THE p-LAPLACIAN ON DISCRETE GRAPHS AND THE TORSION FUNCTION

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Given a discrete measure space (X, m) and a graph (b, c) over (X, m), for $p \in (1, \infty)$, we study the *p-Laplacian* $\mathcal{L}^{(p)}_{b,c,m}$: this is an operator formally acting on functions in $C(X)$ by
(1)

$$\mathcal{L}^{(p)}_{b,c,m} f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))|f(x) - f(y)|^{p-2} + \frac{c(x)}{m(x)}|f(x)|^{p-2} f(x).$$

For $p = 2$, $\mathcal{L}^{(p)}_{b,c,m}$ is nothing but the formal Laplacian on the graph; but for $p \neq 2$ (1) defines a non-linear operator.
Studying this new object, we will extend many results we encountered in the ISem-Lectures [1] in the special case $p = 2$, first on finite and then on infinite graphs: we will explain the details of the introduction of these operators and then sketch the construction of the associated semigroups and perhaps discuss (a nonlinear version of) the Markov property for such semigroups.
Among the main goals of this project is a gentle introduction to the theory of the *p-torsion function*, i.e., of the (positive) solution τ of

$$\mathcal{L}^{(p)}_{b,c,m} \tau = \mathbf{1};$$

for the students of the ISem, this is a new topic already for $p = 2$. It turns out that the p-torsion function play an interesting role in the spectral theory of p-Laplacians: not only do its 1-norm and ∞-norm deliver upper and lower estimates on the lowest strictly positive eigenvalue of $\mathcal{L}^{(p)}_{b,c,m}$, but the p-torsion function also induces pointwise bounds on *all* eigenfunctions [5]. Our methods will be mostly variational, in view of a known characterization of the torsion function that goes back to Pólya.
If time allows we will also compare our results with recent results for the torsional rigidity on metric graphs in [3] and further structures.

This advanced project is suited for 4 to 5 students, ideally with a good understanding of variational methods.

References

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] D. Mugnolo. Parabolic theory of the discrete p-Laplace operator. *Nonlinear Anal., Theory Methods Appl.*, 87:33–60, 2013.

[3] D. Mugnolo and M. Plümer. *On torsional rigidity and ground-state energy of compact quantum graphs.* Calc. of Var. and Part. Diff. Eq. **62**, 27 (2023)

[4] P. Bifulco and D. Mugnolo. *On the torsion of discrete graphs.* (work in progress)

[5] D. Mugnolo. Pointwise eigenvector estimates by landscape functions: some variations on the Filoche–Mayboroda–van den Berg bound. arXiv:2301.06126, 2023.