau_{b_0}^{(p)}\right\|_{\ell^1(X,m)}^{p-1}$$

It is possible to show that all the results discussed so far holds for $\tau_{b_0}^{(p)}$. In particular

$$T_p(b, X_0) = \max_{\substack{u\in\mathcal{D}_p\cap\ell^2(X,m)\\ u|_{\partial X_0}=0}} \frac{\|u\|_{\ell^1(X,m)}^p}{\mathcal{Q}_b^{(p)}(u)}$$

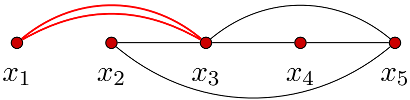
Outline

Surgery principle for $|X| \in \mathbb{N},\ c = 0$, η-fold edge connected

Assume $u^* \in \mathcal{D}_{p,c} \cap \ell^2(X,m)$ maximizes $u \mapsto \dfrac{\|u\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_b(u)}$

There is an ordering of X s.t. $u^*(x_1) \leq \ldots \leq u^*(x_{|X|})$.

- Assume there is x_i, x_j s.t. $b(x_i, x_j) > 0$, $j - i \geq 2$



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- Assume there is x_i, x_j s.t. $b(x_i, x_j) > 0$, $j - i \geq 2$
- Modify b to b' by $b'(x_i, x_{i+1}) = \ldots = b'(x_{j-1}, x_j) = b(x_i, x_j)$, then

$$\left(\sum_{l=i}^{j-1} b'(x_l, x_{l+1})|u^*(x_l) - u^*(x_{l+1})|^p\right)^{\frac{1}{p}}$$

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$$b(x_i, x_j)^{\frac{1}{p}} \left(\sum_{l=i}^{j-1} |u^*(x_l) - u^*(x_{l+1})|^p \right)^{\frac{1}{p}} \leq \ b(x_i, x_j)^{\frac{1}{p}} \sum_{l=1}^{j-1} |u^*(x_l) - u^*(x_{l+1})|$$

Surgery principle for $|X| \in \mathbb{N},\ c = 0,\ \eta$-fold edge connected

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- Take p-th power and add all terms for the remaining edges, then $\mathcal{Q}_{b'}(u^*) \leq \mathcal{Q}_b(u^*)$

Surgery principle for $|X| \in \mathbb{N},\ c = 0$, η-fold edge connected

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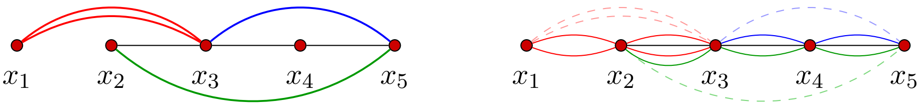
- Take p-th power and add all terms for the remaining edges, then $\mathcal{Q}_{b'}(u^*) \leq \mathcal{Q}_b(u^*)$, repeat to pumpkin chain



Surgery principle for $|X| \in \mathbb{N}, \ c = 0, \ \eta$-fold edge connected

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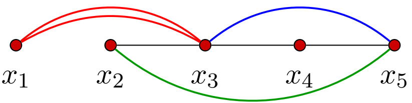
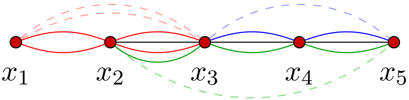
- Take p-th power and add all terms for the remaining edges, then $\mathcal{Q}_{b'}(u^*) \leq \mathcal{Q}_b(u^*)$, repeat to pumpkin chain





Surgery principle: Rewiring to a chain-graph reduces the energy value for a maximizer of the p-Polya quotient

$|X| \in \mathbb{N}$ - induced path graphs upper-bound torsional rigidity

Theorem

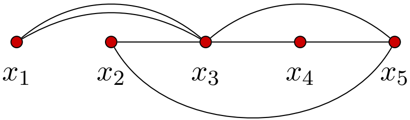
Let b be η-fold edge connected graph with standard weights, $\emptyset \neq X_0 \subsetneq X$, $\tilde{b}$ a path graph on $\#X_0$ vertices with Dirichlet conditions at one endpoint. Then:

$$T_p(b; X_0) \leq \frac{1}{\eta} T_p(\tilde{b}; X_0).$$

Equality holds if and only if b is an η-regular pumpkin chain with Dirichlet conditions at one endpoint.

