26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 09

CHAPTER 7

Intrinsic Metrics and Spectral Estimates

In this chapter we turn to spectral geometry of graphs. This means that we study the interplay between spectral theory of graph Laplacians L and the geometry of the graphs. Here, geometry is captured by intrinsic metrics. These are (pseudo)metrics which are adapted in some sense to the graph structure. A remarkable feature of graphs is that they will in general admit several intrinsic metrics which are not comparable. In particular, in general there is no natural canonical intrinsic metric.

The spectral theory we are interested in is the infimum $\lambda_0(L^{(D)})$ of the spectrum of $L^{(D)}$. Specifically, we will give lower and upper bounds of the form

$$\frac{h^2}{2} \leq \lambda_0(L^{(D)}) \leq \frac{\mu^2}{8},$$

where h is an isoperimetric or Cheeger constant and μ is an exponential volume growth rate.

7.1. Intrinsic metrics

We first discuss some generalities on pseudometrics and then turn to specific pseudometrics associated to graphs. Let X be a set. We will be interested in maps on X, which can be thought of as weak versions of metrics. A *pseudometric* on X is a map $\varrho : X \times X \longrightarrow [0, \infty)$ with

- $\varrho(x, x) = 0$,
- $\varrho(x, y) = \varrho(y, x)$, ("Symmetry")
- $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ ("Triangle inequality")

for all $x, y, z \in X$.

Whenever ϱ is a pseudometric the inequality

$$|\varrho(o, x) - \varrho(o, y)| \leq \varrho(x, y)$$

holds for all $x, y, o \in X$. It is known as *reverse triangle inequality.*

Let now ϱ be a pseudometric on X. Let $C \geq 0$ be given. A function f on X is said to be C*-Lipschitz* (with respect to ϱ) if

$$|f(x) - f(y)| \leq C\varrho(x, y)$$

holds for all $x, y \in X$. A *Lipschitz function* is a function, that is C-Lipschitz for some $C \geq 0$. The set of 1-Lipschitz functions with respect to ϱ is denoted by $\mathrm{Lip}_{1,\varrho}(X)$. Clearly, f is C-Lipschitz (for some $C > 0$) if and only if $\frac{1}{C}f$ belongs to $\mathrm{Lip}_{1,\varrho}(X)$.

From the reverse triangle inequality we see that the function $\varrho_x := \varrho(x, \cdot)$, that gives the ($\varrho$-)distance to $x \in X$, is 1-Lipschitz for any $x \in X$.

In fact, this can be generalized as follows: Let $U \subseteq X$ be non-empty and define the *(ϱ-)distance* to U by

$$\varrho_U : X \longrightarrow [0,\infty], \quad \varrho_U(y) := \inf_{x\in U} \varrho(x,y).$$

Then, ϱ_U can easily be seen to belong to $\mathrm{Lip}_{1,\varrho}(X)$. So, the pseudometric ϱ gives rise to a wealth of 1-Lipschitz functions.

Conversely, any 1-Lipschitz function f gives rise to a pseudometric σ_f defined by $\sigma_f(x,y) := |f(x) - f(y)|$ with $\sigma_f \le \varrho$. One can recover ϱ from $\mathrm{Lip}_{1,\varrho}(X)$ as

$$\varrho = \sup\{\sigma_f : f \in \mathrm{Lip}_{1,\varrho}(X)\}.$$

Here, the inequality $\ge$ is immediate from $\sigma_f \le \varrho$. The converse inequality follows by considering $f = \varrho_x$, $x \in X$. So, $\mathrm{Lip}_{1,\varrho}(X)$ determines ϱ. In fact, this set has some special features. Specifically, the set of 1-Lipschitz functions is closed under taking maxima, minima, adding constants and multiplication with -1. Also it is closed under pointwise limits and suprema.

Let X be a countable set and m a measure on X with full support. Any graph (b,c) over (X,m) gives rise to a special class of pseudometrics.

Definition 7.1 (Intrinsic metric). A pseudometric ϱ is called an *intrinsic metric* for a graph b over (X,m) if

$$\sum_{y\in X} b(x,y)\varrho(x,y)^2 \le m(x)$$

for all $x \in X$. We call a pseudo metric ϱ an *intrinsic metric* for a graph (b,c) over (X,m) if ϱ is intrinsic for b over (X,m).

