

26th Internet Seminar
Graphs and Discrete Dirichlet Spaces
Final Workshop

Analysis on self-similar and non self-similar fractals

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Wuppertal, 18. Juli 2023

Motivation: Analysis on fractals

Plan of the lecture

1. (Self-similar) fractals
2. Analysis on (self-similar) fractals
The Dirichlet form approach (Kusuoka, Kigami)
Model case: Sierpinski gasket
3. Spectral asymptotics
4. The V -variable model
5. Spectral asymptotics in the V -variable model
6. Stretched fractals (= quantum graphs with vanishing edge lengths)

1. Introduction: Self similar fractals

1.1. Definition and Examples

$K \subseteq \mathbb{R}^n$ is called **self similar**, if

$$K = \bigcup_{i=1}^M S_i(K)$$

where $M \geq 2$ and $S_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ similitudes.

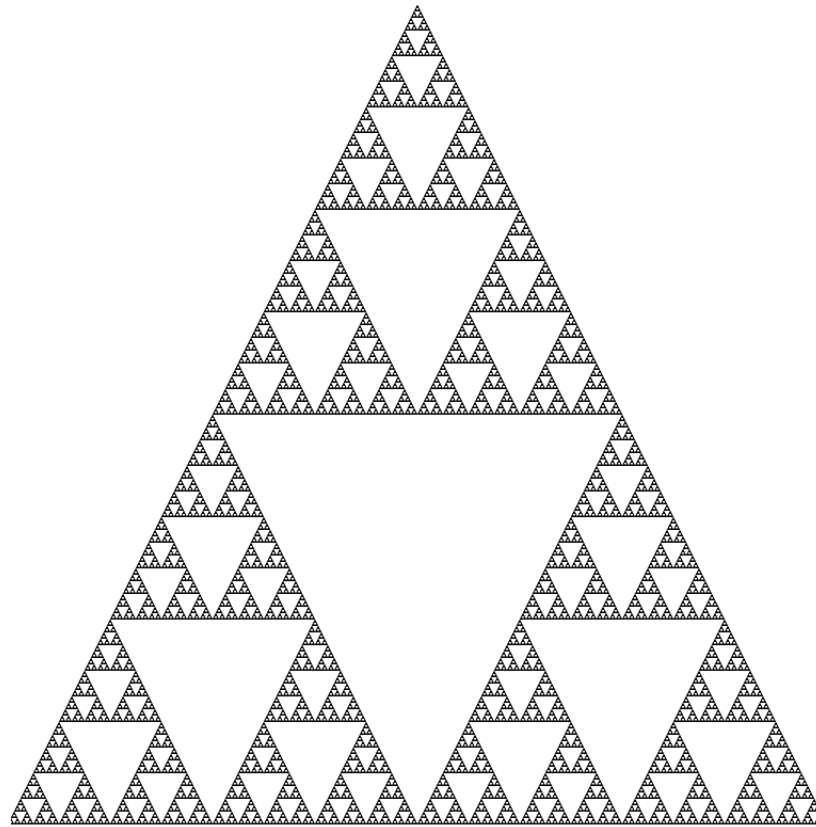
Exp. **Sierpinski gasket**:

A, B, C vertices of a unilateral triangle

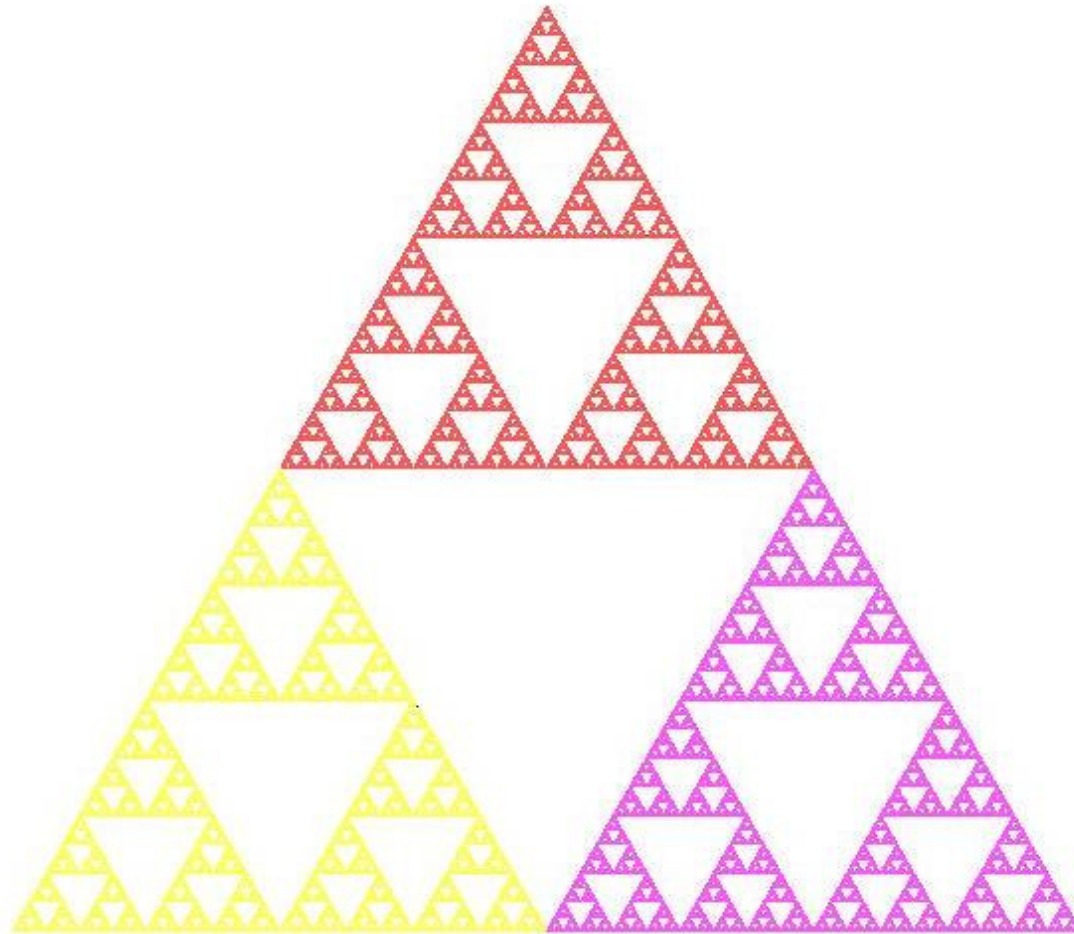
Family $\mathfrak{S} = \{S_1, S_2, S_3\}$ of contractions on \mathbb{R}^2 , where

$$S_1(x) = \frac{1}{2}(x - A) + A, S_2(x) = \frac{1}{2}(x - B) + B, S_3(x) = \frac{1}{2}(x - C) + C$$

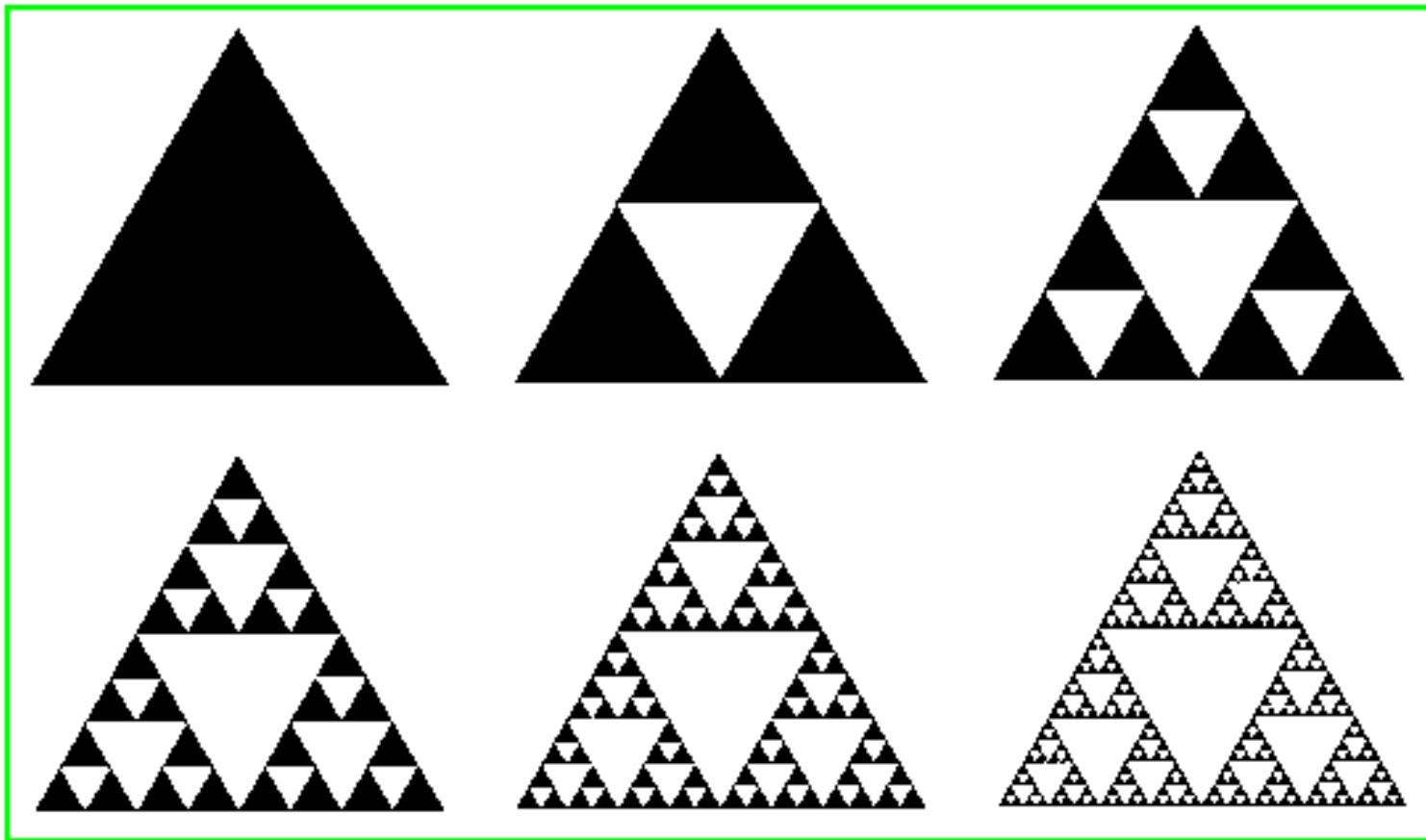
There is a unique (non empty and compact) set K , the so-called **Sierpinski gasket**:



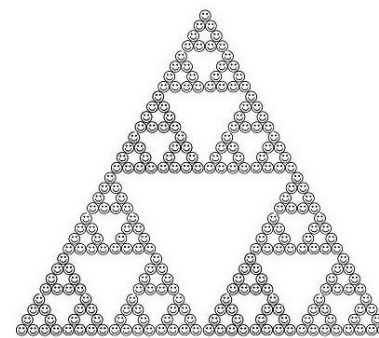
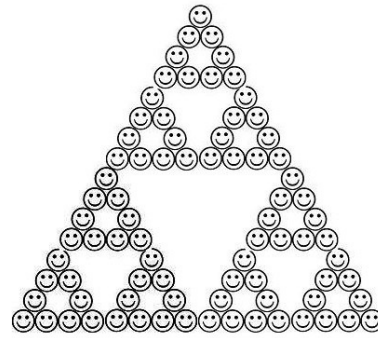
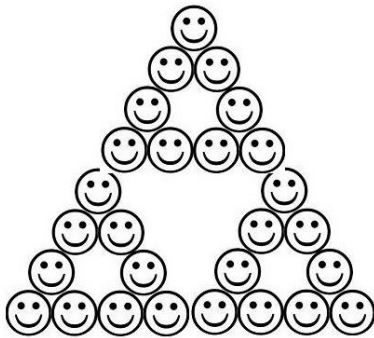
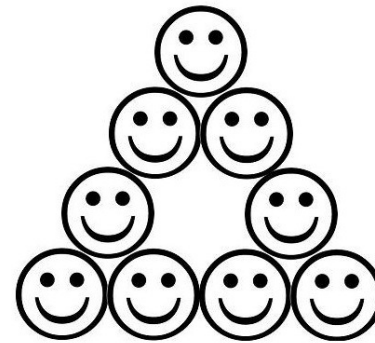
Again:



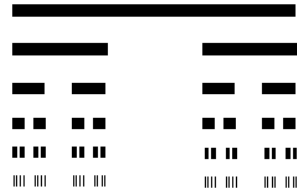
It can be obtained by iteration of the three mappings:



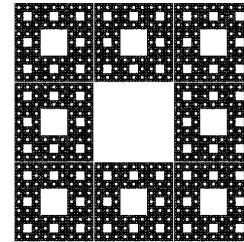
Hereby, you can start with any set:



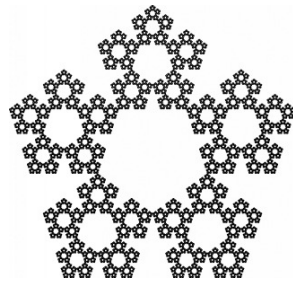
Further examples for self-similarity:



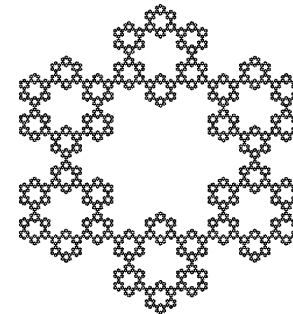
a) Cantor set



b) Sierpinski carpet



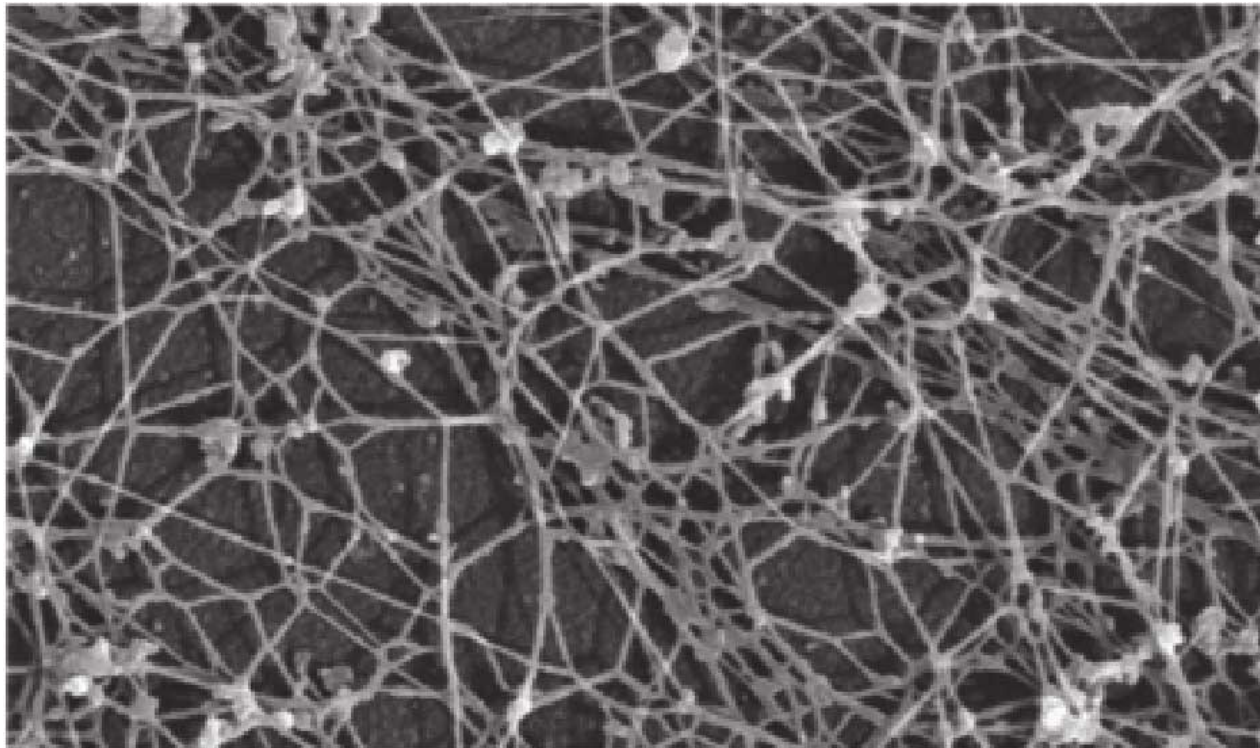
c) Pentagasket



d) Snowflake

1.2. What can we model with the help of fractals?

Application in **medicine** pulmonary tissue,
tumor cells (and their behavior under stress)



electron microscope

picture of the zytoskeleton in a tumor cell in human pancreas

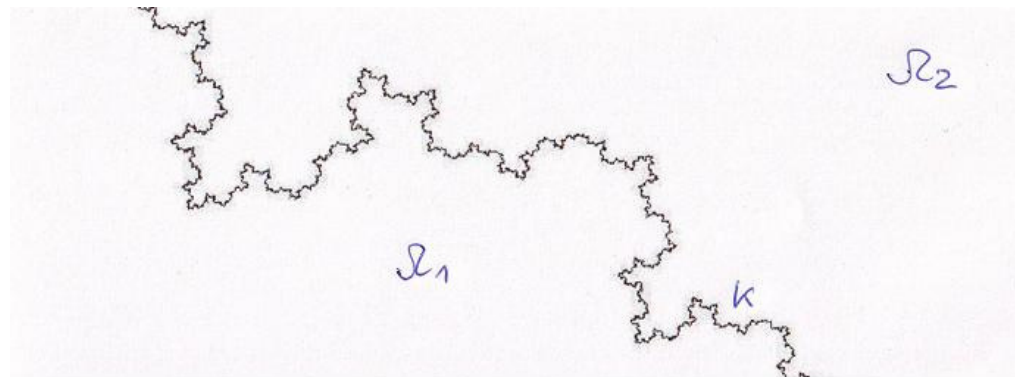
Application in **Physics, Material science**

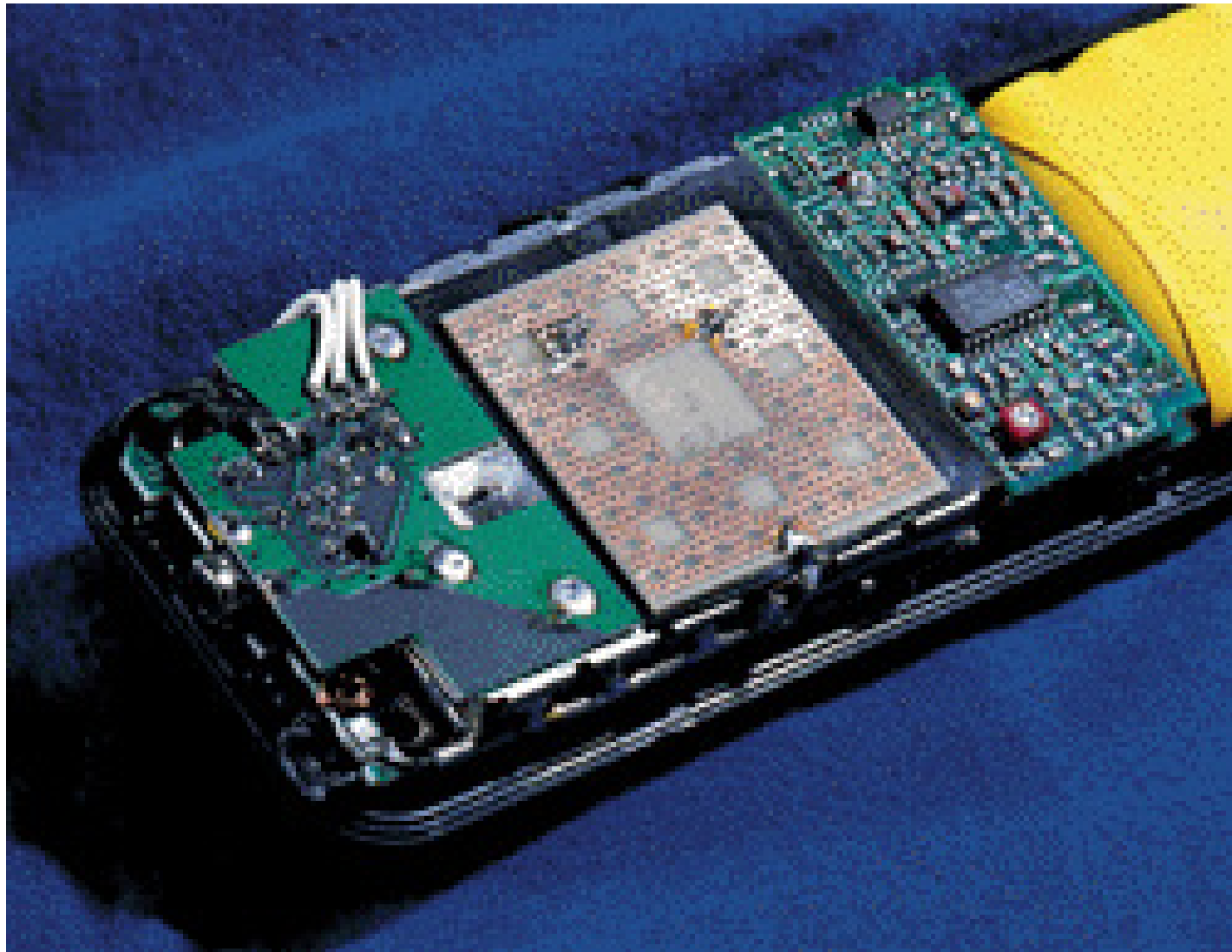
fractal antenna, fractal conductor plates; porous materials

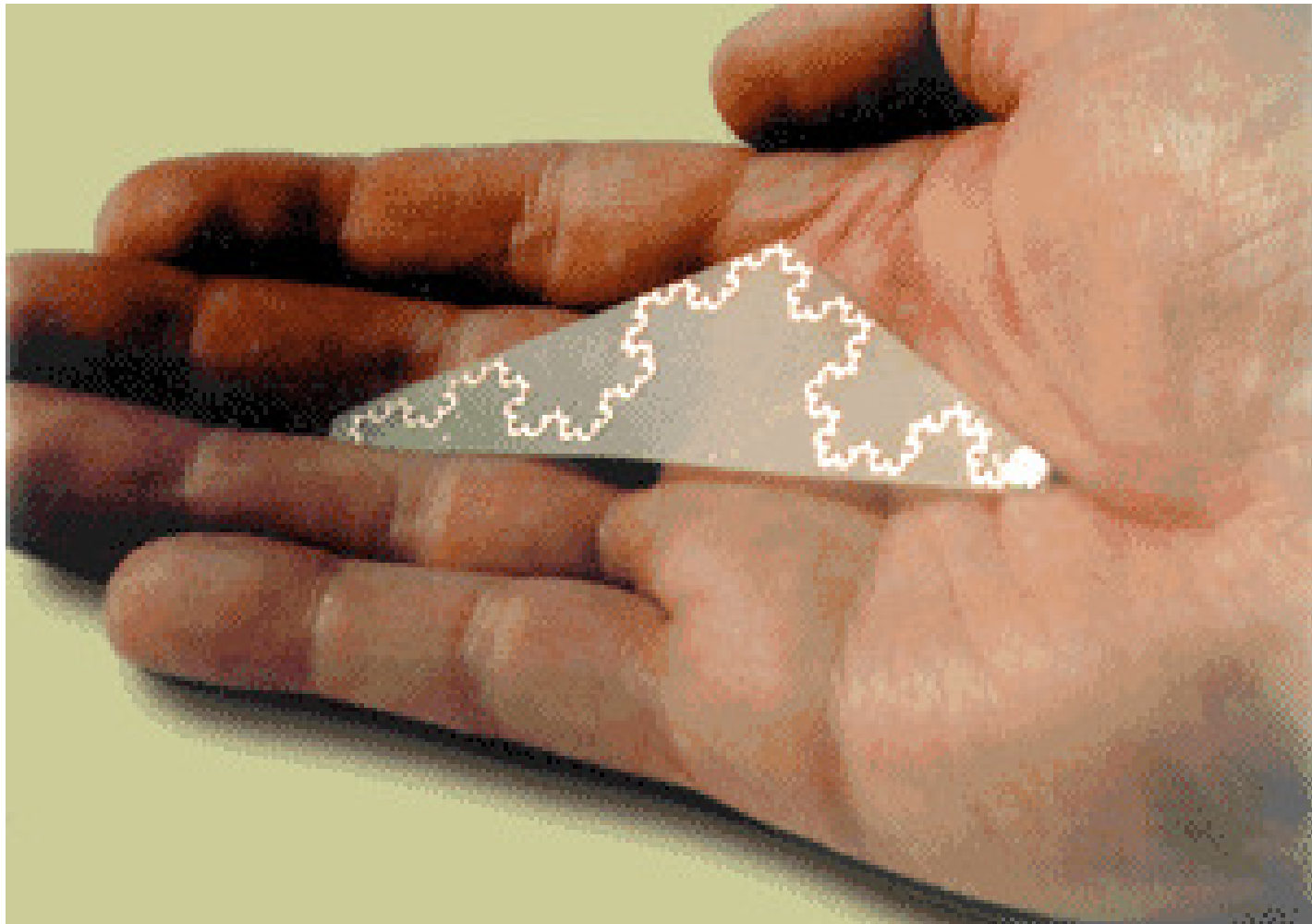
- Transport *on* such structures
- „Transmission problems *across* fractal layers“

$$\begin{cases} -\Delta_{\mathbb{R}^n} u = g & \text{in } \Omega_i, i = 1, 2 \\ \Delta_K u = C(\frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2}) & \text{on } K \end{cases}$$