$|X| \in \mathbb{N}$ - induced path graphs upper-bound torsional rigidity - Pf.

Suppose τ maximizes $u \mapsto \frac{\|u\|^p_{\ell^1(X,m)}}{\mathcal{Q}^p_b(u)}$

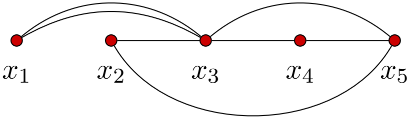


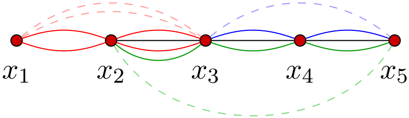
$|X| \in \mathbb{N}$ - induced path graphs upper-bound torsional rigidity - Pf.

Suppose τ maximizes $u \mapsto \frac{\|u\|^p_{\ell^1(X,m)}}{\mathcal{Q}^p_b(u)}$

Repeat surgery principle until modified graph $\hat{b}$ is a pumpkin chain, then

$$T_p(b, X_0) \leq \frac{\|\tau\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_b(\tau)} \leq \frac{\|\tau\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_{\hat{b}}(\tau)}$$



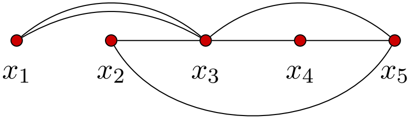


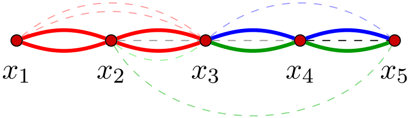
$|X| \in \mathbb{N}$ - induced path graphs upper-bound torsional rigidity - Pf.

Suppose τ maximizes $u \mapsto \frac{\|u\|^p_{\ell^1(X,m)}}{\mathcal{Q}^p_b(u)}$

Repeat surgery principle until modified graph $\hat{b}$ is a pumpkin chain, and remove edges to reduce $\mathcal{Q}$, repeat till η-regular pumpkin chain $\tilde{b}$, then

$$T_p(b, X_0) \leq \frac{\|\tau\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_b(\tau)} \leq \frac{\|\tau\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_{\hat{b}}(\tau)} \leq \frac{\|\tau\|^p_{\ell^1(X,m)}}{\mathcal{Q}^{(p)}_{\tilde{b}}(\tau)} \leq T_p(\tilde{b}, X_0)$$





$|X| \in \mathbb{N}$ - the ground state energy is bounded from below

Theorem

Let b be η-fold edge connected graph with standard weights, $\emptyset \neq X_0 \subsetneq X$, b_0 the subgraph of b induced by X_0, $\tilde{b}_0$ a path graph on $\# X_0$ with Dirichlet conditions at one endpoint. The ground state energy $\lambda_0(L_{b_0}^{(D)})$ is bounded from below as follows:

$$\lambda_0(L_{b_0}^{(D)}) \geq \eta \lambda_0(L_{\tilde{b}_0}^{(D)}) \geq \eta \left(1 - \cos\left(\frac{\pi}{2\# X_0 + 1}\right)\right)$$

Equality holds if and only if b is an η-regular pumpkin chain with Dirichlet conditions at one endpoint.

$|X| \in \mathbb{N}$ - the ground state energy is bounded from below

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Equality holds if and only if b is an η-regular pumpkin chain with Dirichlet conditions at one endpoint.

- Proof again uses surgery principle
- *Variational* proof of Fiedler's theorem on algebraic connectivity of graphs

A geometrical perspective

We have shown that

$$T_p(b; X_0) \leq \frac{1}{\eta} T_p(\tilde{b}; X_0)$$
$$\lambda_0(L^{(D)}_{b_0}) \geq \eta \lambda_0(L^{(D)}_{\tilde{b}_0})$$

This is remarkably similar to the classical inequalities

$$T(\Omega) \leq |\Omega|^2 \, T(B) \qquad \text{(Saint-Venant)}$$
$$\lambda_0(\Omega) \geq \frac{1}{|\Omega|} \lambda_0(B) \qquad \text{(Faber-Krahn)}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and B is the unit ball of $\mathbb{R}^N$

