Laplacians on Infinite Graphs: discrete vs. continuous

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joint work with A. Kostenko (Ljubljana&Vienna) and M. Malamud (Donetsk)

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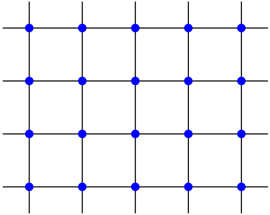
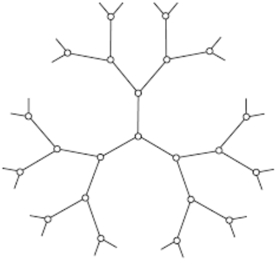
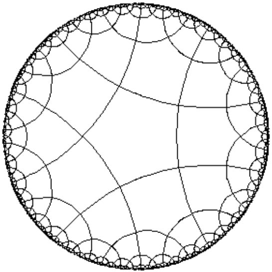
(Combinatorial) Graphs

Definition

A graph $\mathcal{G}_d$ is a (countable) vertex set $\mathcal{V}$ together with an edge set $\mathcal{E}$.







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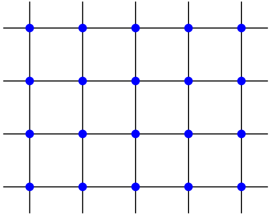
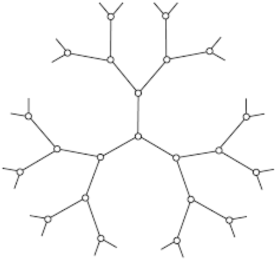
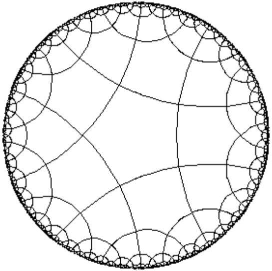






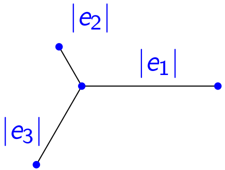
Assumptions

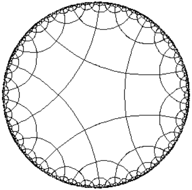
- $\mathcal{G}_d$ is **connected**
- $\mathcal{G}_d$ is **locally finite**, i.e. all vertices have only finitely many neighbors

(Weighted) metric graphs

Definition

- If we assign each edge $e \in \mathcal{E}$ of a combinatorial graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ a finite length $0 < |e| < \infty$, we get a metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$.





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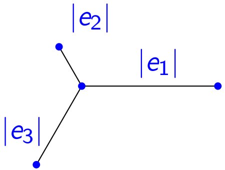
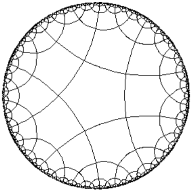
- If $\mu, \nu \colon \mathcal{E} \to (0, \infty)$ are **edge weights**, then $(\mathcal{G}, \mu, \nu)$ is called a weighted metric graph. This gives "weighted Lebesgue measures"

$$\mu = \sum_{e \in \mathcal{E}} \mu_e dx_e \qquad \nu = \sum_{e \in \mathcal{E}} \nu_e dx_e.$$

(Here, $\mu_e := \mu(e), \nu_e := \nu(e) \in \mathbb{R}_{>0}$ are the weights of $e \in \mathcal{E}$)





Kirchhoff Laplacians ("quantum graphs")

For a weighted metric graph $(\mathcal{G}, \mu, \nu)$, consider...

- the L^2-space $L^2(\mathcal{G}; \mu)$ for the measure $\mu = \sum_{e \in \mathcal{E}} \mu_e dx_e$
- the edgewise defined **differential expression**

$$H_e f := -\frac{1}{\mu_e} \frac{\mathrm{d}}{\mathrm{d}x_e} \nu_e \frac{\mathrm{d}}{\mathrm{d}x_e} f$$

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- and the **(weighted) Kirchhoff conditions**

$$\text{for every vertex } v \in \mathcal{V} : \left\{ \begin{array}{c} f \text{ is continuous in } v \\ \sum_{e \sim v} \nu_e f_e'(v) = 0 \end{array} \right\} \tag{0.1}$$