& boundary and continuity conditions



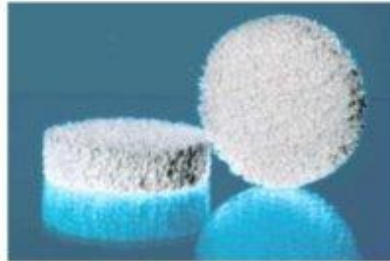




Zirkoniumoxid
(ZrO_2)



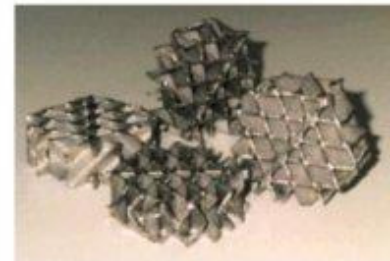
Aluminiumoxid
(Al_2O_3)



Siliziumkarbid
(SiC)



Fe-Cr-Al-
und Ni-Basis
Legierungen



Porous materials

2. Dirichlet form and Laplacian on the Sierpinski gasket

2.1. Interludium: Analysis on fractals

aim: Definition of the Laplacian Δ
(wave/heat/Poisson/Schrödinger equation)

Problem: Fractals are too „rough“

⇒ no tangent space

⇒ new notion of derivative necessary

Classical approaches:

- limit of difference operators ([Dirichlet form theory](#))
Kusuoka, Kigami, Lapidus, Mosco, Hambly, Teplyaev, Strichartz,...
- Construction of the „natural“ [Brownian motion](#) as the limit of a sequence of appropriate renormalized random walks
Kusuoka, Barlow, Bass, Perkins, Lindstrøm; Sabot, Metz,...
- [Martin boundary theory](#) on the Code space
Denker, Sato, Koch,...
- (fractal dimensional) traces of [function spaces](#) (for exp. Sobolev spaces) or via Riesz potentials
Triebel, Haroske, Schmeißer,...; Zähle

New approaches:

- Generalized Laplacians (Δ -Beltrami, Hodge- Δ , Dirac- Δ)
M. Hinz, Teplyaev, Rogers,...
- Non-commutative Geometry: Interpretation of the fractal in terms of **spectral triple**
Bellissard, Falconer, Samuel, Lapidus; Cipriani, Guido, Isola, ...
- Theory of **resistance forms**
Kigami, Kajino, Alonso-Ruiz, F. ,...
- Approximation by **quantum graphs**
Teplyaev, Kelleher, Alonso-Ruiz, F. ...; Mugnolo, Lenz, Keller, Post, Kuchment, ...

2.2. Kusuoka' s approach

Aim: Define Δ_K Laplacian on K

Idea:

- Define „fractal analogue“ $\mathcal{E}_K[u]$ of $\mathcal{E}[u] = \int_{\Omega} |\nabla u|^2 dx$
- $\mathcal{E}_K(u, v) := \frac{1}{2} (\mathcal{E}_K[u + v] - \mathcal{E}_K[u] - \mathcal{E}_K[v])$ bilinear form
- Δ_K via Gauß–Green–formula:

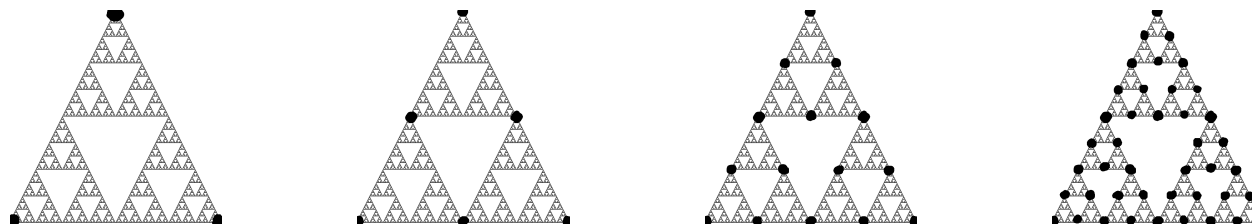
$$\int_K (\Delta_K u) v d\mu = \text{boundary terms} - \mathcal{E}_K(u, v)$$

$$(\text{cf. } \int_{\Omega} \Delta u \cdot v = \text{boundary terms} - \int_{\Omega} \nabla u \cdot \nabla v)$$

via: Dirichlet forms on Graphs

Approximation of K :

$$V_0 := \{A, B, C\}, \quad V_n := \bigcup_{i=1}^3 S_i(V_{n-1}), n \geq 1$$



V_0, V_1, V_2 and V_3

$$(V_n) \uparrow, \quad V_* := \bigcup_{n \geq 0} V_n = \sup_{n \geq 0} V_n, \quad K = \overline{V_*}$$

Let $u : V_* \longrightarrow \mathbb{R}$

$$\text{Ansatz: } E_n[u] := \varrho^n \sum_{p \in V_n} \sum_{|p-q|=2^{-n}} (u(p) - u(q))^2, \quad n \geq 0.$$

ϱ scaling number (defined and obtained by „Gaussian principle“)

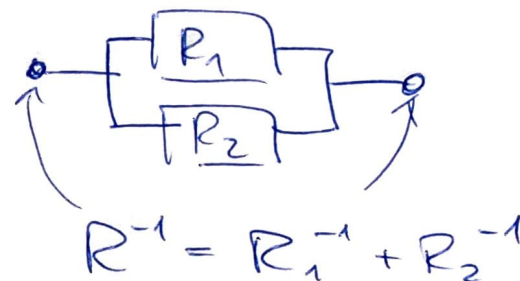
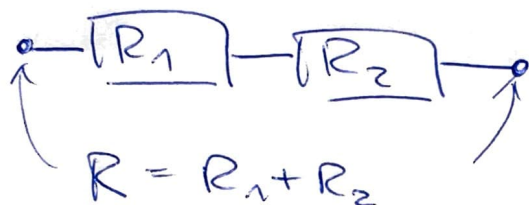
It turns out that $\varrho_F = 5/3$.

①

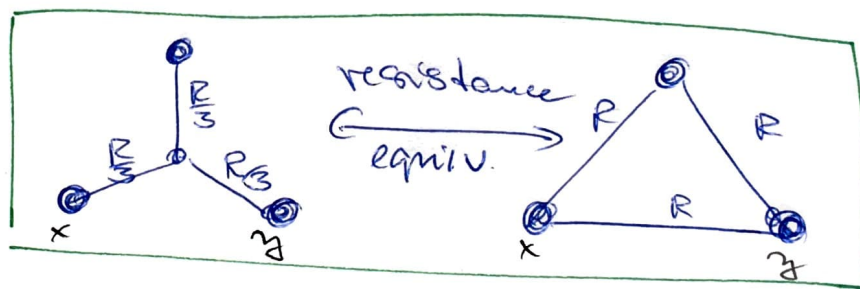
Finding the right conductance renormalization

① Way: conductance equivalent networks ($C_{xy} = \chi_{xy}^{-1}$)

resistances



Claim (Δ -Y-rule)



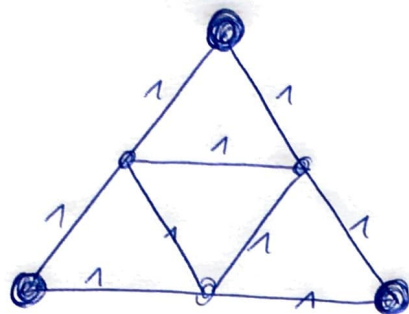
Proof l.h.s. $R_{xy} = \frac{R}{3} + \frac{R}{3} = \frac{2R}{3}$

r.h.s. $R_{xy}^{-1} = \frac{1}{R} + \frac{1}{2R} = \frac{3}{2R} \quad \neq$

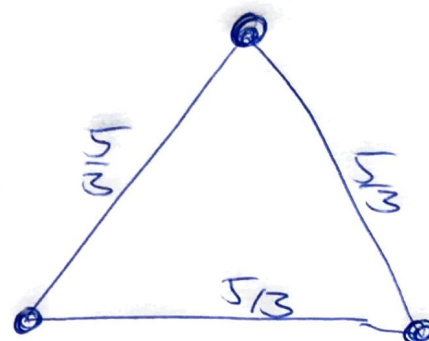
Remark \exists Δ -Y-rule for unbalanced case,
much more complicated!

2

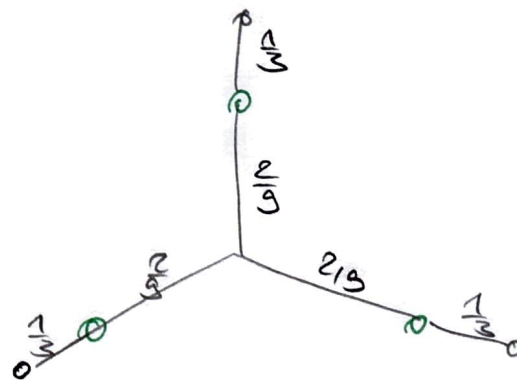
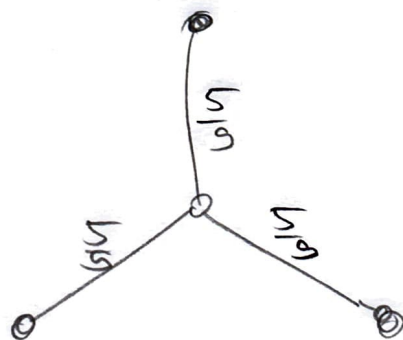
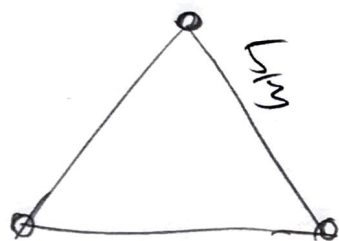
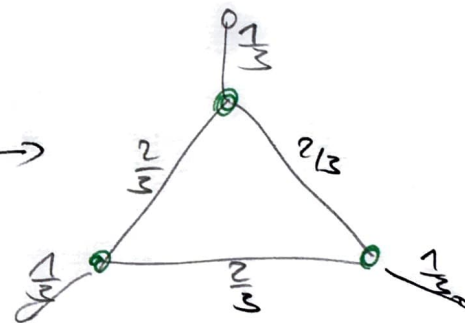
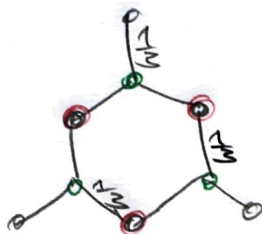
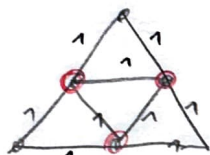
Corollary



resist.
equiv.



