INDEX THEOREMS AND TRACE FORMULAS FOR QUANTUM GRAPHS

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In the ISem lectures discrete graphs and difference operators have been considered. A different graph-like setting are metric graphs where the edges are identified with intervals filling the "space between the vertices". Operators on such metric graphs then consist of one-dimensional ordinary differential operators on the edges (that is a collection of intervals) and coupling boundary conditions on the vertices. For such operators – sometimes called *quantum graphs* – the spectral theory is studied in this project.

One theme of the ISem lecture [1] has been the interplay between geometry and spectral properties. Here, we carry over this theme to the setting of metric graphs discussing *index theorems* and *trace formulas* for quantum graphs. For differential operators on manifolds the so-called index theorems provide a connection between the topology of the manifold and the Fredholm index of certain differential operators, where the Fredholm index describes the difference between dimension of the kernel and the co-range of an operator, that is, it quantifies information on the solvability of the corresponding differential equations. In this project we will work through the corresponding *index theorems for quantum graphs* given in [2] by Fulling, Kuchment, and Wilson. Metric graphs can be seen also as singular one-dimensional manifolds, and therefore – if there is interest – also relations to the famous Atiyah-Singer index theorem for manifolds can be discussed, cf. e.g. [3]. Another instance where spectral quantities and geometry are related are *trace formulas*. Here, we will study the work [4] of Kostrykin, Potthoff, and Schrader where the trace of the heat kernel is related to the Euler characteristic of the graph and closed cycles on the graph. Connections to the celebrated Selberg trace formula for manifolds can be discussed if time permits.

This project is suited for 3 to 4 students.

References

[1] ISem 26, Lecture Notes, 2023. https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf

[2] Fulling, S. A. and Kuchment, P. and Wilson, J. H., Index theorems for quantum graphs, J. Phys. A, 40(47): 14165–14180, 2007. https://doi.org/10.1088/1751-8113/40/47/009

[3] Gilkey, P. B., Invariance theory, the heat equation, and the Atiyah-Singer index theorem, CRC Press, Boca Raton, FL, 1995. https://pages.uoregon.edu/gilkey/dirPDF/InvarianceTheory1Ed.pdf

[4] Kostrykin, V. and Potthoff, J. and Schrader, R. , Heat kernels on metric graphs and a trace formula, In *Adventures in mathematical physics* Amer. Math. Soc., Providence, RI, 447: 175–198, 2007. https://doi.org/10.1090/conm/447/08691 Preprint https://arxiv.org/pdf/math-ph/0701009.pdf