(i) The maximal Kirchhoff Laplacian $H\colon \operatorname{dom}(H) \subseteq L^2(\mathcal{G}; \mu) \to L^2(\mathcal{G}; \mu)$ acts edgewise as H_e with domain

$$\operatorname{dom}(H) = \{f \in L^2(\mathcal{G}; \mu) |\ f \text{ is edgewise } H^2, (0.1) \text{ holds } \& \ Hf \in L^2(\mathcal{G}; \mu)\}.$$

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Relation: $H = (H^0)^*$

Discrete graphs and their Laplacians

Definition: (locally finite) graphs in the sense of ISEM26

A **(weighted) discrete graph** $(\mathcal{V}, m, b)$ consists of ...

- a countable set $\mathcal{V}$,
- a function $m \colon \mathcal{V} \to (0, \infty)$ ("vertex weight"), and
- a function $b \colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$, which vanishes on the diagonal and is symmetric and locally finite, that is,

$$\#\{u \in \mathcal{V} |\, b(u, v) \neq 0\} < \infty \quad \text{for all } v \in \mathcal{V}.$$

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Consider the **difference expression**

$$(\tau f)(v) := \frac{1}{m(v)} \sum_{u \in \mathcal{V}} b(u, v)\big(f(v) - f(u)\big), \qquad v \in \mathcal{V},$$

for a function $f : \mathcal{V} \to \mathbb{C}$ in the weighted ℓ^2-space $\ell^2(\mathcal{V}; m)$.

Discrete Laplacians

Definition

Let $(\mathcal{V}, m, b)$ be a discrete graph.

(i) The (maximal) discrete Laplacian $h\colon \operatorname{dom}(h) \subseteq \ell^2(\mathcal{V}; m) \to \ell^2(\mathcal{V}; m)$ is given by the difference expression τ on the domain

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- How to **connect discrete and metric graph Laplacians**?

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- How to **connect discrete and metric graph Laplacians**?
- What are the **applications**?

Connecting metric and discrete graphs

Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. Consider the **discrete graph** $(\mathcal{V}, m, b)$ with the following weights $b(u, v)$, $u, v \in \mathcal{V}$, and $m(v)$, $v \in \mathcal{V}$:

$$b(u, v) = \sum_{\substack{e \text{ edges} \\ \text{between } u \text{ and } v}} \frac{\nu_e}{|e|}, \qquad m(v) = \sum_{e \sim v} \mu_e |e|,$$

where loop edges are counted twice in the latter sum.

- We sometimes denote $(\mathcal{V}, m, b) = \mathcal{D}(\mathcal{G}, \mu, \nu)$

Motivation

- m relates L^2-**spaces** $L^2(\mathcal{G}; \mu)$ and $\ell^2(\mathcal{V}; m)$ ("discretize" the measure μ)
- b relates **quadratic forms** of Kirchhoff / discrete Laplacian (see Katharina's previous talk)

Discrete vs. continuous graph Laplacians

The KH Laplacian H and h share **many basic properties**, for example,

- Spectral properties (Exner–Kostenko–Malamud–Neidhart '18; Kostenko–N '21)
  - Self-adjointness properties
  - Spectral gap estimates
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  - Markovian uniqueness ("form uniqueness") (Kostenko–N '21)
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Goal: Apply these connections to "transfer" results!

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h and H on **finite graphs** are always **self-adjoint**

$\Rightarrow$ from now an, assume that the graphs are infinite (i.e. $\#\mathcal{V} = +\infty$)

Equivalence of self-adjointness problem

Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and $(\mathcal{V}, m, b)$ its discrete graph. From now on, we assume that $\sup_{e\in\mathcal{E}} |e|^2 \frac{\mu_e}{\nu_e} < \infty$.

Theorem (*) (Exner-Kostenko-Malamud-Neidhardt 2018; Kostenko-N. 2021)

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Exner, Kostenko, Malamud and Neidhardt, Ann. Henri Poincaré (2018).