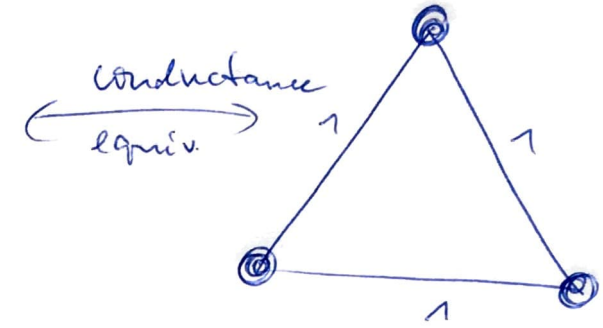
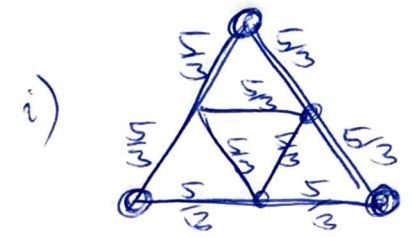
Proof



Q

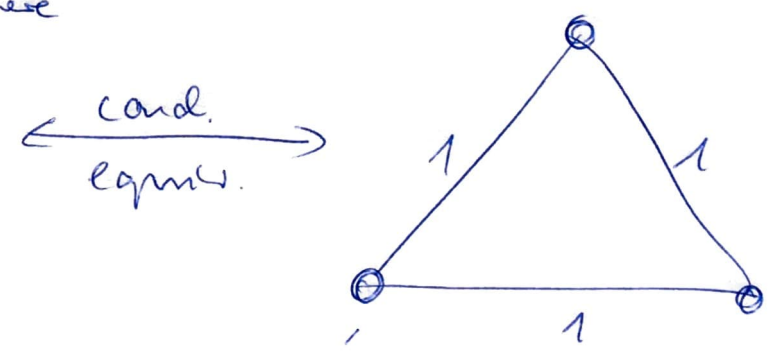
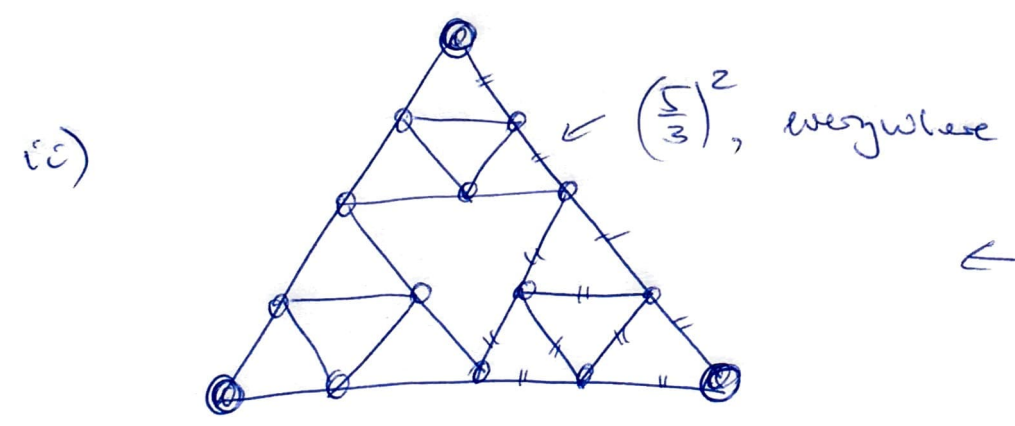
3

Corollary



$\Rightarrow g = \frac{5}{3}$

by induction, it follows that:



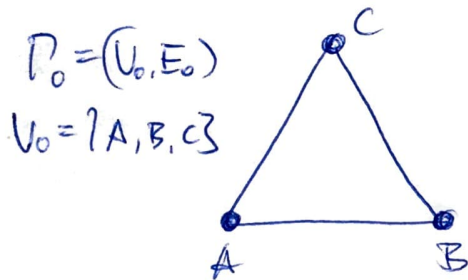
thanks to \rightarrow finite renormalization
 $\&$ \rightarrow self-similarity

$\Rightarrow g_n = \left(\frac{5}{3}\right)^n$

2nd way (in finding g) : seeking for harmonic extensions

from graph Γ_0 to Γ_1

"fourier principle"

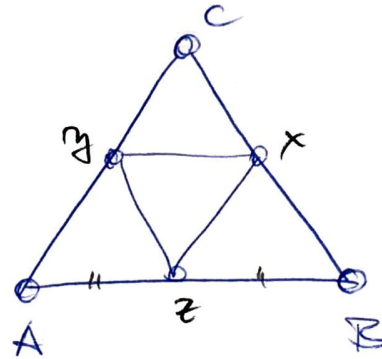


$$u: V_0 \rightarrow \mathbb{R}, \quad u(A) =: u_A, \dots$$

$$E_0[u] = (u_A - u_B)^2 + (u_B - u_C)^2 + (u_A - u_C)^2$$

now: $\Gamma_1 = (V_1, E_1)$
 $V_1 = \{A, B, C, x, y, z\}$

$$V_0 \subseteq V_1$$



Seeking for $u: V_1 \rightarrow \mathbb{R}$
 st. $u|_{V_0}$ as above

st.

$$E_1[u] \rightarrow \min$$

where

$$E_1[u] = (u_A - u(z))^2 + (u(z) - u_B)^2 + \dots \quad (9 \text{ summands})$$

This argmin $E_1[u]$ is the harmonic extension of $u|_{V_0}$!

⑤

Calculation shows: $u^+(x) = \frac{u_A + 2u_B + 2u_C}{5}, \dots$

and $E_1[u]_{\min} = 3.5 E_0[u]$ vs "correction" factor $= 5/3$

again: self sim & finite ramification $\sim (5/3)^n$

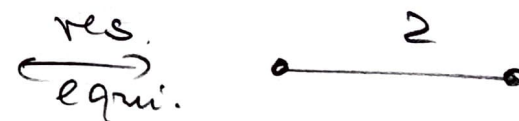
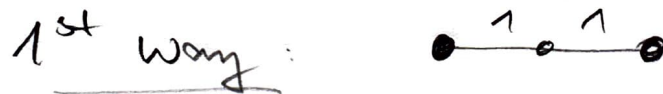
1st & 2nd way give the same result!

Reason: Voltage is a harmonic function

↑ Doyle & Snell: "Random walks + Electric networks"

Why is this the right way to think about it?

Exp $[0,1]$ unit interval



$\boxed{f=2}$

(6)

goal $E[u] = \int_0^1 |u'(x)|^2 dx$

Kusuoka ansatz:

$$E_n[u] = \sum_{\substack{|x-y|=2^{-n} \\ x, y \in D_n}} 2^n (u(x) - u(y))^2$$

where $D_n = \left\{ \frac{x}{2^n} : x = 0 \dots 2^n \right\}$

$$E_n[u] = \sum_{x \in D_n \setminus \{1\}} \underbrace{2^n \left(u\left(x + \frac{1}{2^n}\right) - u(x) \right)^2}_{\left[\frac{u\left(x + \frac{1}{2^n}\right) - u(x)}{2^{-n}} \right]^2 \cdot \frac{1}{2^n}}$$

" \rightarrow " $\int_0^1 [u'(x)]^2 dx$

$u'(x)$

↑ Stochastic "PDE on fractals"

(for exp.
for $u \in C^1[0,1]$)

The
Carus
Mathematical
Monographs

22

Random Walks and Electric Networks

Peter G. Doyle
and
J. Laurie Snell



Self similarity and finite ramification $\Rightarrow (\mathcal{E}_n[u])_{n \geq 0} \nearrow$

Limit form $\mathcal{E}_K[u] := \lim_{n \rightarrow \infty} \mathcal{E}_n[u]$

on $\mathcal{D}_* := \{u : V_* \longrightarrow \mathbb{R} : \mathcal{E}_K[u] < \infty\}$

Extension from $u \in \mathcal{D}_*$ to $u \in \mathcal{C}(K)$

$\mathcal{D} := \overline{\mathcal{D}_*}$ completion wrt. $(\|\cdot\|_{L_2(K, \mu)}^2 + \mathcal{E}_K[\cdot])^{1/2}$

$(\mathcal{E}, \mathcal{D})$ is Dirichlet form on $L_2(K, \mu) \Rightarrow \Delta_K$ Laplace operator

$(\mathcal{E}, \mathcal{D})$ ist regular and local \Rightarrow ex. associated diffusion process
 $(X_t)_{t \geq 0}$ (Brownian motion on K)

3. Spectral asymptotics in the deterministic model

3.1. Spectral dimension – Definition

Consider $\Delta_n = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ Laplace operator of \mathbb{R}^n .

H. Weyl, 1915: The eigenvalue counting function

$$N_n(x) := \# \{ \lambda_k \leq x \quad : \quad -\Delta_n u = \lambda_k u \text{ for some } u \neq 0 \},$$

(Counting according multiplicities) is well-defined, and for any $n \in \mathbb{N}$ it holds that

$$N_n(x) = (2\pi)^{-n} c_n \text{vol}^n(\Omega) x^{n/2} + o(x^{n/2}), \quad \text{as } x \rightarrow \infty,$$

Hence, $d_S(\Omega) = n$.

3.2. Spectral asymptotics in the self similar deterministic model

Spectral dimension d_S

$$N(x) \sim x^{d_S/2}, \quad x \rightarrow \infty.$$

Kigami, Lapidus 1993

μ = Hausdorff measure on $K \Rightarrow \exists C_1, C_2, x_0$ such that
 $C_1 x^{d_S/2} \leq N_{D/N}(x) \leq C_2 x^{d_S/2}, \quad x \geq x_0,$

where $\frac{d_S}{2} = \frac{\ln M}{\ln(\varrho M)}$.

Hereby, M = number of copies, ϱ = energy scaling factor of \mathcal{E} .
We set $T := \varrho M$ (**Einstein-Relation!**). Hence, $\frac{d_S}{2} = \frac{\ln M}{\ln T}$.

Further remarks:

- **Berry's conjecture, 80's:** There is a Weyl type asymptotics also valid for fractals K , i.e.

$$N_K(x) = c_d \mathcal{H}^d(K) x^{d/2} + o(x^{d/2}), \quad \text{as } x \rightarrow \infty$$

with $d := d_H(K)$, c_d a constant independent of K . **Is wrong!**
i.g. $d_H \neq d_S$

- In general we do **not** have that $\mathcal{E}[u] \preceq \mathcal{H}^d$, i.e. we do **not** have that

$$\mathcal{E}[u] = \int |\nabla u|^2 d\mathcal{H}^d$$

- Second derivatives are easier to define than first derivatives!

Lecture Notes in Mathematics

András Telcs

The Art of Random Walks

1885

$$E \simeq \rho V$$

 Springer

Theorem 7.1 $(VD) + (TC) \Rightarrow E(x, 2r) \simeq p(x, r, 2r) V(x, 2r)$

4. The V -variable model

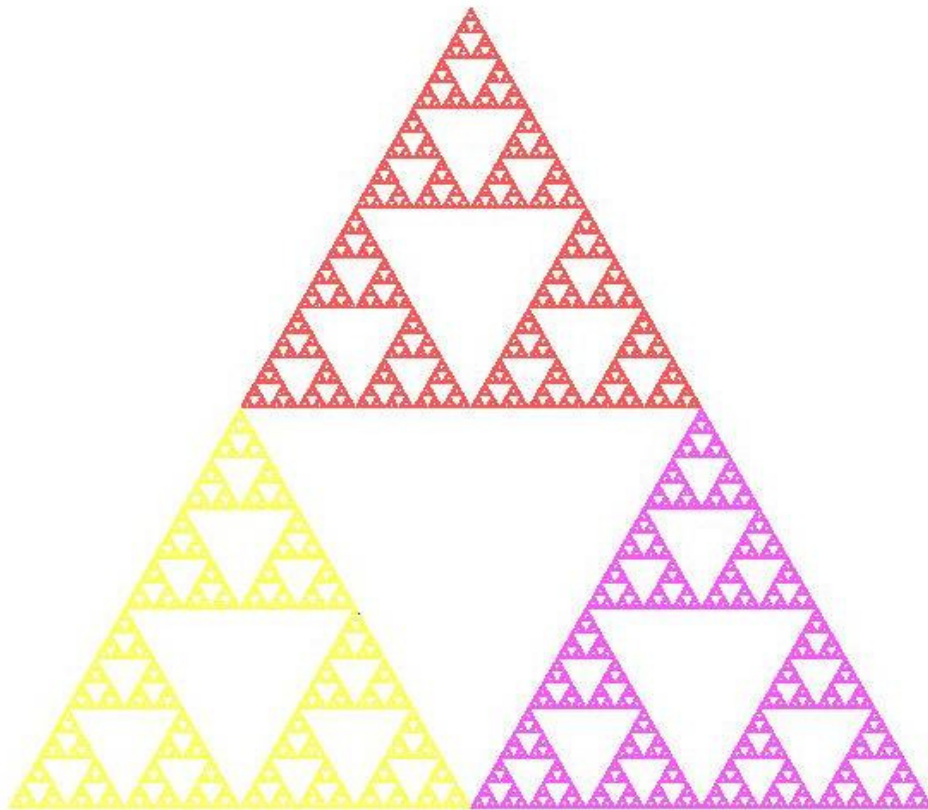
The assumption of strict self similarity can be too restricting (in order to model real world fractal sets).