We will now put this definition in perspective. To do so, we note that a differentiable functions f on the real line $\mathbb{R}$ satisfies

$$|f(s) - f(t)| \le |s-t| \text{ for all } s,t \in \mathbb{R}$$

if and only if

$$|f'(s)| \le 1 \text{ for all } s \in \mathbb{R}.$$

The basic idea is that similarly the 1-Lipschitz functions with respect to an intrinsic metric should be related to the functions whose derivative is bounded by 1. Of course, we first need a notion of derivative to make this precise. In this context, we define for $f \in C(X)$ the norm of the *gradient* by

$$|\nabla f| := |\nabla f|_{b,m} : X \longrightarrow [0,\infty]$$

$$|\nabla f|(x) := \left(\frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y))^2\right)^{1/2}$$

for $x \in X$ and

$$A_1(X) := \{f \in C(X) \mid |\nabla f|(x)^2 \le 1 \text{ for all } x \in X\}.$$

Then, for $c = 0$,

$$\mathcal{Q}(f) = \frac{1}{2}\sum_{x\in X} |\nabla f|(x)^2 m(x)$$

holds for all $f \in C(X)$, and any $f \in A_1(X)$ satisfies the inequality

$$\frac{1}{2}\sum_{x\in K} |\nabla f|^2(x)m(x) \leq m(K)$$

for any finite $K \subseteq X$.

We now characterize intrinsic metrics as follows.

LEMMA 7.2 (Characterization intrinsic metrics). *Let b be a graph over (X, m) and let ϱ be a pseudometric. Then, the following statements are equivalent:*

(i) *ϱ is an intrinsic metric.*
(ii) $\mathrm{Lip}_{1,\varrho}(X) \subseteq A_1(X)$.
(iii) *$|\nabla \varrho(o,\cdot)|^2 \leq 1$, i.e., $\varrho(o,\cdot) \in A_1(X)$ for all $o \in X$.*
(iii)' *$|\nabla \rho_U|^2 \leq 1$, i.e., $\varrho_U \in A_1(X)$ for all $U \subseteq X$.*

In particular, if $\eta \in C(X)$ is C-Lipschitz with respect to an intrinsic metric ϱ and $C \geq 0$, then

$$|\nabla \eta|^2 \leq C^2.$$

PROOF. (i) $\Longrightarrow$ (ii): Let $f \in \mathrm{Lip}_{1,\varrho}(X)$ be given, where ϱ is an intrinsic metric. Then, the Lipschitz property of f and the defining feature of an intrinsic metric give

$$\begin{aligned}|\nabla f|(x)^2 &= \frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y))^2 \leq \frac{1}{m(x)}\sum_{y\in X} b(x,y)\varrho(x,y)^2 \\ &\leq 1\end{aligned}$$

for all $x \in X$. Hence, f belongs to $A_1(X)$.

(ii) $\Longrightarrow$ (iii)': As discussed above, we have $\varrho_U \in \mathrm{Lip}_{1,\varrho}(X)$. Since $\mathrm{Lip}_{1,\varrho}(X) \subseteq A_1(X)$ by (ii), we conclude $\varrho_U \in A_1(X)$ and (iii)' follows.

(iii)' $\Longrightarrow$ (iii): This follows immediately (as we can take $U = \{o\}$ for any $o \in X$).

(iii) $\Longrightarrow$ (i): The assumption $|\nabla \varrho(o,\cdot)|^2 \leq 1$ for all $o \in X$ gives

$$\begin{aligned}1 \geq |\nabla \varrho(o,\cdot)|(x)^2 &= \frac{1}{m(o)}\sum_{y\in X} b(o,y)(\varrho(o,o)-\varrho(o,y))^2 \\ &= \frac{1}{m(o)}\sum_{y\in X} b(o,y)\varrho(o,y)^2 \leq 1.\end{aligned}$$

Hence,

$$\sum_{y\in X} b(o,y)\varrho(o,y)^2 \leq m(o)$$

holds for all $o \in X$. Thus, ϱ is an intrinsic metric.

The "in particular" statement follows immediately as C-Lipschitz means that

$$|\eta(x)-\eta(y)| \leq C\varrho(x,y)$$

for $x, y \in X$. $\square$

EXAMPLE 7.3. For a graph b over (X,m), ρ defined by

$$\rho(x,y) := \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=1}^{n} (\mathrm{Deg}(x_{i-1} \vee \mathrm{Deg}(x_i)))^{-1/2}$$

for $x,y \in X$ can be easily seen to be an intrinsic metric. We call it the *degree path metric*.

REMARK 7.4 (Why there is no equality between $\mathrm{Lip}_{1,\varrho}(X)$ and $A_1(X)$). For graphs there will, in general, not exist a pseudometric ϱ with $\mathrm{Lip}_{1,\varrho}(X) = A_1(X)$. This is different from our motivating example of differentiable functions on the real line. To understand this better, we note that the set $\mathrm{Lip}_{1,\varrho}(X)$ is always closed under taking maxima. However, the set $A_1(X)$ is in general not closed under taking maxima (as can already be seen by simple examples of graphs with three vertices).