- Goal: Study self-adjointness using discrete-continuous connections!

Before: use results **from discrete Laplacians** for QGs (with $\mu = \nu \equiv 1$)

Inverse direction?

Let $(\mathcal{V}, m, b)$ be a discrete graph. A **cable system** for $(\mathcal{V}, m, b)$ is a weighted metric graph $(\mathcal{G}_d, |\cdot|, \mu, \nu)$ such that

$$(\mathcal{V}, m, b) = \mathcal{D}(\mathcal{G}_d, |\cdot|, \mu, \nu),$$

that is, the previous construction gives $(\mathcal{V}, m, b)$.

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Every (locally finite!) discrete graph $(\mathcal{V}, m, b)$ **has a cable system** (which is also explicitly constructable).

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Looking at **weighted** metric graphs is **necessary**!
Not all discrete Laplacians are covered by "usual" metric graphs!
(i.e. $\mu = \nu \equiv 1$)

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Folz, *Volume growth and stochastic completeness of graphs*, TAMS (2014).

Self-adjointness: from discrete to continuous criteria

Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and $(\mathcal{V}, m, b)$ its discrete graph.

Proposition

The Laplacians H and h are **s.a.** under each of the following conditions:

(i) The weighted degree/expected waiting time is **uniformly bounded**,

$$\sup_{v \in \mathcal{V}} \mathrm{Deg}(v) = \sup_{v \in \mathcal{V}} \frac{1}{m(v)} \sum_{u \sim v, u \neq v} b(e_{u,v}) < +\infty.$$

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... Gaffney-type theorems, i.e. criteria in terms of metric completeness?

Intrinsic metric for Kirchhoff Laplacians

Definition

Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. The **intrinsic metric** is

$$\varrho_\eta(x, y) := \inf_{\mathcal{P}} L(\mathcal{P}), \qquad x, y \in \mathcal{G},$$

where the infimum is over all **paths** $\mathcal{P}$ from x to y, and the length $L(\mathcal{P})$ of a path is obtained from the **intrinsic edge lengths** $\eta(e) = |e|\sqrt{\frac{\mu_e}{\nu_e}}$, $e \in \mathcal{E}$.

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- Background: To each strongly local Dirichlet form, one can associate its **intrinsic metric** $\Rightarrow$ generalize results from Riemannian manifolds!

K.-T. Sturm, *Analysis on local Dirichlet spaces I. Recurrence, conservativeness and L^p-Liouville properties*, J. reine angew. Math. **456**, 173–196 (1994).

Intrinsic metrics for discrete Laplacians

Definition (Frank, Lenz and Wingert, J. Funct. Anal., 2014)

Let $(\mathcal{V}, m, b)$ be a discrete graph.

A metric $\varrho\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ is called **intrinsic**, if

$$\sum_{u \in \mathcal{V}} b(u, v)\varrho(u, v)^2 \leq m(v) \quad \text{for all vertices } v \in \mathcal{V}.$$

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- Intrinsic metrics **recover many results from manifolds** for graphs!
  E.g. spectral estimates (Cheeger, Buser and Brooks), parabolic properties, ...

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- But: Each $(\mathcal{V}, m, b)$ has infinitely many intrinsic metrics!

Intrinsic metrics: discrete versus continuous

Theorem (Kostenko–N., 2021)

Let $(\mathcal{V}, m, b)$ be a discrete graph.

(i) Let $(\mathcal{G}, \mu, \nu)$ be a **cable system** of h (with $\sup_e \eta(e) < \infty$).
Then the **restriction** of the intrinsic metric $\varrho_\eta \colon \mathcal{G} \times \mathcal{G} \to [0, \infty)$,

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The above procedure actually induces a (almost!) a **bijection** between

"nice" cable systems for $(\mathcal{V}, m, b)$ (i.e. suitably normalized)
$$\cong$$
"nice" intrinsic metrics $(\mathcal{V}, m, b)$ (i.e. finite jump size path metric)

Gaffney's theorem for Graph Laplacians

Theorem (Haeseler, 2014; EKMN, 2018; Kostenko–N., 2021)

Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and ϱ_η its **intrinsic metric**.
If $(\mathcal{G}, \varrho_\eta)$ is **complete**, then the Kirchhoff Laplacian H is **self-adjoint**.