Way out: „Weakening“ of the assumption, for exp. by

random mixing of different IFS's

4.1. The model case: SG(2) vs. SG(3)

SG(2)–Sierpinski gasket (F)



measure scaling factor μ

$$\mu_F = 1/3$$

length scaling factor r

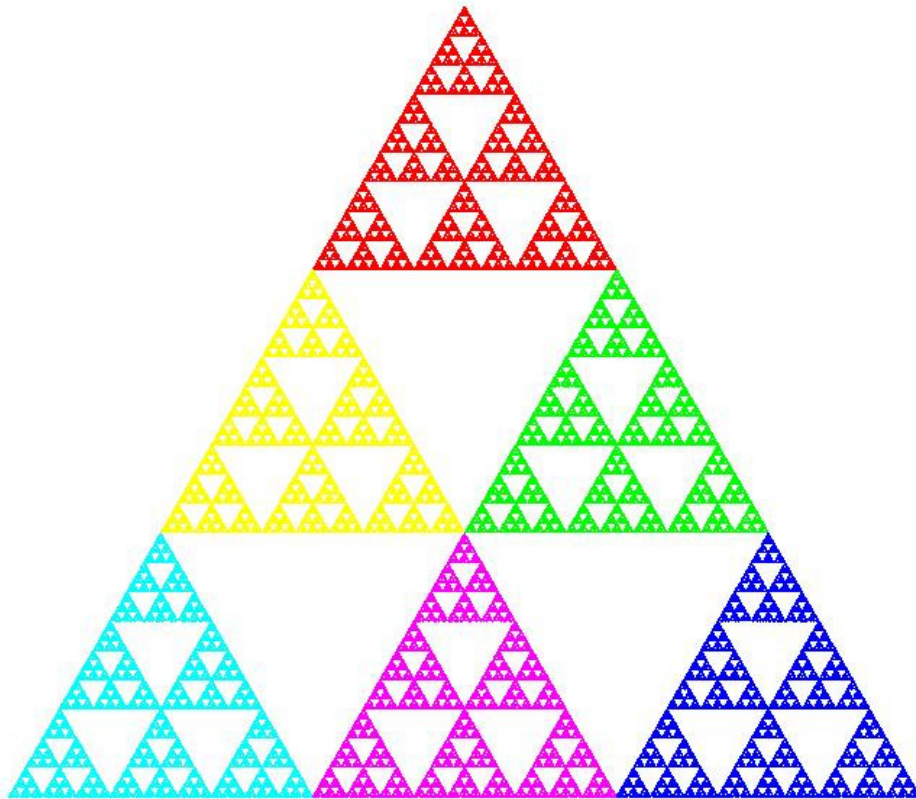
$$r_F = 1/2$$

energy/resistance scaling factor ϱ $\varrho_F = 5/3$

time scaling factor $T = M\varrho$

$$T_F = 5$$

SG(3)–modified Sierpinski gasket (G)



measure scaling factor μ

$$\mu_G = 1/6$$

length scaling factor r

$$r_G = 1/3$$

energy/resistance scaling

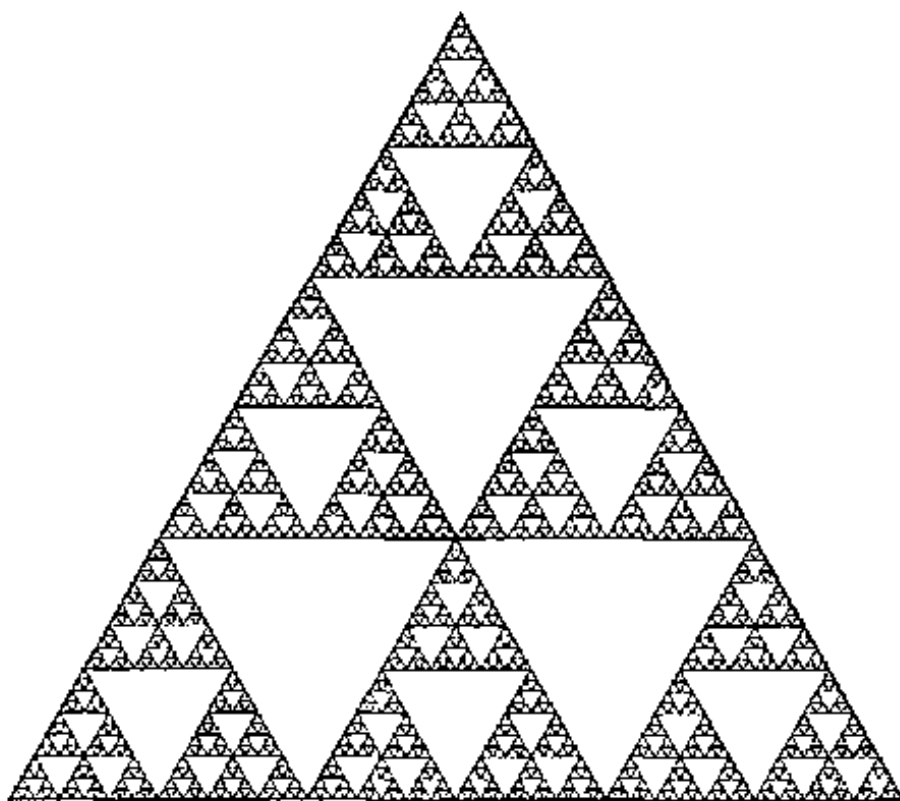
factor ϱ $\varrho_G = 15/7$

time scaling factor $T = M\varrho$

$$T_G = 90/7$$

Random mixing I

„homogenous random“ ($V = 1$)



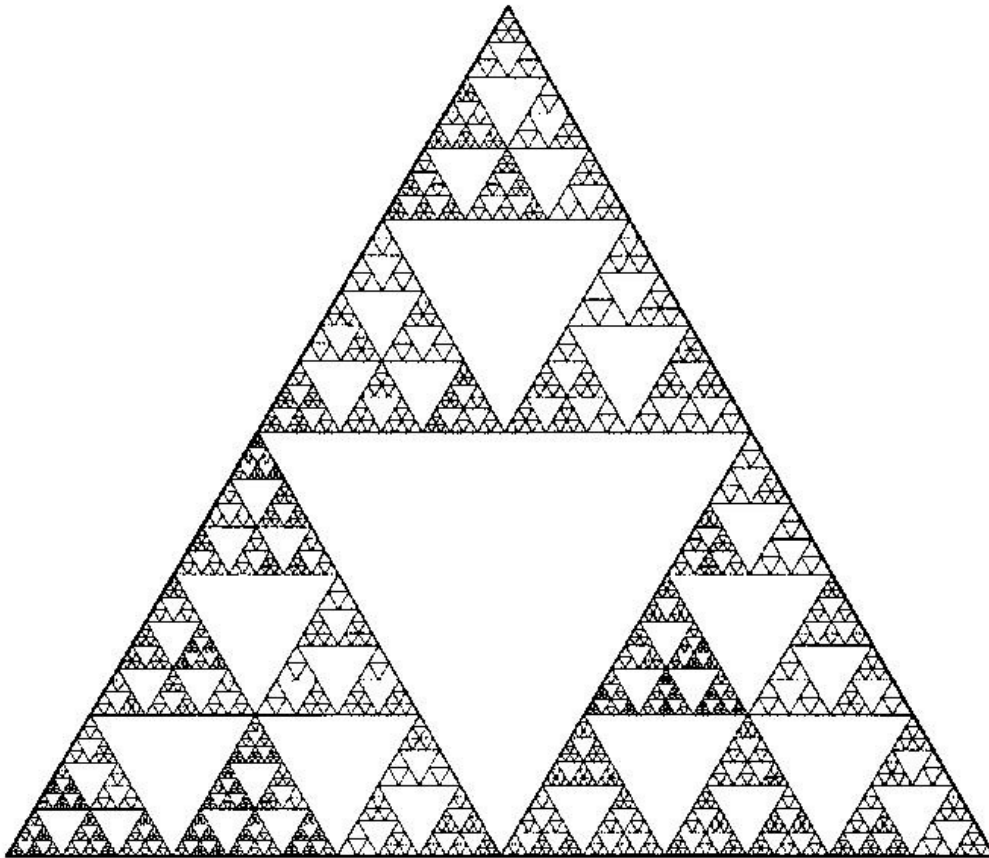
Hambly ('97 BM, '99 heat
kernels, '00 Δ)

(Kifer '95, Stenflo '01)

Coding: random sequence

Random mixing II

„random recursive“, „standard random “ ($V = \infty$)



Hambly '92 (BM), Barlow
Hambly '97 (Δ)

(Falconer '86, Mauldin
Williams '86, Graf '87,
Hutchinson Rüschemdorf
'98, '00)

Coding: random labelled
tree

now: „Interpolation“ between both the methods I and II

\rightsquigarrow „ V -variable fractal“ $V \in \mathbb{N} \cup \{\infty\}$

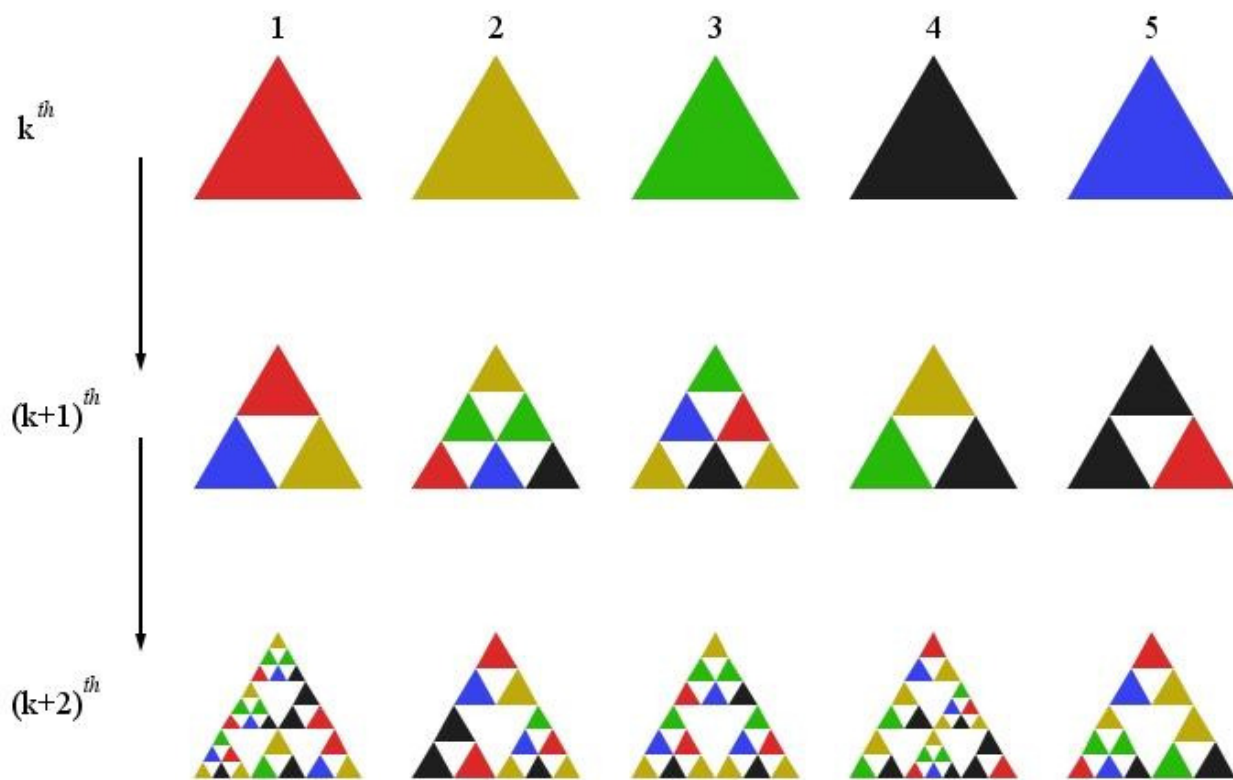
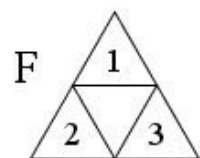
Construction of the V members of the $(k+1)^{st}$ generation from the V members of the k^{th} generation:

1° Choose \mathcal{F} or \mathcal{G} according to probabilities (p_F, p_G) .

2° Choose 3 (or 6, resp.) „parents“ from generation k for the i^{th} child of generation $k+1$.

Run this loop V times.

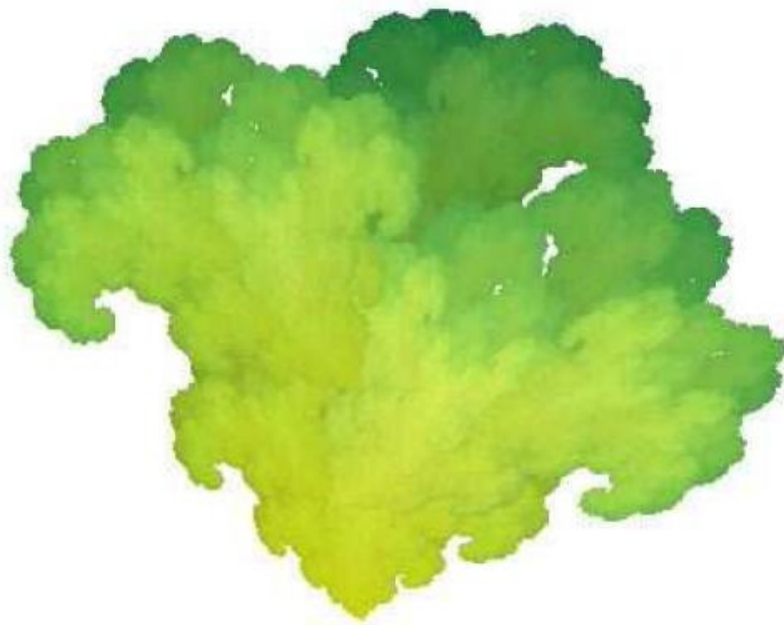
We construct V -tuples of sets instead of single sets!