REMARK 7.5 (What about the combinatorial distance?). Any graph (b,c) over (X,m) comes naturally with the *combinatorial distance* defined by

$$d_{\mathrm{comb}}(x,y) := \inf\{n \in \mathbb{N}_0 : \text{there exist path of length } n \text{ from } x \text{ to } y\}.$$

Here, $x_0,\ldots,x_n$ from X are called a *path of length* n if $x_0 = x$, $x_n = y$ and $b(x_j,x_{j+1}) > 0$ whenever $j \in \{0,\ldots,n-1\}$. Here, as usual, the infimum over the empty set is defined to be ∞. Clearly, $d_{\mathrm{comb}}(x,y) = 1$ whenever $b(x,y) > 0$ holds. Thus,

$$\frac{1}{m(x)} \sum_{y\in X} b(x,y) d_{\mathrm{comb}}(x,y)^2 = \frac{1}{m(x)} \sum_{y\in X} b(x,y).$$

Now, clearly both combinatorial distance and the condition of being an intrinsic metric only depend on b (and not on c). Thus, we restrict now attention to graphs b over (X,m), i.e. $c = 0$. In this case, the expression featured in the last equality is just the weighted degree Deg. Thus, we infer that d_{comb} is an intrinsic metric for the graph b over (X,m) if and only if $\mathrm{Deg} \leq 1$ holds. Now, we already know from Theorem 2.18 that boundedness of Deg is equivalent to boundedness of the associated Laplacian L. Hence, we can conclude that for a graph b over (X,m) the Laplacian is bounded if and only if a suitable multiple αd_{comb} is an intrinsic metric. For general graphs (b,c) still boundedness of the Laplacian implies that the combinatorial distance is an intrinsic metric (up to scaling). So, the advantage of intrinsic metrics over the combinatorial distance comes only about when situations with unbounded Laplacians are considered.

REMARK 7.6 (Why intrinsic metrics?). The inequality

$$\frac{1}{2} \sum_{x\in K} |\nabla f|(x)^2 m(x) \leq m(K)$$

shown above for $f \in A_1(X)$ and any finite $K \subseteq X$ gives a first indication that $A_1(X)$ is a useful set of functions. A pseudometric is then intrinsic if its 1-Lipschitz functions satisfy this inequality. In Sections 7.2 and 7.3 we will see two instances of how this can be used to deal with general graphs without making any boundedness assumption. A particular feature is that intrinsic metrics allow one to exhibit cut-off functions. We will come back to this point below. Here, we already note that whenever ϱ is an intrinsic metric, for any $A \subseteq X$ and $R > 0$ the function

$$\eta := (1 - \frac{1}{R}\varrho_A)_+$$

satisfies the following properties:

- $1_A \leq \eta \leq 1_{B_R(A)}$. Here, 1_S denotes the characteristic function of the set $S \subseteq X$ and $B_R(A)$ denotes the set of points in X with ϱ-distance not exceeding R to A.
- η is $1/R$-Lipschitz with respect to ϱ. In particular,

$$|\nabla \eta|^2 \leq \frac{1}{R^2}.$$

7.2. Cheeger Inequality

In this section we provide a lower bound on the infimum of the spectrum of the Laplacian on a graph in terms of the Cheeger constant. Roughly speaking the Cheeger constant measures the ratio between the size of the boundary of a set and the set itself.

We first give the definition of the Cheeger constant. The *boundary* of a set $W \subseteq X$ is given by those pairs $(x, y) \in X \times X$ that have one element in W and one element in its complement, i.e.

$$\partial W := (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

Moreover, analogously to the measures m on X, any $w\colon X \times X \longrightarrow [0, \infty)$ gives rise to a measure on $X \times X$ and therefore for $U \subseteq X \times X$ we write

$$w(U) := \sum_{(x,y)\in U} w(x, y) \in [0, \infty].$$

Definition 7.7 (Cheeger constant). Let b be a graph over (X, m) and let ϱ be an intrinsic metric. For a finite set $W \subseteq X$, we let the *area of the boundary* be given by

$$A_{b\varrho}(\partial W) := \frac{1}{2} \sum_{(x,y)\in\partial W} b(x, y)\varrho(x, y) = \frac{1}{2}(b\varrho)(\partial W).$$

We define the *Cheeger constant* $h := h_{b\varrho,m}$ (of the graph) by

$$h := \inf_{W\subseteq X, W \text{ finite}} \frac{A_{b\varrho}(\partial W)}{m(W)}.$$

We prove the following theorem.

