HARNACK ESTIMATES FOR THE DISCRETE HEAT AND POROUS MEDIUM EQUATION VIA CURVATURE-DIMENSION CONDITIONS

SEBASTIAN KRÄSS

In order to derive parabolic Harnack estimates for positive solutions u to the heat equation on $(0,\infty)\times\mathbb{R}^n$, the fundamental estimate of Li and Yau

$$-\Delta \log u(t,x) \leq \frac{n}{2t} \tag{1}$$

has turned out to be a very helpful tool. Indeed, by integration over a suitable path in time, one can derive a classical Harnack estimate. In this sense, Li-Yau inequalities are often called differential Harnack estimates. For more details, see [3, Chapter 12]. A discrete version of (1) has been established in [1, Theorem 4.1] in the case of a finite graph. This discrete estimate is based on a so-called curvature dimension inequality, namely the condition $CD(F;0)$ defined in [1, Definition 3.8]. This theory will be the starting point of our project.

We then also have a closer look at the generalization of this theory to the nonlinear case when dealing with positive solutions to the discrete porous medium equation (PME)

$$\partial_t u(t,x) - Lu^m(t,x) = 0 \tag{2}$$

on $(0,\infty)\times X$, where $m>1$, X is a (possibly infinte) discrete set and L denotes the discrete Laplacian as defined in the lecture notes of this year's ISEM. Note, that for $m=1$, equation (2) becomes the heat equation.

Our **main references** for the project are [1] and [2]. In the case of the PME, we will prove estimates of the form

$$-L\left(\frac{m}{m-1}u^{m-1}\right)(t,x) \leq \frac{c}{t}, \tag{3}$$

which are called Aronson-Bénilan estimates in the literature. Note, that (3) becomes a discrete version of the Li-Yau inequality (1) for $m\to 1$ (as the left hand side in (3) converges to $-L\log u$ when $m\to 1$).

Our procedure will be the following: We start by discussing the linear case $m=1$ considered in [1]. We will see how a certain curvature-dimension inequality, namely $CD(F;0)$, is sufficient to prove discrete Li-Yau estimates on finite graphs. We then study the generalization of these results to the nonlinear case $m>1$ by formulating an adapted curvature-dimension condition that is sufficient to prove estimates of the form (3). Several examples, the most important being the one of a general complete graph, will be treated for the PME case. We also have a closer look at the lattice $\mathbb{Z}$ where the proof of an Aronson-Bénilan estimate turns out to be much more difficult. In both, the linear and the nonlinear case, we will use our differential estimates to derive Harnack type estimates by simple integration arguments.

This project is suited for 3 to 4 students.

References

[1] D. Dier, M. Kassmann, R. Zacher: *Discrete versions of the Li-Yau gradient estimate.* Preprint 2017. https://arxiv.org/abs/1701.04807.

[2] S. Kräss, R. Zacher: *Aronson-Bénilan and Harnack estimates for the discrete porous medium equation.* Preprint 2023. https://arxiv.org/abs/2301.07683.

[3] P. Li: *Geometric Analysis.* Cambridge Studies in Advanced Mathematics **134**. Cambridge University Press, Cambridge, 2012.