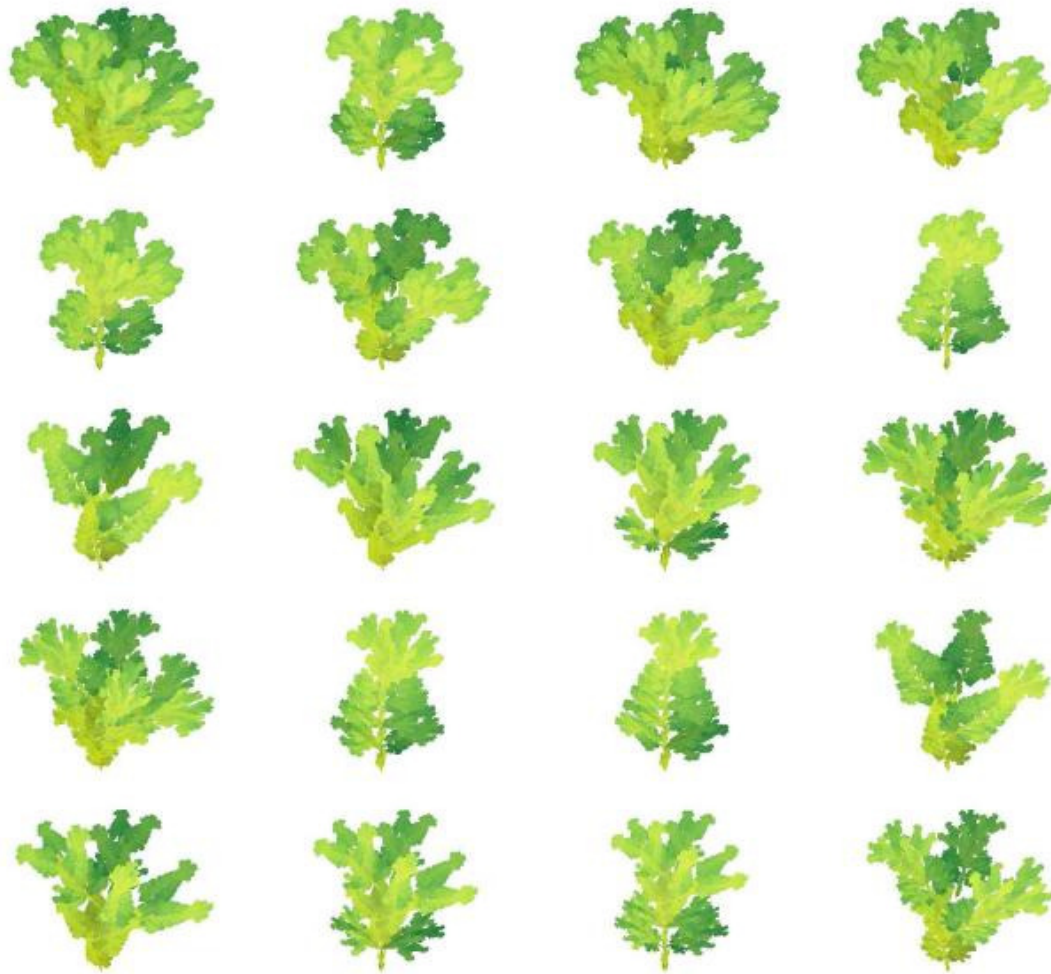
4.2. Applications

z.B. Modeling of species

- for exp: choose parents wrt. their geographical origin
- „breed“ fractals



salad and fern (attractors of IFS's)



$$V = 2$$

fern-lettuce-hybrids

5. Spectral asymptotics in the random case

5.1. Construction of the form

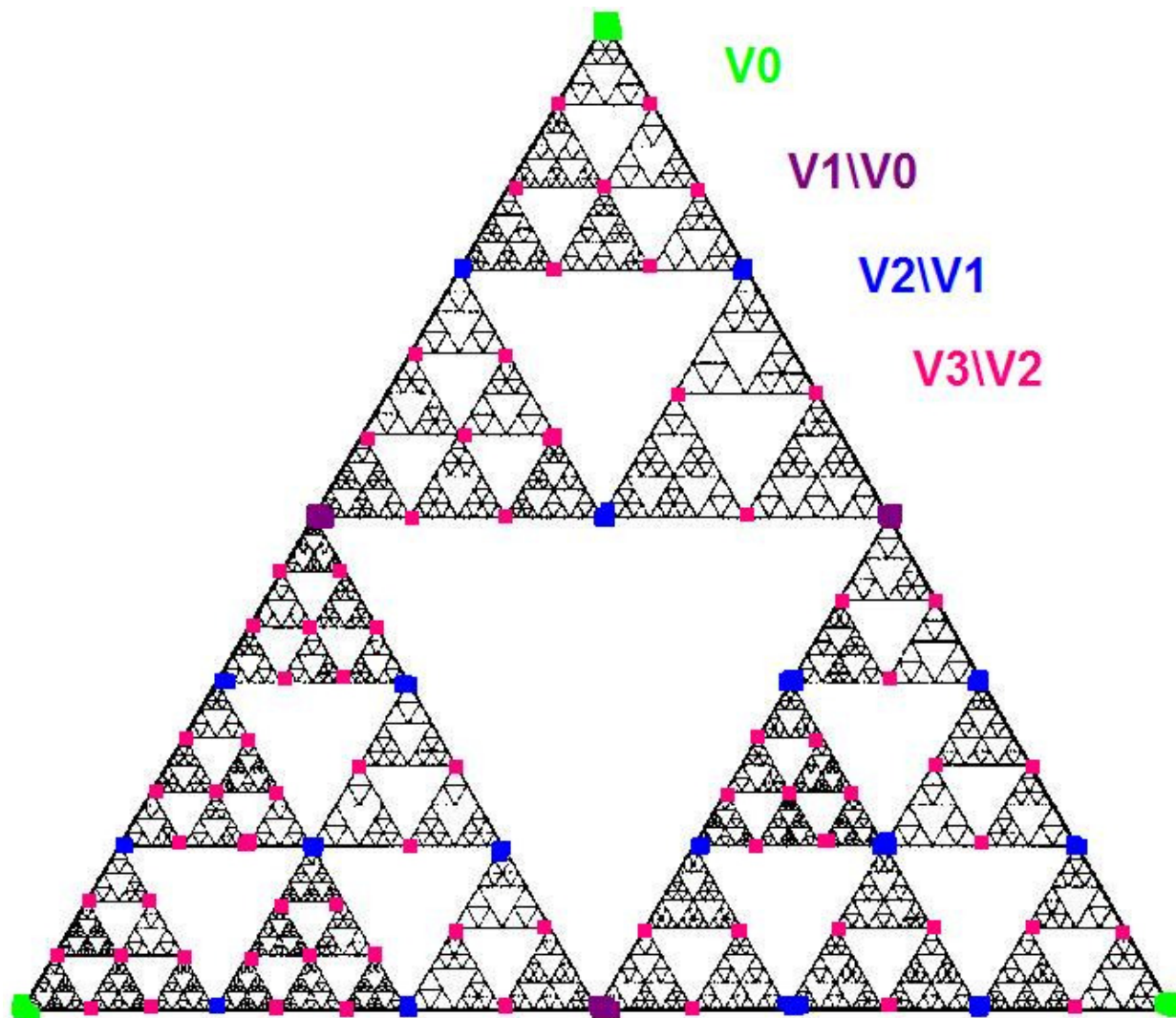
is done hierarchically and ω -wise

(Ω is the set of all V -variable trees)

Define V_0, V_1, V_2, \dots ω -wise

Values of a function given on $V_0 \rightsquigarrow$ harmonic extension to $V_1 \setminus V_0$
(i.e. one calculates the function in 3 (or 7) new points)

harmonic extension to $V_2 \setminus V_1$ on EACH of the 3 (or 6) sub
triangles of V_1 according to an \mathcal{F} - (or \mathcal{G} , reps.)-rule



Proceeding like this one obtains a sequence

$$\mathcal{E}_n^{(\omega)}[f] = \sum_{\bar{i} \in \omega_n} R(\bar{i}) \mathcal{E}_0[f \circ \psi_{\bar{i}}]$$

where

$$R(\bar{i}) = \prod_{j=1}^{|\bar{i}|} \varrho_j, \quad \varrho_j \in \{\varrho_F = 5/3, \varrho_G = 15/7\}$$

From the construction we obtain:

$$\mathcal{E}_n^{(\omega)}[f|_{V_n}] = \inf\{\mathcal{E}_{n+1}^{(\omega)}[g] : g|_{V_n} = f|_{V_n}\}$$

The limit form $(\mathcal{E}^{(\omega)}, \mathcal{D}(\mathcal{E}^{(\omega)}))$ is a Dirichlet form on $L_2(K(\omega), \mu(\omega))$,

where $K(\omega)$ is the realization of the random set, and $\mu(\omega)$ is a random self similar measure on $K(\omega)$ obtained as the Monge–Kantorovich–limit obtained by applying the Markov–operators $\mathcal{M}_{\mathcal{F}}$ or $\mathcal{M}_{\mathcal{G}}$, reps. according to the tree ω .

5.2. Results

- Homogenous case ($V = 1$): Sequences of F's and G's; Strong law of large numbers, Law of iterated logarithm, martingale theory.
- Recursive case ($V = \infty$): branching theory
- V -variable case: we now need products and sums of certain parameters according to the V -variable setting.

IDEA: Products of random $V \times V$ -matrices coding up the information of the construction process

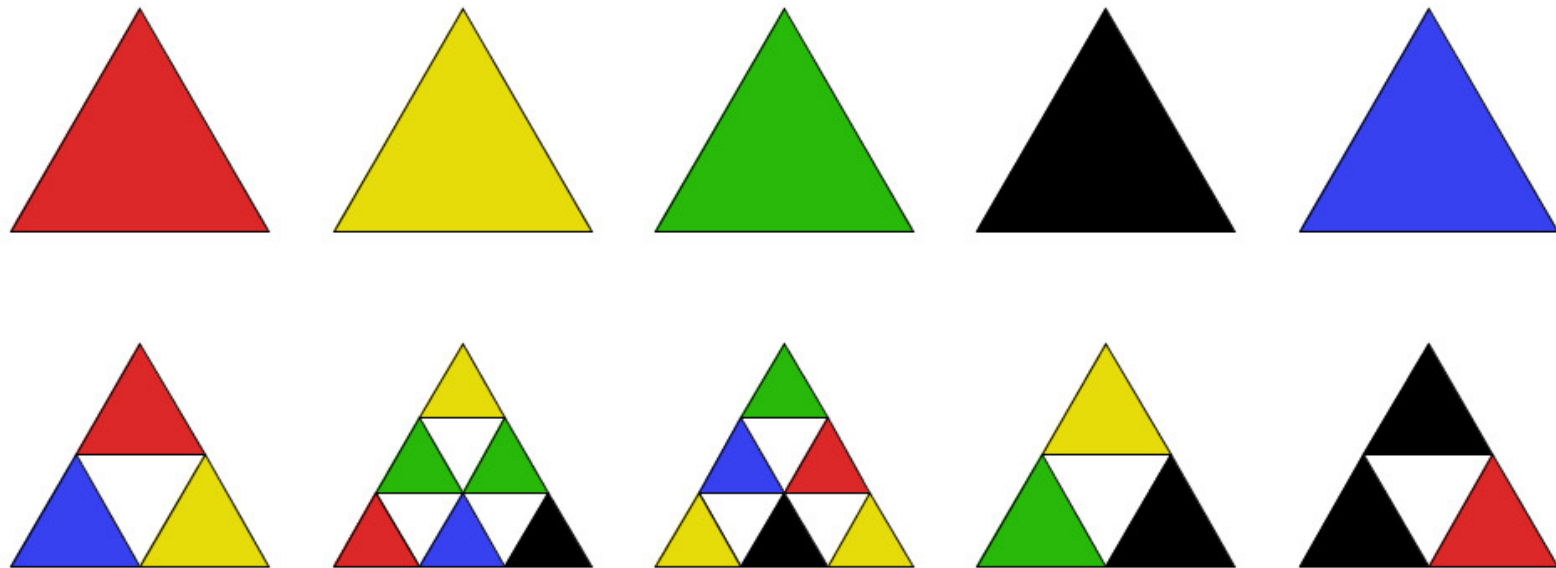
How to get now d_S ?

We now need products and sums of ϱ_F and ϱ_G according to the V -variable setting.

Define $T_F := \varrho_F M_F$ and $T_G := \varrho_G M_G$, i.e. $T_F = 5$ and $T_G = 90/7$.
(Note: These are the mean crossing times through the generating graphs.)