Theorem 7.8 (Cheeger inequality). *Let b be a graph over (X, m) and $Q := Q^{(D)}_{b,0,m}$ the associated form and $L := L^{(D)}$ the induced Dirichlet Laplacian. Let ϱ be an intrinsic metric. Then,*

$$\lambda_0(L) \geq \frac{h^2}{2}.$$

The remaining part of this section is devoted to a proof of the theorem. We will first present two formulas which involve (strict) superlevel sets of functions. For a function $f \in C(X)$ and $t \in \mathbb{R}$, we define the *(strict) superlevel sets*

$$\Omega_t(f) := \{x \in X \mid f(x) > t\}.$$

The first formula relates the differences of a function to an integral over the boundary of the superlevel sets. We refer to this as a *co-area formula.*

LEMMA 7.9 (Co-area formula). *Let $w\colon X \times X \longrightarrow [0, \infty)$ and $f \in C(X)$ be given. Then,*

$$\sum_{x,y\in X} w(x, y)|f(x) - f(y)| = \int_{-\infty}^{\infty} w(\partial\Omega_t(f))\mathrm{d}t,$$

where both sides may take the value ∞.

PROOF. For vertices $x, y \in X$ with $x \neq y$ we define the interval $I_{x,y}$ by

$$I_{x,y} := [f(x) \wedge f(y), f(x) \vee f(y)),$$

and let $|I_{x,y}| = |f(x) - f(y)|$ be the length of $I_{x,y}$. Denote by $1_{x,y}$ the characteristic function of $I_{x,y}$. Then, for $t \in \mathbb{R}$ we have $(x, y) \in \partial\Omega_t(f)$ if and only if $t \in I_{x,y}$ holds. Therefore,

$$w(\partial\Omega_t(f)) = \sum_{x,y\in X} w(x, y)1_{x,y}(t).$$

From these considerations and the monotone convergence theorem we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} w(\partial\Omega_t(f))\mathrm{d}t &= \int_{-\infty}^{\infty} \sum_{x,y\in X} w(x, y)1_{x,y}(t)\mathrm{d}t = \sum_{x,y\in X} w(x, y) \int_{-\infty}^{\infty} 1_{x,y}(t)\mathrm{d}t \\ &= \sum_{x,y\in X} w(x, y)|f(x) - f(y)|.\end{aligned}$$

This proves the statement. □

For the next formula, we assume that the function f is positive and that there exists a measure on the space. The formula then relates the values of the function to the measure of the superlevel sets associated to the function.

LEMMA 7.10 (Area formula). *Let $m\colon X \longrightarrow [0, \infty)$ and $f\colon X \longrightarrow [0, \infty)$ be given. Then,*

$$\sum_{x\in X} f(x)m(x) = \int_0^{\infty} m(\Omega_t(f))\mathrm{d}t,$$

where both sides may take the value ∞.

PROOF. We have $x \in \Omega_t(f)$ if and only if $1_{(t,\infty)}(f(x)) = 1$. From this and the monotone convergence theorem we obtain

$$\begin{aligned}\int_0^{\infty} m(\Omega_t(f))\mathrm{d}t &= \int_0^{\infty} \sum_{x\in\Omega_t(f)} m(x)\mathrm{d}t = \int_0^{\infty} \sum_{x\in X} m(x)1_{(t,\infty)}(f(x))\mathrm{d}t \\ &= \sum_{x\in X} m(x) \int_0^{\infty} 1_{(t,\infty)}(f(x))\mathrm{d}t = \sum_{x\in X} m(x)f(x).\end{aligned}$$

This finishes the proof. □

After these preparations we now turn to the proof of Theorem 7.8.

Proof of Theorem 7.8. Let $\varphi \in C_c(X)$ be given and denote the superlevel sets of φ^2 by

$$\Omega_t := \Omega_t(\varphi^2) = \{x \in X : \varphi(x)^2 > t\}.$$

Then, Ω_t is finite for any $t \geq 0$ as φ has finite support. Hence, by the definition of the Cheeger constant h we infer

$$hm(\Omega_t) \leq A_{b\varrho}(\partial\Omega_t)$$

for all $t \geq 0$. From this and the area and co-area formulae provided in Lemmas 7.10 and 7.9 we find

$$\begin{aligned}
h\|\varphi\|^2 &= h\sum_{x\in X}\varphi(x)^2 m(x)\\
&\overset{\text{Area f.}}{=} h\int_0^\infty m(\Omega_t)\mathrm{d}t \leq \int_0^\infty A_{b\varrho}(\partial\Omega_t)\mathrm{d}t\\
&\overset{\text{Co-area f.}}{=} \frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)|\varphi(x)^2-\varphi(y)^2|\\
&= \frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)|\varphi(x)-\varphi(y)||\varphi(x)+\varphi(y)|\\
&= \frac{1}{2}\sum_{x,y\in X} b(x,y)|\varphi(x)-\varphi(y)|\varrho(x,y)|\varphi(x)+\varphi(y)|.
\end{aligned}$$

Now, the Cauchy–Schwarz inequality on the space $\ell^2(X \times X, b)$ gives

$$\left|\sum_{x,y\in X} b(x,y)F(x,y)G(x,y)\right|^2 \leq \left(\sum_{x,y\in X} b(x,y)F(x,y)^2\right)\left(\sum_{x,y\in X} b(x,y)F(x,y)^2\right)$$

for all $F, G \in \ell^2(X \times X, b)$. Hence, we infer

$$h\|\varphi\|^2 \leq Q(\varphi)^{1/2}\left(\frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)^2(\varphi(x)+\varphi(y))^2\right)^{1/2}.$$