A geometrical perspective

$$\begin{aligned} \eta\text{-regular Pumpkin Chains} \quad &\longleftrightarrow \quad \text{Balls in } \mathbb{R}^N \\ \text{Surgery principle} \quad &\longleftrightarrow \quad \text{Symmetrization} \end{aligned}$$

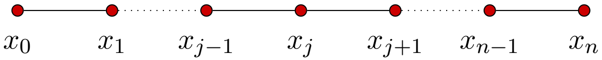
Torsional rigidity of a path graph

- b is some torsion admissible path graph on $X_0 = \{x_0, x_1, \ldots, x_n\}$
- For now, let $p = 2$ (linear case)
- Dirichlet condition at x_0, so $\tau(x_0) = 0$

Torsional rigidity of a path graph

The equation $L_b\tau = 1$ reads

$$\frac{1}{m(x_j)}\left[b(x_j,x_i)(\tau(x_j)-\tau(x_{j-1}))+b(x_{j+1},x_j)(\tau(x_j)-\tau(x_{j+1}))\right]=1$$
$$\frac{1}{m(x_n)}b(x_n,x_{n-1})(\tau(x_n)-\tau(x_{n-1}))=1$$

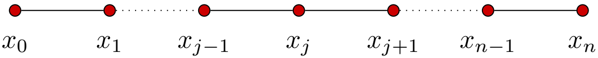


Torsional rigidity of a path graph

Isolating $\tau(x_j)$ we get

$$\begin{aligned}(b(x_{j-1},x_j)+b(x_j,x_{j+1}))\tau(x_j)-b(x_j,x_{j-1})\tau(x_{j-1})-b(x_j,x_{j+1})\tau(x_{j+1}) &= m(x_j)\\ -b(x_n,x_{n-1})\tau(x_{n-1})+b(x_n,x_{n-1})\tau(x_n) &= m(x_n)\end{aligned}$$

This is a linear system!



Torsional rigidity of a path graph

The solution to the system reads

$$\tau(x_j) = \sum_{\ell=1}^{j} \frac{1}{b(x_{\ell-1}, x_\ell)} \sum_{k=\ell}^{n} m(x_k)$$
$$\tau(x_0) = 0$$

In particular, with standard weigths $(b \equiv 1)$ and $m = \mathbf{1}$, we obtain

$$\tau(x_j) = j\left((n+1) - \frac{j+1}{2}\right)$$

So

$$T(b) = \frac{n(n+1)(2n+1)}{6}$$

An explicit upper bound

Corollary

Let b be a finite graph with $n+1$ vertices $x_0, \ldots, x_n$, standard weights over X and Dirichlet vertex condition at x_0. Assume that b is η-fold edge connected. Furthermore, let $\emptyset \neq X_0 \subsetneq X$, such that the subgraph induced by X_0 is connected. Then

$$T(b; X_0) \leq \frac{n(n+1)(2n+1)}{6\eta}$$

p-Torsional rigidity of a path graph

With $L_b\tau = 1$ and $\tau(x_0) = 0$:

$$\frac{1}{m(x_j)}\Big[b(x_j, x_{j-1})(\tau(x_j) - \tau(x_{j-1})) \\ + b(x_{j+1}, x_j)(\tau(x_j) - \tau(x_{j+1}))\Big] = 1$$

$$\frac{1}{m(x_n)} b(x_n, x_{n-1})(\tau(x_n) - \tau(x_{n-1})) = 1$$

p-Torsional rigidity of a path graph

With $L_b^{(p)}\tau = 1$ and $\tau(x_0) = 0$:

$$\frac{1}{m(x_j)}\Big[b(x_j, x_{j-1})|\tau(x_j) - \tau(x_{j-1})|^{p-2}(\tau(x_j) - \tau(x_{j-1})) + b(x_{j+1}, x_j)|\tau(x_{j+1}) - \tau(x_j)|^{p-2}(\tau(x_j) - \tau(x_{j+1}))\Big] = 1$$

$$\frac{1}{m(x_n)}b(x_n, x_{n-1})|\tau(x_n) - \tau(x_{n-1})|^{p-2}(\tau(x_n) - \tau(x_{n-1})) = 1$$

p-Torsional rigidity of a path graph

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$$\frac{1}{m(x_n)}b(x_n, x_{n-1})|\tau(x_n) - \tau(x_{n-1})|^{p-2}(\tau(x_n) - \tau(x_{n-1})) = 1$$