The transformation from level k to level $k + 1$ of V -tuples of triangles we code up with the help of an $V \times V$ -matrix $M^{(k)}(\alpha)$ as follows: (hereby $\alpha > 0$ is a free parameter)

Remember:



$$M^{(k)}(\alpha) = \begin{pmatrix} \left(\frac{1}{T_F}\right)^\alpha & \left(\frac{1}{T_F}\right)^\alpha & 0 & 0 & \left(\frac{1}{T_F}\right)^\alpha \\ \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & 2\left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha \\ \left(\frac{1}{T_G}\right)^\alpha & 2\left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha \\ 0 & \left(\frac{1}{T_F}\right)^\alpha & \left(\frac{1}{T_F}\right)^\alpha & \left(\frac{1}{T_F}\right)^\alpha & 0 \\ \left(\frac{1}{T_F}\right)^\alpha & 0 & 0 & 2\left(\frac{1}{T_F}\right)^\alpha & 0 \end{pmatrix}$$

pressure function

$$\gamma_V(\alpha) := \lim_{k \rightarrow \infty} \frac{1}{k} \log \left(\frac{1}{V} \left\| M^{(k)}(\alpha) \dots M^{(1)}(\alpha) \right\| \right),$$

where the norm $\|A\|$ is the sum of all the entries in the matrix A .

Theorem (F+Hambly+Hutchinson, 2010)

$\gamma_V(\alpha)$ is a well defined function of α and independent of the realization of the experiment.(Furstenberg/Kesten 1960).

Moreover it holds that $\gamma_V(\cdot)$ is strictly monotone decreasing and $\exists! d : \gamma_V(d) = 0$.

For this zero d of $\gamma_V(\cdot)$ it holds a.s. that $N(x) \sim x^d$.

More precisely, it holds that

$$N(x)x^{-\alpha} \longrightarrow 0 \quad \mathbf{P} - a.s. \quad \text{for } \alpha > d.$$

and

$$N(x)x^{-\alpha} \longrightarrow \infty \quad \mathbf{P} - a.s. \quad \text{for } \alpha < d.$$

[FHaHu, 2011] Refinement of the result on the spectral asymptotics:

- For any self similar measure it holds that

$$\lim_{x \rightarrow \infty} \frac{\log N(x)}{\log x} = \frac{d_s}{2} \quad \mathbf{P} - a.s.$$

- In the „flat-measure-case“ it holds that

$$c^{-1}x^{d_s/2}/\phi(x) \leq N(x) \leq cx^{d_s/2}\phi(x), \quad x \rightarrow \infty, \mathbf{P} - a.s.,$$

where $\phi(x) := \exp(-c\sqrt{\log x \log \log \log x})$.

- In the „flat-measure-case“ it holds for $V = \infty$ that

$$\lim_{x \rightarrow \infty} \frac{N(x)}{x^{d_s/2}} = C, \quad \mathbf{P} - a.s.$$

[FHaHu, 2011] heat-kernel-estimates (on-diagonal)

In the „flat-measure-case“ exist constants such that \mathbf{P} -a.s. for μ -almost every $x \in K$ it holds that

$$c_1 \phi(1/t)^{-c_2} t^{-d_s/2} \leq p_t(x, x) \leq c_3 \phi(1/t)^{c_4} t^{-d_s/2}, \quad 0 < t < 1,$$

where $\phi(x) = \exp(-c\sqrt{\log x \log \log \log x})$.

Remark: The measure is not „doubling“!! Existence of heat kernels: [Croydon, 2007]

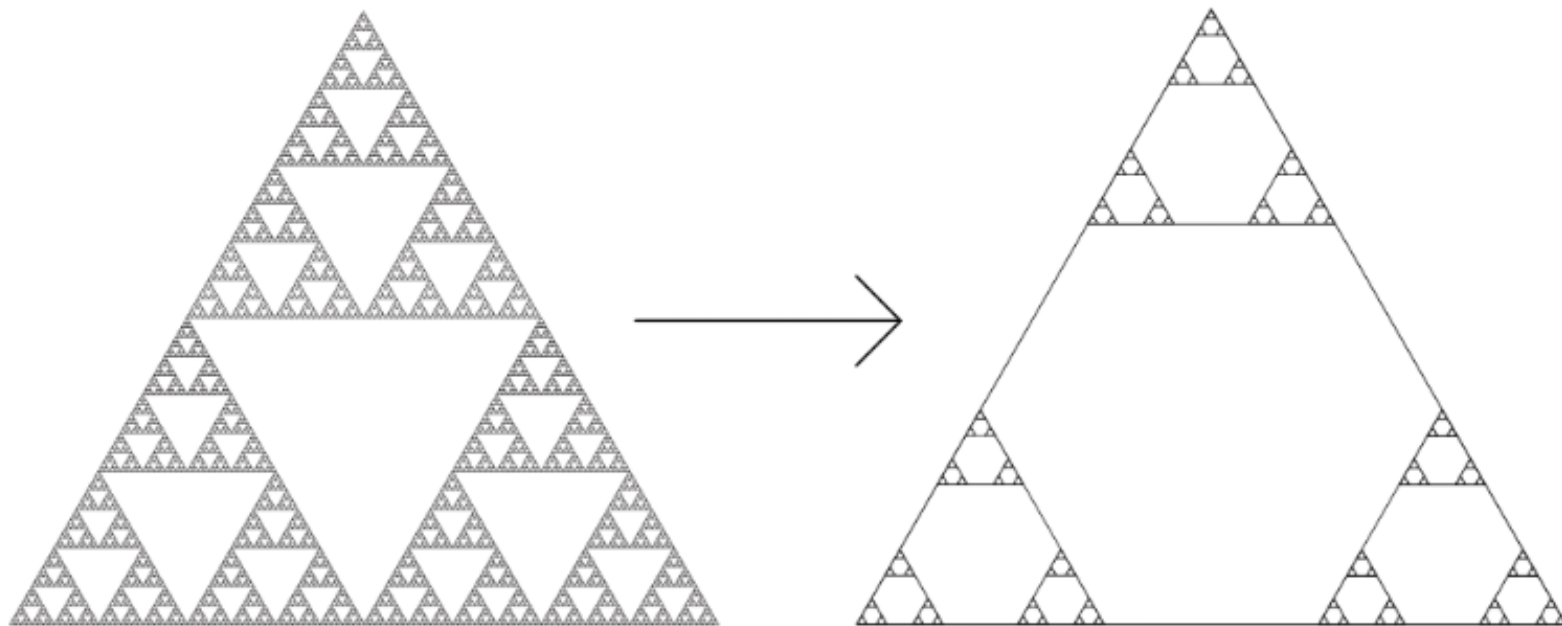
- Well known results for $V = 1$ are contained now as special cases.
- For $V \geq 2$: no explicit expression for d_S (simulation)

Reference:

U. Freiberg, B.M. Hambly and J.E. Hutchinson
[Spectral asymptotics for \$V\$ -variable Sierpinski gaskets.](#)
Ann. Inst. H. Poincaré Probab. Stat. 53, 2162–2213, 2017

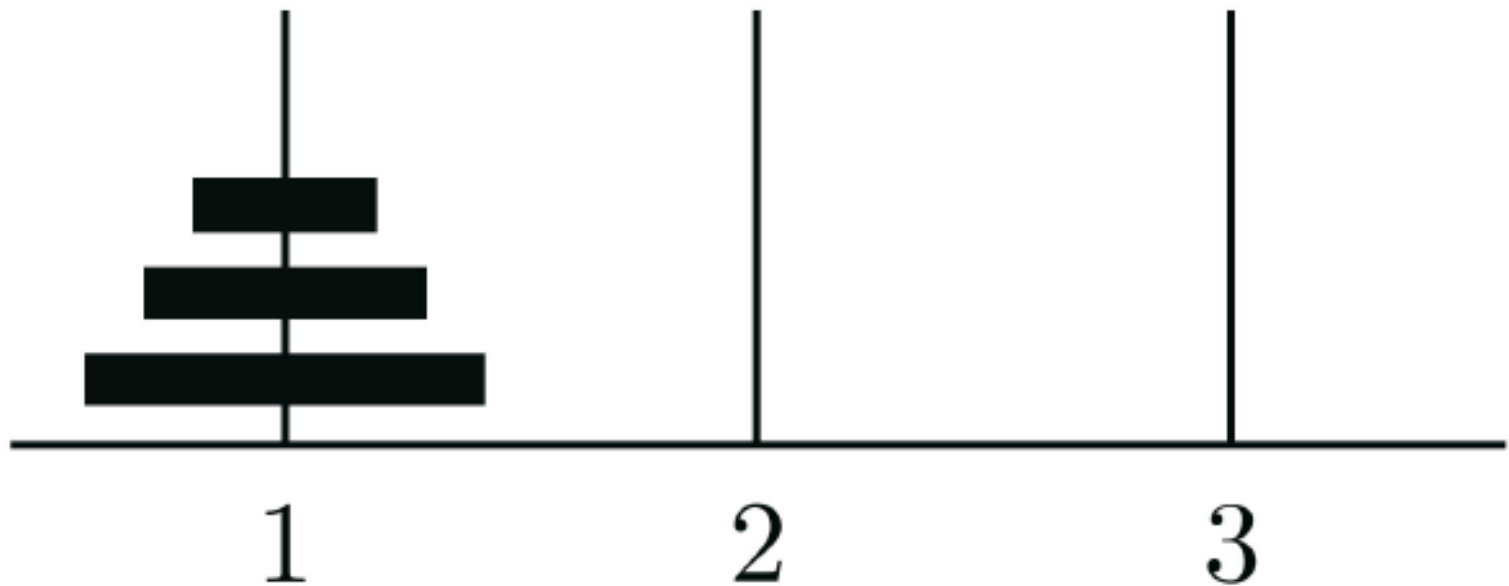
(or, Ben's homepage)

6. Stretched fractals

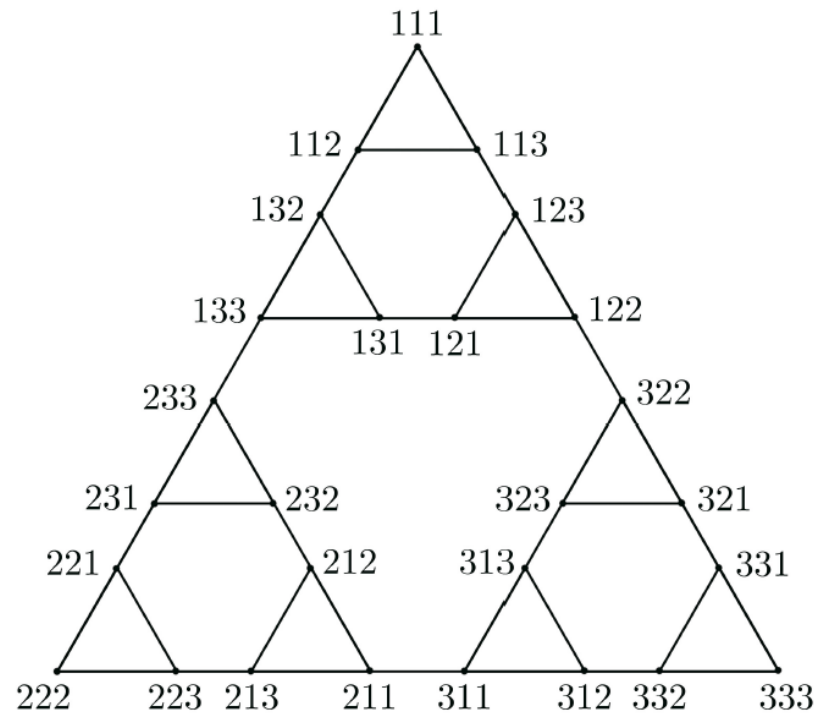


Sierpinski gasket and stretched Sierpinski gasket

Stretched Sierpinski gasket is often called **Hanoi graph**

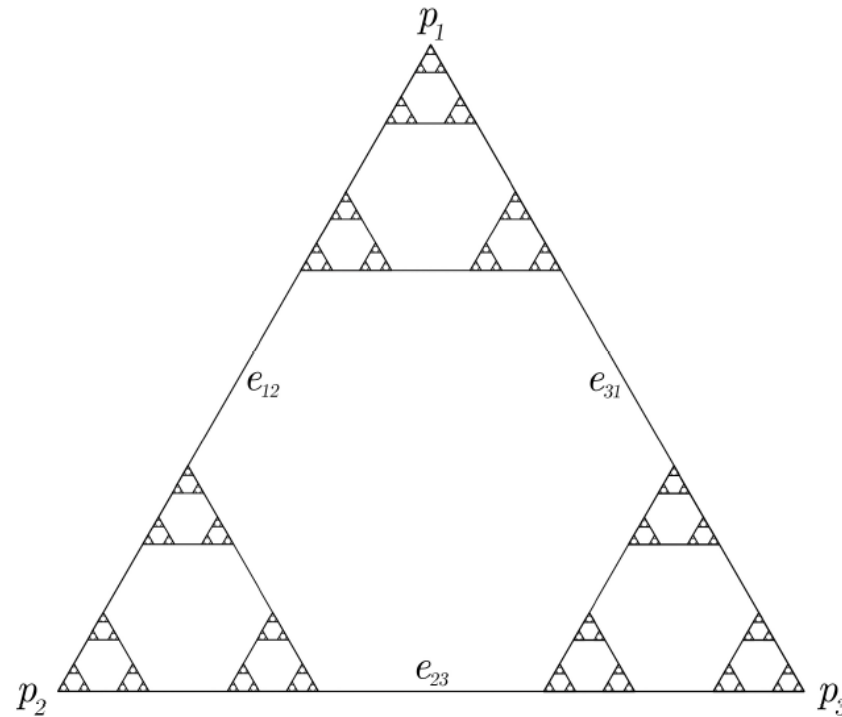


Tower of Hanoi game



Legal moves

see: Hinz, Klavzar, Petr: The tower of Hanoi – myths and maths, 2nd ed.
2018



SSG is an **inhomogenous** self similar fractal

- Let p_1, p_2, p_3 be the vertex points of an equilateral triangle with side length 1 and for $\alpha \in (0, 1)$ define

$$G_i(x) := \frac{1 - \alpha}{2}(x - p_i) + p_i, \quad i = 1, 2, 3$$

- $e_{ij} :=$ line segment between $G_i(p_j)$ and $G_j(p_i)$.