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Theorem (Huang–Keller–Masamune–Wojciechowski, 2014)

Let $(\mathcal{V}, m, b)$ be a discrete graph which is **locally finite**.
Assume there exists an **intrinsic metric** ϱ which generates the discrete topology on $\mathcal{V}$ and such that $(\mathcal{V}, \varrho)$ is **complete**.

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Then the discrete Laplacian h is **self-adjoint**.

Graph Laplacians: Schrödinger operators?

Let $(\mathcal{V}, m, b)$ be a discrete graph and $\alpha\colon \mathcal{V} \to \mathbb{R}$.

The maximal discrete Schrödinger operator h_α is defined in $\ell^2(\mathcal{V}; m)$ by

$$(h_\alpha f)(v) := \frac{1}{m(v)}\Big(\sum_{u\in\mathcal{V}} b(u,v)(f(v) - f(u)) + \alpha(v)f(v)\Big), \quad v \in \mathcal{V}.$$

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$$H_e f = -\frac{1}{\mu_e}\frac{\mathrm{d}}{\mathrm{d}x_e}\nu_e\frac{\mathrm{d}}{\mathrm{d}x_e} f$$

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$$\text{For every vertex } v \in \mathcal{V}: \left\{ \begin{array}{c} f \text{ is continuous in } v \\ \sum_{e \text{ edges at } v} \nu_e f_e'(v) = \alpha(v) f(v) \end{array} \right\}$$

Self-adjointness of Schrödinger operators

Theorem (EKMN, 2018; Kostenko–N., 2021)

(i) Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph with discrete graph $(\mathcal{V}, m, b)$. Consider a function $\alpha \colon \mathcal{V} \to \mathbb{R}$. Then H_α has the same **"basic properties"** as the discrete Schrödinger operator h_α with potential α on $(\mathcal{V}, m, b)$.

E.g. the **self-adjointness connections** from Theorem (*) hold.

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- Question: Are the operators H_α and h_α **self-adjoint**?

Self-adjointness of Schrödinger operators II

<u>Recall:</u> the **Laplacians** H and h are **self-adjoint** if one of these holds:

(i) The weighted degree function $\mathrm{Deg}(v)$ is bounded. ($\Leftrightarrow$ h is bounded)

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Q: Which conditions are **"stable"**, i.e., work for H_α & h_α for $\alpha \colon \mathcal{V} \to \mathbb{R}$?

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Let $(\mathcal{V}, m, b)$ be a weighted, locally finite graph. Assume there exists an intrinsic metric ϱ such that $(\mathcal{V}, \varrho)$ is complete and the generated topology is discrete. Consider the discrete Schrödinger operator h_α^0 for a $\alpha\colon \mathcal{V} \to \mathbb{R}$.

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... "Glazman-Povzner-Wienholtz for graphs" (Povzner '52; Wienholtz '58)
Let $H_V = -\Delta + V$ on $\mathbb{R}^N$ for $V \in \mathcal{C}(\mathbb{R}^N)$. H_V lower semibounded $\Rightarrow$ H_V s.a.

Parabolic properties: from continuous to discrete?

Principle (eg Varopoulos; Barlow; EKMN; Folz; Huang; Haeseler; Kostenko-N;...)

H and h **"share many basic parabolic properties"**!

(recurrence/transience, heat semigroup estimates, stochastic completeness, ...)

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Varopoulos, *Long range estimates for Markov chains*, Bull. Sci. Math. (1985)

Barlow&Murugan, *Stability of the elliptic Harnack inequality*, Ann. of Math. (2018)

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Huang, *A Note on the Volume Growth Criterion for Stochastic Completeness [...]*, Potential Anal (2014).

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- It also provides **alternative proofs/perspective** on other results for discrete Laplacians!

Keller–Lenz–Wojciechowski, *Graphs and Discrete Dirichlet Spaces*, Springer, 2021.

Thank you for your attention!

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