Now, Young's inequality $(\alpha+\beta)^2 \leq 2\alpha^2 + 2\beta^2$ for $\alpha, \beta \in \mathbb{R}$, symmetry of both b and ϱ and the intrinsic metric property give

$$\begin{aligned}
&\left(\frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)^2(\varphi(x)+\varphi(y))^2\right)^{1/2} \\
&\leq \left(2\sum_{x,y\in X} b(x,y)\varrho(x,y)^2\varphi(x)^2\right)^{1/2} \\
&= \left(2\sum_{x\in X}\varphi(x)^2\left(\sum_{y\in X} b(x,y)\varrho(x,y)^2\right)\right)^{1/2} \\
&\overset{\varrho \text{ intrinsic}}{\leq} \sqrt{2}\left(\sum_{x\in X}\varphi(x)^2 m(x)\right)^{1/2} = \sqrt{2}\|\varphi\|.
\end{aligned}$$

Putting everything together we arrive at

$$h\|\varphi\|^2 \leq \sqrt{2}Q(\varphi)^{1/2}\|\varphi\|$$

and this gives

$$\frac{h}{\sqrt{2}} \leq \frac{Q(\varphi)^{1/2}}{\|\varphi\|}.$$

By squaring both sides, we find

$$\frac{h^2}{2} \leq \frac{Q(\varphi)}{\|\varphi\|^2}$$

for all $\varphi \in C_c(X)$ with $\varphi \neq 0$. This gives the statement by the variational characterization of $\lambda_0(L)$, see Theorem 6.3. $\square$

Remark 7.11. Whenever (b,c) is a graph over (X,m) we clearly have $Q^{(D)}_{b,c,m}(\varphi) \geq Q^{(D)}_{b,0,m}(\varphi)$ for all $\varphi \in C_c(X)$. Hence, we find for the Laplacian $L^{(D)}$ associated to $Q^{(D)}_{b,c,m}$ as well the estimate $\frac{h^2}{2} \leq \lambda_0(L^{(D)})$.

7.3. Brooks–Sturm Inequality

In this section we show a version of Brooks–Sturm inequality. This inequality gives an upper bound on the infimum of the spectrum in terms of exponential volume growth of balls.

Let b be a graph over (X,m) and ϱ an intrinsic metric. The distance balls $B_r(x)$ of radius $r > 0$ about a vertex $x \in X$ with respect to ϱ are given by

$$B_r(x) := \{y \in X \mid \varrho(x,y) \leq r\}.$$

If the distance balls are finite, we define the *exponential volume growth rate with variable center*, which we call μ, by

$$\mu = \liminf_{r\to\infty} \inf_{x\in X} \frac{1}{r}\log\frac{m(B_r(x))}{m(B_1(x))}.$$

So, for any $\varepsilon > 0$ there exists an $R > 0$ with

$$m(B_r(x)) \geq m(B_1(x))e^{r(\mu-\varepsilon)}$$

for all $r \geq R$ and all $x \in X$.

THEOREM 7.12 (Theorem of Brooks–Sturm). *Let b be a connected graph over (X, m) and let ϱ be an intrinsic metric such that the distance balls are finite and $L := L^{(D)}$ the associated Dirichlet Laplacian. Then,*

$$\lambda_0(L) \leq \frac{\mu^2}{8}.$$

Brooks original result considered the bottom of the essential spectrum by a similar exponential volume growth constant.

The basic idea is to use test functions as follows.

LEMMA 7.13 (Test function). *Let b be a graph over (X, m) and ϱ and intrinsic metric. Assume that $f, g \in C(X)$ satisfy*

$$(f(x) - f(y))^2 \leq C(g(x)^2 + g(y)^2)\varrho(x, y)^2$$

for all $x, y \in X$, for some $C \geq 0$. Then,

$$\mathcal{Q}(f) \leq C \sum_{x \in X} g(x)^2 m(x)$$

holds (where the value ∞ is possible).

PROOF. This follows by a direct computation, invoking the symmetry of b and ϱ as follows

$$\begin{aligned}
\mathcal{Q}(f) &= \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 \\
&\leq \frac{C}{2} \sum_{x,y \in X} b(x, y) \left(g(x)^2 + g(y)^2\right) \varrho(x, y)^2 \\
&= C \sum_{x \in X} g(x)^2 \left(\sum_{y \in X} b(x, y)\varrho(x, y)^2 \right) \\
&\overset{\varrho \text{ intrinsic}}{\leq} C \sum_{x \in X} g(x)^2 m(x).
\end{aligned}$$

This finishes the proof. □

The test functions will be constructed via intrinsic metrics and exponential functions. To construct such functions we use the following proposition.