Some inductive reasoning

$$\tau(x_0) = 0 \qquad \tau(x_j) = \sum_{l=1}^{j}\left(\frac{1}{b(x_{l-1}, x_l)}\sum_{k=l}^{n} m(x_k)\right)^{\frac{1}{p-1}}$$

p-Torsional rigidity of a path graph

Corollary

Suppose (X, m) is a discrete measure space with a graph b. Let $X_0 \cup \partial X_0 = \{x_0, \ldots, x_n\} \subset X$ and let b' denote the path graph with edge weights $(b(x,y))_{x,y}$ and Dirichlet vertex conditions in $x_0 \in \partial X_0$. Then for $j = 1, \ldots, n$:

$$\tau_p(x_0) = 0 \qquad \tau_p(x_j) = \sum_{l=1}^{j} \left(\frac{1}{b(x_{l-1}, x_l)} \sum_{k=l}^{n} m(x_k) \right)^{\frac{1}{p-1}}$$

The p-Cheeger constant

Let b be a graph over (X, m). We say that $\rho : X \times X \to [0, \infty)$ is a pseudometric if

- ρ is symmetric
- ρ is p-intrinsic, that is,

$$\sum_{y \in X} b(x, y)\rho(x, y)^{\frac{p}{p-1}} \leq m(x) \quad \forall x \in X.$$

Now call

$$h_\rho(b) := \inf_{\substack{W \subseteq X \\ W \text{ finite}}} \frac{A(\partial W)}{m(W)},$$

where $A(\partial W)$ is the area of the boundary of W. Then the p-Cheeger constant associated to the graph is defined as

$$h_p(b) := \sup_{\rho} h_\rho(b)$$

Torsional rigidity and the Cheeger constant

Theorem

Let b be a graph over (X, m), and $\emptyset \neq X_0 \subsetneq X$. Then $T_p(b; X_0)$ fulfills

$$\lambda_0(L_b^{(p)})T_p(b; X_0) < m(X_0)^{p-1},$$

and the p-Cheeger constant $h_p(b)$ of b fulfills

$$h_p(b)^p T_p(b; X_0) < \tfrac{p^p}{2^{p-1}} m(X_0)^{p-1},$$

Torsional rigidity and the Cheeger constant - Proof

Suppose $\tau_b^{(p)}$ denotes the torsion function, then $\mathcal{Q}_b^{(p)}(\tau_b^{(p)}) = \left\|\tau_b^{(p)}\right\|_{\ell^1(X,m)}$. Then

$$T_p(b, X_0) = \left\|\tau_b^{(p)}\right\|_{\ell^1(X,m)}^{p-1} = \frac{\left\|\tau_b^{(p)}\right\|_{\ell^1(X,m)}^{p}}{\mathcal{Q}_b^{(p)}(\tau_b^{(p)})} < \frac{m(X_0)^{p-1}\left\|\tau_b^{(p)}\right\|_{\ell^p(X,m)}^{p}}{\mathcal{Q}_b^{(p)}(\tau_b^{(p)})} \leq \frac{m(X_0)^{p-1}}{\lambda_0(L_b^{(p)})}$$

Cheeger inequality for p-Laplacian $\frac{2^{p-1}}{p^p} h_p(b)^p \leq \lambda_0(L_b^{(p)})$ implies final result.

Outline

Conclusions and open problems

- We defined the p-Laplacian on discrete graphs
- Several classical properties of the p-Laplacian can be translated to discrete graphs
- There is an interesting connection between torsional rigidity and ground state

A quick round up about spaces

Recall the *weak* equation $L^{(p)}_{b,c} u = \mathbf{1}$.

