SURJECTIVITY OF FORMAL GRAPH LAPLACIANS AND MAGNETIC SCHRÖDINGER OPERATORS

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Let b be a graph on an at most countable set X as in the lectures. In this project we study surjectivity of the formal Laplacian (and related formal magnetic Schrödinger operators)

$$\mathcal{L}\colon \mathcal{F} \to C(X), \quad \mathcal{L}f(x) = \sum_{y\in X} b(x,y)(f(x)-f(y))$$

with

$$\mathcal{F} = \{f \in C(X) \mid \sum_{y\in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\}.$$

If X is finite, then $\mathcal{L}$ is not injective because $\mathcal{L}1 = 0$ and hence it cannot be surjective. On infinite connected graphs one can show with the help of local Harnack inequalities that any $f \in \mathcal{F}$ with $\mathcal{L}f = 0$ must have infinite support, making the restriction $\mathcal{L}|_{C_c(X)}$ injective.

It was observed in [3] that on locally finite graphs injectivity of $\mathcal{L}|_{C_c(X)}$ leads to surjectivity of $\mathcal{L}$. This is based on an old surjectivity critertion of Eidelheit [2], which relies on some abstract surjectivity results for spaces of type (B_0) due to Mazur and Orlicz (Fréchet spaces in modern terminology, the name had not yet been coined in 1936). A self-contained account on the surjectivity of $\mathcal{L}$ on locally finite graphs without much reference to abstract Fréchet space theory can be found in [4]. There it is also shown that on infinite connected graphs which are not locally finite surjectivity of $\mathcal{L}$ need not hold anymore. A different approach to surjectivity of $\mathcal{L}$ with a Mittag-Leffler type argument is given in [1]. Indeed, it preceeded and inspired the previously mentioned [3].

The first aim of this project is to study and present the two different apporoaches to surjectivity of $\mathcal{L}$ in [3, 4] and [1]. Eidelheit's surjectivity criterion should lead to surjectivity results on some not locally finite graphs, a case which apart from a counterexample has not been treated in the literature. The second aim of this project is to explore and in the best case characterize those not locally finite graphs whose formal Laplacian is surjective with the help of Eidelheit's results from [2].

This project is suited for 3 to 4 students. Since one of the references is in German, at least one German speaker is required.

References

[1] Tullio Ceccherini-Silberstein, Michel Coornaert, and Józef Dodziuk. The surjectivity of the combinatorial Laplacian on infinite graphs. *Enseign. Math. (2)*, 58(1-2):125–130, 2012.

[2] M. Eidelheit. Zur Theorie der Systeme linearer Gleichungen. *Studia Math.*, 6:139–148, 1936.

[3] T. Kalmes. A short remark on the surjectivity of the combinatorial Laplacian on infinite graphs. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 110(2):695–698, 2016.

[4] J. Koberstein, M.Schmidt A note on the surjectivity of operators on vector bundles over discrete spaces. *Arch. Math. (Basel)*, 114 (2020), no. 3, 313–329.