PROPOSITION 7.14 (Estimates for differences of exponentials). *For any $u, v \in \mathbb{R}$ the estimate*

$$|e^u - e^v| \leq \frac{|u - v|}{2}(e^u + e^v)$$

holds.

PROOF. For $w \geq 0$ we find by direct computation

$$\sum_{n=2}^{\infty} \frac{2}{n!} w^{n-1} = \sum_{n=1}^{\infty} \frac{2}{(n+1)!} w^n \leq \sum_{n=1}^{\infty} \frac{1}{n!} w^n = e^w - 1.$$

This gives

$$e^w - 1 = \sum_{n=0}^{\infty} \frac{w^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{w^n}{n!} = \frac{w}{2}\left(2 + \sum_{n=2}^{\infty} \frac{2}{n!} w^{n-1}\right) \le \frac{w}{2}(1 + e^w).$$

Without loss of generality, we can now assume $u \le v$. The preceding estimate for $w := v - u \ge 0$ gives

$$e^{v-u} - 1 \le \frac{v-u}{2}(1 + e^{v-u}).$$

Multiplication with e^u then gives the desired statement. □

Let b be a graph over (X, m) and let ϱ be an intrinsic metric. Consider $\beta > 0$, $s \ge 0$ and $U \subseteq X$ non-empty, and define

$$h := h_{s,U,\beta} : X \longrightarrow (0, \infty), \quad h(x) := e^{\beta(s - \varrho_U(x))}.$$

By Proposition 7.14 the function h satisfies

$$\begin{aligned} |h(x) - h(y)| &\le \frac{|\beta(s - \varrho_U(x)) - \beta(s - \varrho_U(y))|}{2}(h(x) + h(y)) \\ &\le \frac{\beta\varrho(x,y)}{2}(h(x) + h(y)) \end{aligned}$$

for all $x, y \in X$, where we used ϱ_U is 1-Lipschitz. Taking squares and using $(a+b)^2 \le 2a^2 + 2b^2$ we obtain

$$|h(x) - h(y)|^2 \le \frac{\beta^2}{2}(h(x)^2 + h(y)^2)\varrho(x,y)^2$$

for all $x, y \in X$. The function h satisfies $h = e^{\beta s}$ on U and decays exponentially outside of U.

For our considerations we will need more than exponential decay viz vanishing outside of balls. Thus, we will construct modifications f and g from $h_{r,B_r(o),\beta}$, both of which can be seen as restrictions of h to balls $B_{2r}(o)$. Specifically, for the parameters $r \ge 0$, $o \in X$ and $\beta > 0$, we define the function $f := f_{r,o,\beta} \colon X \longrightarrow [0, \infty)$ by

$$f := (h_{r,B_r(o),\beta} - 1)_+,$$

i.e.

$$f(x) = (e^{\beta(r - \varrho_{B_r(o)}(x))} - 1)_+, \quad x \in X.$$

We observe the following basic properties of f, where $B_r := B_r(o)$,

- $f|_{B_r} = e^{\beta r} - 1$
- $f|_{B_{2r} \setminus B_r} = e^{\beta(r - \varrho(B_r(o), \cdot))} - 1$
- $f|_{X \setminus B_{2r}} = 0.$

Furthermore, for the parameters $r \ge 0$, $o \in X$ and $\beta > 0$, we define the auxiliary functions $g := g_{r,o,\beta} \colon X \longrightarrow [0, \infty)$ by

$$g := f + 2 \cdot 1_{B_{2r}(o)} = (h_{r,B_r(o),\beta} + 1)1_{B_{2r}(o)}.$$

We observe the following basic properties for g

- $g|_{B_r} = e^{\beta r}$
- $g|_{B_{2r} \setminus B_r} = e^{\beta(r - \varrho_{B_r(o)}(\cdot))} + 1$
- $g|_{X \setminus B_{2r}} = 0.$

Hence,

$$f \le h \le g$$

on $B_{2r}(o)$. The relevant features of f and g are provided in the next lemma.

LEMMA 7.15 (Properties of f and g). *Let b be a graph over (X, m) and let ϱ be an intrinsic metric. Let $o \in X$, $r \ge 0$ and $\beta > 0$.*

(a) *For $f := f_{r,o,\beta}$ and $g := g_{r,o,\beta}$ the inequality*

$$(f(x) - f(y))^2 \le \frac{\beta^2}{2}\left(g(x)^2 + g(y)^2\right)\varrho(x,y)^2,$$

holds for all $x, y \in X$.