- By definition,

$$\tau^{(p)}_{b,c} \in C(X)$$

- However, for the equation to make sense, we need at least that

$$\tau^{(p)}_{b,c} \in \mathcal{D}_p \cap \ell^2(X, m)$$

- But then, since $\mathbf{1} \in \ell^2(X, m)$, it must be $m(X) < \infty$, and so

$$\tau^{(p)}_{b,c} \in \ell^1(X, m)$$

$\Rightarrow$ **The p-torsional rigidity is always a finite number!** (when the p-torsion function exists)

Strong solutions and open problems

If we look for strong (read: *pointwise*) solutions to $L_{b,c}^{(p)} u = \mathbf{1}$, the picture changes:

If I find a solution $\tau_{b,c}^{(p)}$, does it always belong to ℓ^1?

Answer: **Not necessarily!**

The general characterization of graphs with finite torsional rigidity is an open problem even for $p = 2$.

Thank you for your attention!

Evolution Equations where art ye?

26th Internet Seminar "Graphs and Discrete Dirichlet Spaces"

Description of the Course

The 26th Internet Seminar on Evolution Equations is devoted to the treatment of graphs and discrete Dirichlet spaces. A graph is a geometric structure on a set of vertices and comes with both a Dirichlet form and a Laplacian defined on the set of functions on its vertices. More precisely, given a discrete and countable set X of *vertices* and a measure m on X of full

The discrete p-heat equation

Let $(b, 0)$ be a graph over X. For $T > 0$ we want to make sense of the following equation

$$\begin{cases} \dot{u}(t) = -L_b^{(p)} u(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

We need to define

- Suitable boundary conditions
- Underlying functional spaces
- Precise notion of solution

The discrete p-heat equation

- Suitable boundary conditions ✓
- Underlying functional spaces
- Precise notion of solution

We can work either with Neumann or Dirichlet boundary conditions

The discrete p-heat equation

- Suitable boundary conditions ✓
- Underlying functional spaces ✓
- Precise notion of solution

In the Neumann case we look for solutions in

$$w^{1,p,2}(X,m) := \mathcal{D}_p \cap \ell^2(X,m)$$

In the Dirichlet case we look for solution in

$$\dot{w}^{1,p,2}(X,m) := \overline{C_c(X)}^{\|\cdot\|_{w^{2,p}}}$$

Where $C_c(X)$ are the finitely supported functions from X to $\mathbb{R}$

The discrete p-heat equation

- Suitable boundary conditions ✓
- Underlying functional spaces ✓
- Precise notion of solution ✓

Let $f \in L^2(0,T;\ell^2(X,m))$. Then a solution to the Neumann Cauchy problem is $u \in H^1(0,T,\ell^2(X,m)) \cap L^\infty(0,T;w^{1,p,2}(X,m))$ such that

- $u(t,\cdot) \in w^{1,p,2}(X,m)$ for a.e. $t \in [0,T]$
- $\dot{u}(t) = -L_b^{(p)}u(t) + f(t)$ is satisfied for a.e. $t \in [0,T]$
- $u(0) = u_0 \in w^{1,p,2}(X,m)$

The discrete p-heat equation

- Suitable boundary conditions ✓
- Underlying functional spaces ✓
- Precise notion of solution ✓

Let $f \in L^2(0,T;\ell^2(X,m))$. Then a solution to the Dirichlet Cauchy problem is $u \in H^1(0,T,\ell^2(X,m)) \cap L^\infty(0,T;\dot{w}^{1,p,2}(X,m))$ such that

◇ $u(t,\cdot) \in \dot{w}^{1,p,2}(X,m)$ for a.e. $t \in [0,T]$

◇ $\dot{u}(t) = -L_b^{(p)}u(t) + f(t)$ is satisfied for a.e. $t \in [0,T]$

◇ $u(0) = u_0 \in \dot{w}^{1,p,2}(X,m)$

Wellposedness result

Theorem

Let $p \in (1, +\infty)$, $T > 0$ *and* $f \in \ell^2(X, m)$. *Then*

(i) The Dirichlet-Cauchy problem admits a unique solution for all initial data $u_0 \in \dot{w}^{1,p,2}(X, m)$

(ii) The Neumann-Cauchy problem admits a unique solution for all initial data $u_0 \in w^{1,p,2}(X, m)$

(iii) If $f \equiv 0$, *the solution to either problem is a given* C_0*-semigroup of nonlinear contractions*