Project A – Positivity preserving and improving C_0-semigroups

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Definition

A strongly continuous semigroup $(e^{-tA})_{t\geq 0}$ on a L^p space is called **positivity preserving** if each operator e^{-tA} is positivity preserving, i.e., if $0 \leq f \in L^p$ implies $0 \leq e^{-tA} f$ for each $t \geq 0$, or equivalently, if

$$|e^{-tA} f| \leq e^{-tA}|f| \quad \text{holds for each } f \in L^p \text{ and all } t \geq 0.$$

Example

The matrix exponential $(e^{-tA})_{t\geq 0}$ is positivity preserving for all $t \geq 0$ if and only if $A + \mathrm{diag}(-A) \leq 0$, i.e., $a_{ij} \leq 0$ for $i \neq j$.

Example

The left translation semigroup given by

$$e^{-tA}\colon \mathrm{L}^p(\mathbb{R}) \to \mathrm{L}^p(\mathbb{R});$$

$$e^{-tA}f(x) = f(x+t)$$

is positivity preserving.

Example

Consider the Gaussian semigroup that solves the heat equation

$$\begin{cases} u_t = u_{xx} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(t,+\infty) = u(t,-\infty) = 0 \quad t \in \mathbb{R} \\ u(0,x) = f(x) \quad x \in \mathbb{R}. \end{cases}$$

Let $p \in [1,\infty)$. The explicit formula is given by

$$(e^{-tA}f)(x) = \frac{1}{\sqrt[2]{4\pi t}} \int_{\mathbb{R}} f(z) e^{\frac{-(z-x)^2}{4\pi}} dz$$

For $f \in L^p(\mathbb{R})$ and $t > 0$.

The semigroup is positivity preserving.

Definition (Ideal)

A subspace $I \subseteq \mathrm{L}^p$ is called an **ideal** if for all $f \in I$ and $g \in \mathrm{L}^p$:

$$|g| \leq |f| \implies g \in I.$$

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   Without proof: All closed ideals are of this kind.

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   If $f \geq 0$, then

$$\mathrm{L}^p_f = \bigcup_{k \in \mathbb{N}} \{g \in \mathrm{L}^p \colon |g| \leq kf\}.$$

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hence $g \cdot 1_{A_j} \in \mathrm{L}^p_f$. By $g = \sum_{j \in \mathbb{N}} g \cdot 1_{A_j}$, we obtain $g \in \overline{\mathrm{L}^p_f}$. As $\mathrm{L}^p \cap \mathrm{L}^\infty$ is dense in L^p, the claim follows. $\square$

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Remark

If I is A-invariant, then the same is true for $\overline{I}$.

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Proposition (Characterization of irreducibility)

Let $(e^{-tA})_{t\geq 0}$ be a positivity preserving semigroup on L^p. The following are equivalent:

(i) $(e^{-tA})_{t\geq 0}$ is irreducible.

(ii) For some/every $\lambda > s(-A)$ and for every $0 \neq f \geq 0$, $(\lambda + A)^{-1} f > 0$ a.e.

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&= k e^{-tA} \int_0^\infty e^{-\lambda s} e^{-sA} f \,\mathrm{d}s \\
&= k \int_0^\infty e^{-\lambda s} e^{-(s+t)A} f \,\mathrm{d}s \\
&= k \int_t^\infty e^{-\lambda(s-t)} e^{-sA} f \,\mathrm{d}s \\
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\end{aligned}$$

Proposition (Characterization of irreducibility)

(e^{-tA}) positivity preserving; denote $\lambda > s(-A)$, $0 \neq f \geq 0$.

(i) $(e^{-tA})_{t\geq 0}$ irred. $\quad\Longrightarrow\quad$ (ii) $(\lambda + A)^{-1} f > 0$ a.e. $\forall f$

Proof:

Let $g \in \mathrm{L}^p_{(\lambda+A)^{-1}f}$. For any $t \geq 0$, we find

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Thus, $I = \mathrm{L}^p$.

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$\overset{(\mathrm{iii})}{\Longrightarrow}$ I is trivial. $\square$

Theorem

Suppose $(e^{-tA})_{t\geq 0}$ is a holomorphic positive semigroup.
If $(e^{-tA})_{t\geq 0}$ is irreducible then (e^{-tA}) is positivity improving, for all $t \geq 0$.
Equivalently, given $f \in L^p(\Omega,\mu)$, $\phi \in L^{p'}(\Omega,\mu)$ such that $0 \neq f \geq 0$, $0 \neq \phi \geq 0$, then $\langle \phi, e^{-tA} f\rangle > 0$ for all $t > 0$.

Aim

We will prove that for an arbitrary holomorphic positive semigroup $(e^{-tA})_{t\geq 0}$ the following holds:

Given $0 \neq f \geq 0$, $0 \neq \phi \geq 0$ then either $\langle \phi, e^{-tA} f\rangle = 0$ or $\langle \phi, e^{-tA} f\rangle > 0$ for all $t \geq 0$

Since $(e^{-tA})_{t\geq 0}$ is irreducible, then always the second case occurs. For $0 \neq f \geq 0$, $0 \neq \phi \geq 0$, assume that $\langle \phi, e^{-t_0 A} f \rangle = 0$ for some $t_0 > 0$.

We consider a null sequence (t_n), $0 < t_n < t_0$ such that

$$\| e^{-t_n A} f - f \| \leq 2^{-n}$$

and define $f_n := e^{-t_n A} f$, $g_n := f - \sum_{k=n}^{\infty} (f - f_k)^+$. Then we have

$$g_n \leq (f - f_m)^+ = \inf(f, f_m) \leq f_m.$$

For $n \in \mathbb{N}$ fixed and $m \geq n$ we obtain

$$0 \leq \langle \phi, e^{-(t_0 - t_m)A} g_n^+ \rangle \leq \langle \phi, e^{-(t_0 - t_m)A} f_m \rangle = \langle \phi, e^{-t_0 A} f \rangle = 0.$$

Thus the function $t \mapsto \langle \phi, e^{-tA} g_n^+ \rangle$ is identically zero by the uniqueness theorem for analytic functions. Since $f = \lim_{n\to\infty} g_n^+$ we have $\langle \phi, e^{-tA} h \rangle = 0$ for all $t \in \mathbb{R}_+$.