(b) *Assume $\mu < \infty$ and choose $\beta > \frac{\mu}{2}$. Then, there exist sequences (o_k) in X and (r_k) in $[0,\infty)$ such that*

$$\lim_{k\to\infty} \frac{\|g_{r_k,o_k,\beta}\|}{\|f_{r_k,o_k,\beta}\|} = 1.$$

PROOF. (a) We distinguish three cases:

Case 1: $x, y \in B_{2r}(o)$. Then, $f = (h_{r,B_r(o),\beta} - 1)$ and we obtain from the calculation above

$$\begin{aligned} |f(x) - f(y)|^2 &= |h_{r,B_r(o),\beta}(x) - h_{r,B_r(o),\beta}(y)|^2 \\ &\le \frac{\beta^2}{2}|h_{r,B_r(o),\beta)}(x)^2 + h_{r,B_r(o),\beta)}(y)^2|\varrho(x,y)^2 \\ &\le \frac{\beta^2}{2}\left(g(x)^2 + g(y)^2\right)\varrho(x,y)^2, \end{aligned}$$

where we used $g = h_{r,B_r(o),\beta} + 1$ on $B_{2r}(o)$.

Case 2: $x, y \in X \setminus B_{2r}(o)$. Then, $f(x) = 0 = f(y)$ and the estimate clearly follows.

Case 3: $x \in B_{2r}(o)$ and $y \in X \setminus B_{2r}(o)$ or $y \in B_{2r}(o)$ and $x \in X \setminus B_{2r}(o)$. By symmetry it suffices to consider the case $x \in B_{2r}(o)$ and $y \in X \setminus B_{2r}(o)$. By $y \in X \setminus B_{2r}(o)$ we have $\varrho_{B_r(o)}(y) \ge r$. By $x \in B_{2r}(o)$ we have

$$t := r - \varrho_{B_r(o)}(x) \ge 0.$$

Altogether we find

$$t = r - \varrho_{B_r(o)}(x) \le \varrho_{B_r(o)}(y) - \varrho_{B_r(o)}(x) \le \varrho(x,y).$$

Using this we find from Proposition 7.14

$$\begin{aligned} |f(x) - f(y)| = |e^{\beta t} - 1| = |e^{\beta t} - e^{\beta\cdot 0}| &\le \frac{\beta}{2}(e^{\beta t} + 1)t \\ &\le \frac{\beta}{2}(g(x) + g(y))\rho(x,y). \end{aligned}$$

Now, the desired estimate follows after taking squares.

(b) Let $\beta > \frac{\mu}{2}$ and

$$0 < \varepsilon < \left(\beta - \frac{\mu}{2}\right) \wedge 1.$$

By definition of μ there exist a sequence (r_k) with $r_k \to \infty$ and a sequence (o_k) in X with

$$\frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))} \le e^{(2\mu+\varepsilon)r_k}$$

for $k \in \mathbb{N}$. With $f_k := f_{r_k,o_k,\beta}$ and $g_k := g_{r_k,o_k,\beta}$ for $k \in \mathbb{N}$ we have $g_k = (f_k + 2)1_{B_{2r_k}(o_k)}$ for all $k \in \mathbb{N}$, so we estimate using Cauchy–Schwarz and Young's inequality $(s+t)^2 \leq (1-\varepsilon)^{-1}s^2 + \varepsilon^{-1}t^2$,

$$\|g_k\|^2 \leq \left(\|f_k\| + 2\sqrt{m(B_{2r_k}(o_k))}\right)^2 \leq \frac{1}{1-\varepsilon}\|f_k\|^2 + \frac{4}{\varepsilon}m(B_{2r_k}(o_k)).$$

On the other hand we have

$$\|f_k\|^2 \geq m(B_{r_k}(o_k))(e^{\beta r_k} - 1)^2$$

for all $k \in \mathbb{N}$. Hence, for sufficiently large k, say $k \geq k_1$, we find

$$\|f_k\|^2 \geq \frac{1}{2}m(B_{r_k}(o_k))e^{2\beta r_k}.$$

Thus, for all $k \geq k_1$,

$$\frac{\|g_k\|^2}{\|f_k\|^2} \leq \frac{1}{1-\varepsilon} + \frac{8}{\varepsilon}e^{-2\beta r_k}\frac{m(B_{2r_k}(o_k))}{m(B_{r_k}(o_k))}.$$

Moreover, given ε as above, there exists $k_2 \geq k_1$ such that for all $k \geq k_2$,

$$\frac{m(B_{r_k}(o_k))}{m(B_1(o_k))} \geq \inf_{o\in X}\frac{m(B_{r_k}(o))}{m(B_1(o))} \geq e^{(\mu-\varepsilon)r_k}.$$

Therefore,

$$\frac{m(B_{2r_k}(o_k))}{m(B_{r_k}(o_k))} = \frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))}\frac{m(B_1(o_k))}{m(B_{r_k}(o_k))} \leq e^{(\mu+2\varepsilon)r_k}.$$

Since $0 < \varepsilon < \beta - \frac{\mu}{2}$, we can combine this with the estimate above to conclude

$$\frac{\|g_k\|^2}{\|f_k\|^2} \leq \frac{1}{1-\varepsilon} + \frac{8}{\varepsilon}e^{(\mu-2\beta+2\varepsilon)r_k} \to \frac{1}{1-\varepsilon}$$

as $k \to \infty$. Since ε was chosen arbitrarily and $0 \leq f_k \leq g_k$, statement (b) follows. □

After these preparations we can now provide the proof of the main result of this section.