- Then, K_α is the unique compact set with

$$K_\alpha = G_1(K_\alpha) \cup G_2(K_\alpha) \cup G_3(K_\alpha) \cup e_{12} \cup e_{23} \cup e_{31}.$$

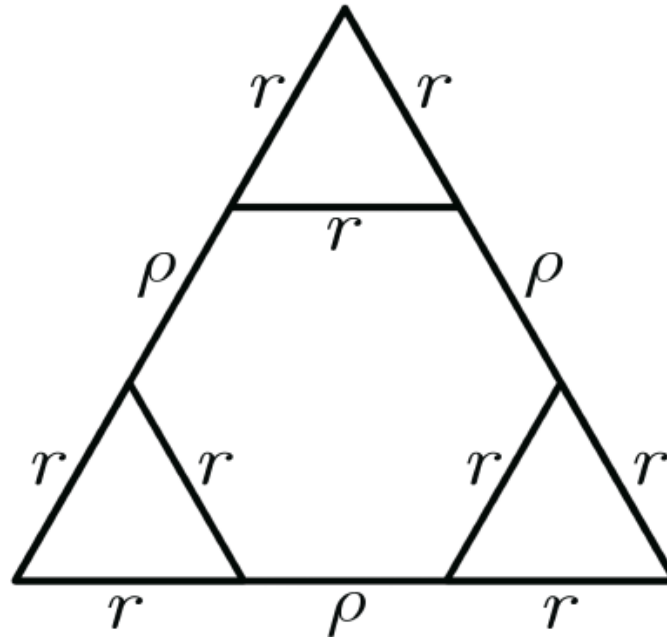
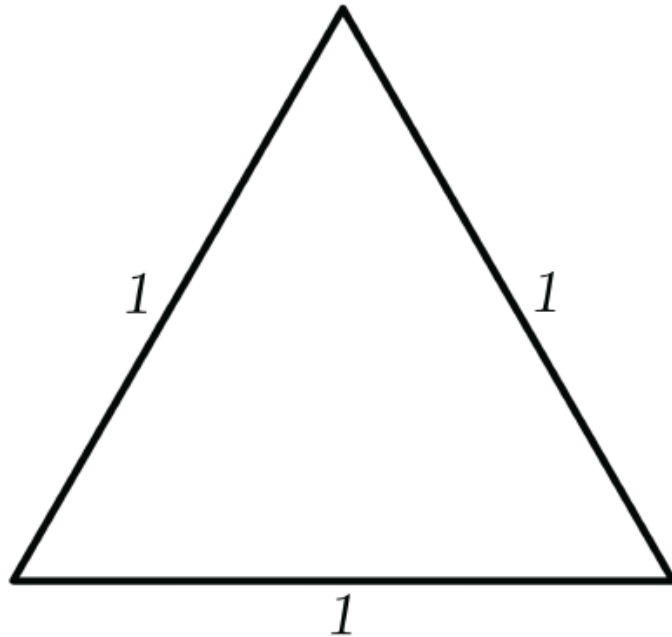
- Unique (in the Hausdorff space) solution of

$$\Sigma_\alpha = G_1(\Sigma_\alpha) \cup G_2(\Sigma_\alpha) \cup G_3(\Sigma_\alpha)$$

we call **fractal part**, the rest $K_\alpha \setminus \Sigma_\alpha$ **line part**.

- **Observation**: $\alpha \downarrow 0 : K_\alpha \rightarrow K$ Sierpinski gasket
in Hausdorff distance, in Hausdorff dimension
(P. Alonso–Ruiz, URF: Hanoi attractors and the Sierpinski gasket, 2012)

- **Question**: Does the analysis converge? In which sense?
(P. Alonso–Ruiz, URF: Weyl asymptotics for Hanoi attractors, 2017)



- $\rho + 5/3r = 1$, (ρ, r) **matching pair**
- more general: $(\rho_n, r_n)_{n \geq 1}$ **matching sequence**, see: P. Alonso–Ruiz, URF and J. Kigami: [Completely symmetric resistance forms on the stretched Sierpinski gasket.](#), J. Fractal Geom. 5 (2018), no. 3, 227–277

- Weyl asymptotics: 3 preprints, on arXiv:

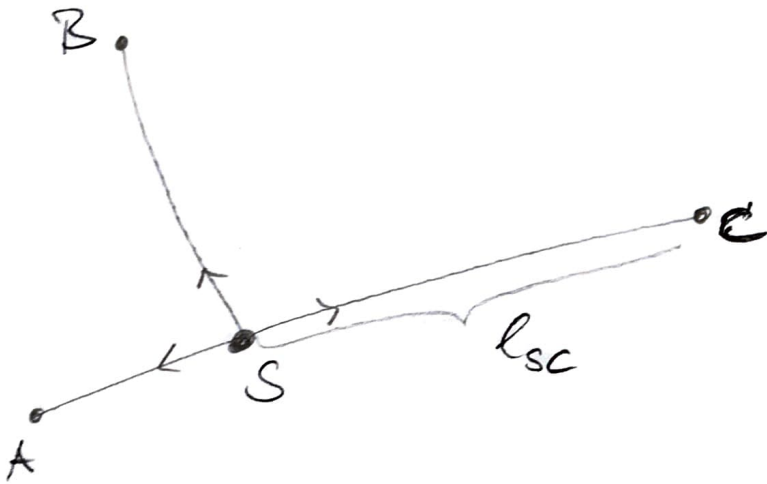
Elias Hauser: Spectral asymptotics on the Hanoi attractor, 2017

Elias Hauser: Oscillations on the Stretched Sierpinski Gasket, 2018.

Elias Hauser: Spectral Asymptotics for Stretched Fractals, 2018.

- many open problems, in particular on the associated stochastic process...

Problem: transition probabilities



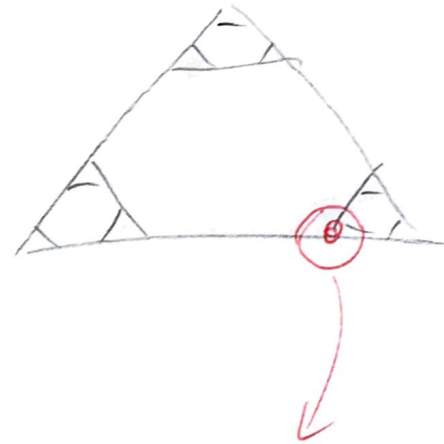
$l_{AS}, l_{BS}, l_{CS} \hat{=}$ lengths

$r_{ij} = l_{ij}$ "resistances"

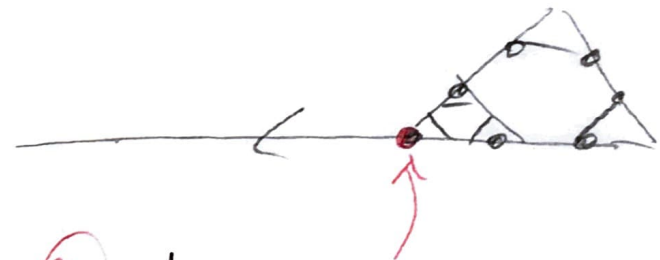
$C_{ij} = r_{ij}^{-1}$ "conductances"

natural:

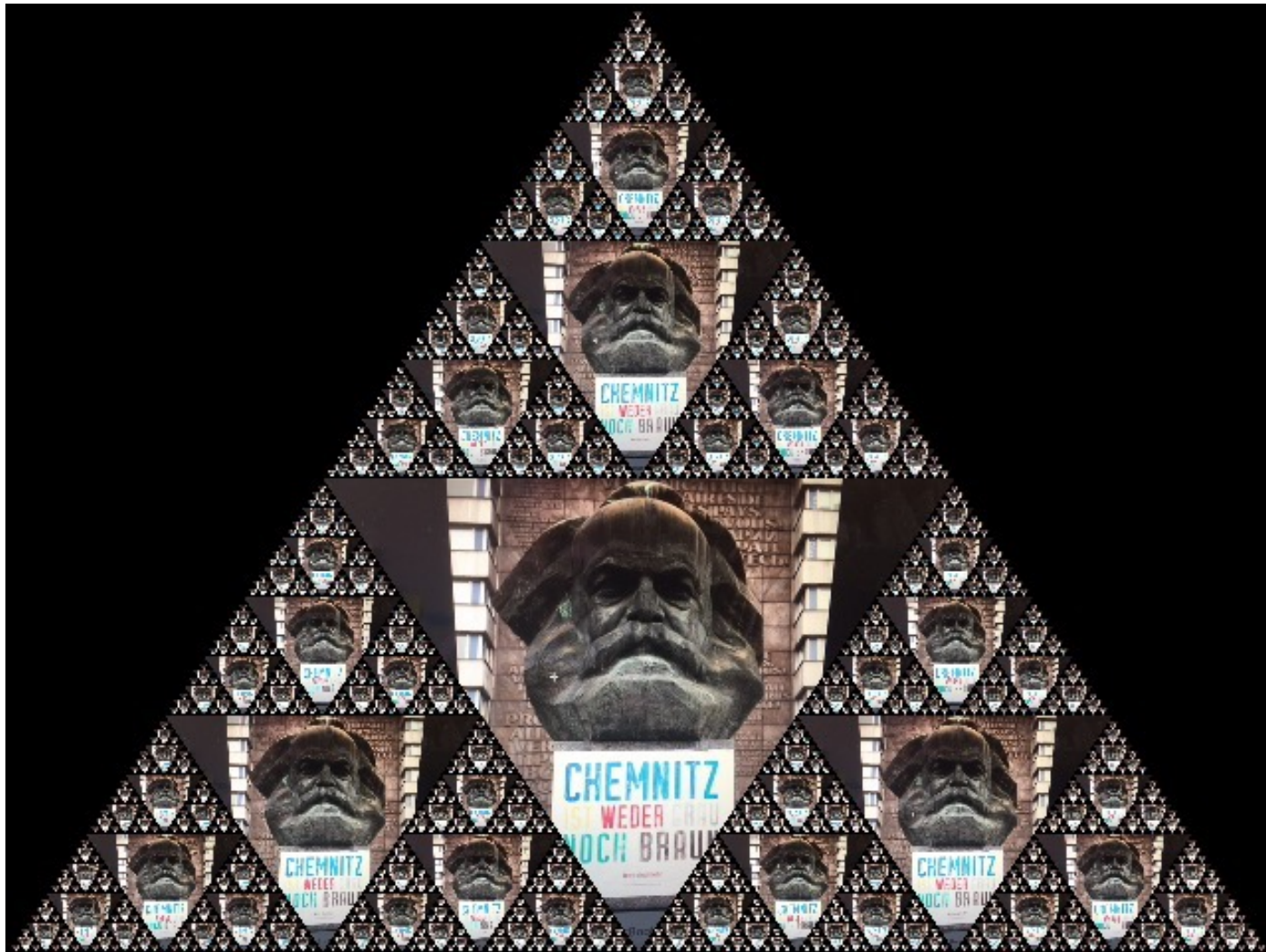
$$P_{SA} = \frac{C_{SA}}{C_{SA} + C_{SB} + C_{SC}}$$



Hanoi graph



(?) transition prob
in this point ?



Thank you for your attention!