Proof of Theorem 7.12. Chose $\beta > \frac{\mu}{2}$. Let the sequences (o_k) and (r_k) be taken from Lemma 7.15 (b) and set $f_k := f_{r_k,o_k,\beta}$ and $g_k := g_{r_k,o_k,\beta}$ for $k \in \mathbb{N}$. By the finiteness of balls we have $f_k, g_k \in C_c(X) \subseteq D(Q^{(D)})$ for all $k \in \mathbb{N}$. By Lemma 7.13 combined with Lemma 7.15 (a) we find

$$Q^{(D)}(f_k) \leq \frac{\beta^2}{2}\|g_k\|^2$$

for all $k \in \mathbb{N}$. By the variational characterization of $\lambda_0(L)$, Theorem 6.3, we then get for any $k \in \mathbb{N}$ the estimate

$$\lambda_0(L) = \inf_{f\in C_c(X)}\frac{Q^{(D)}(f)}{\|f\|^2} \leq \inf_k \frac{Q^{(D)}(f_k)}{\|f_k\|^2} \leq \inf_k \frac{\beta^2}{2}\frac{\|g_k\|^2}{\|f_k\|^2} = \frac{\beta^2}{2},$$

where we used Lemma 7.15 (b) in the last equality. As this hold for any $\beta > \frac{\mu}{2}$ we arrive at the desired statement. □

We finish this section by discussing a class of examples. A connected graph (b, c) over (X, m) is said to have *subexponential growth* with respect to the intrinsic metric ϱ if we find $o \in X$ such that for all $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ with

$$m(B_r(o)) \leq C_\varepsilon e^{r\varepsilon}$$

for all $r \geq 1$. It is not hard to see that this property does not depend on o. Specifically, if this property holds for o then for any other $o' \in X$ we clearly have $B_r(o') \subseteq B_{r+\varrho(o,o')}(o)$ for any $r > 0$ and this implies

$$m(B_r(o')) \leq m(B_{r+\varrho(o,o')}(o)) \leq C_\varepsilon e^{\varepsilon\varrho(o,o')} e^{\varepsilon r}$$

for any $r \geq 1$. For connected graphs with subexponential growth we clearly have $\mu = 0$. Thus, we arrive at the following corollary.

COROLLARY 7.16 (Subexponential growth implies $\lambda_0(L) = 0$). *Let (b, c) be a connected graph over (X, m) with subexponential growth with respect to the intrinsic metric ϱ and $L := L^{(D)}$ the Dirichlet Laplacian. Then, $\lambda_0(L) = 0$.*

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Sheet 9

Intrinsic metrics and Cheeger inequality

Exercise 1 (Recovering the combinatorial metric for trees) 4 points

Let b be a tree with standard weights over (X,m) and let m be the counting measure. Let

$$\sigma(x,y) = \sup\{f(x) - f(y) \mid f \in A_1(X)\},$$

where $A_1(X) = \{f \in C(X) \mid |\nabla f|^2 \leq 1\}$ and let d_{comb} denote the combinatorial graph distance. Show that

$$\sigma = d_{\mathrm{comb}}/\sqrt{2}.$$

Moral: It might seem that σ is a natural candidate for the definition of an intrinsic metric (at least from the perspective of strongly local Dirichlet forms). The exercise shows that one only recovers the combinatorial metric (at least in the case of trees).

Exercise 2 (Finite boundary area) 4 points

Let b be a graph over (X,m) and let ϱ be an intrinsic metric. Show that for all finite sets the area of the boundary is finite. More specifically, if W is a finite subset of X, then

$$A_{b\varrho}(\partial W) \leq (mn)(W)^{1/2},$$

where n denotes the normalizing measure $n(x) = \sum_{y \in X} b(x,y)$.

Exercise 3 (Upper bound via h) 4 points

Let b be a graph over (X,m). Let ϱ be an intrinsic metric such that $\varrho(x,y) \geq C > 0$ for all $x \sim y$. Show that for the Cheeger constant h we have

$$\lambda_0(L) \leq \frac{h}{C}.$$

Exercise 4 (Example) 4 points

(a) Give an example of a graph b over (X,m) with positive Cheeger constant with $m = 1$ and bounded Deg.

(b) Let $m'(X) < \infty$, show that $Q^{(D)}_{b,m'} \neq Q^{(N)}_{b,m'}$.