26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 06

CHAPTER 4

Infinite Graphs II – Graphs and Regular Dirichlet Forms

In this chapter we are going to combine our setup of infinite graphs from Chapter 2 and the general theory of operators and forms and resolvents and semigroups from Chapter 3 in order to set up a Hilbert space theory of operators, forms, resolvents and semigroups associated to graphs. This theory will run in parallel to what has been developed in Chapter 1 for finite graphs.

The relevant Hilbert space will be $\ell^2(X, m)$, where X is a countable set and m is a measure on X with full support. Recall that a graph (b, c) over (X, m) consists of a symmetric map $b \colon X \times X \longrightarrow [0, \infty)$ with zero diagonal satisfying

$$\sum_{y \in X} b(x, y) < \infty, \qquad x \in X,$$

and a non-negative function $c \colon X \longrightarrow [0, \infty)$. Here m is a strictly positive function $X \longrightarrow (0, \infty)$ which extends to a measure via $m(A) := \sum_{x \in A} m(x)$, $A \subseteq X$.

4.1. Forms on Hilbert Spaces

In this section we introduce two closed forms arising from a graph.

Let a graph (b, c) over (X, m) be given. We have already met the space of functions of finite energy

$$\mathcal{D} := \{f \in C(X) \mid \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f^2(x) < \infty\}$$

and the energy form $\mathcal{Q} := \mathcal{Q}_{b,c} \colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{Q}(f, g) := \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x) f(x) g(x)$$

and extended on the diagonal to functions not in $\mathcal{D}$ by ∞. We will also need the norm $\| \cdot \|_{\mathcal{Q}} \colon \mathcal{D} \cap \ell^2(X, m) \longrightarrow [0, \infty)$ given by

$$\|f\|_{\mathcal{Q}} := \left(\mathcal{Q}(f) + \|f\|^2\right)^{1/2},$$

where $\|f\|$ is the $\ell^2(X, m)$ norm of f.

We define the form $Q^{(N)} := Q^{(N)}_{b,c,m}$ as the — so to speak — maximal restriction of $\mathcal{Q}$ to $\ell^2(X, m)$. Specifically, we define

$$D(Q^{(N)}) := \mathcal{D} \cap \ell^2(X, m)$$

and

$$Q^{(N)}(f,g) := \mathcal{Q}(f,g)$$

for $f, g \in D(Q^{(N)})$. Then, clearly $Q^{(N)}$ is symmetric and positive as $\mathcal{Q}$ has these properties. As above, we set

$$Q^{(N)}(f) := Q^{(N)}(f,f)$$

and extend $Q^{(N)}$ to all of $\ell^2(X,m)$ by setting it to be ∞ outside of $\mathcal{D} \cap \ell^2(X,m)$. We think of $Q^{(N)}$ as arising from some sort of Neumann boundary conditions and this is the reason for the superscript (N). We will refer to $Q^{(N)}$ as the *Neumann form*.

If a sequence (f_n) from $\ell^2(X,m)$ converges to f in $\ell^2(X,m)$, then it clearly converges pointwise and from Proposition 2.4 we obtain

$$Q^{(N)}(f) = \mathcal{Q}(f) \leq \liminf_{n} \mathcal{Q}(f_n) = \liminf_{n\to\infty} Q^{(N)}(f_n).$$

Thus, $Q^{(N)}$ is a lower semi-continuous map on a subspace of $\ell^2(X,m)$. By standard theory, see Theorem 3.33, $Q^{(N)}$ is then closed, i.e., $D(Q^{(N)})$ is complete with respect to $\|\cdot\|_{\mathcal{Q}}$.

In some sense, $Q^{(N)}$ is the "maximal" form associated to a graph. We will be even more concerned with the "minimal" form. This form comes about by considering all symmetric closed forms which are restrictions of $Q^{(N)}$ (or $\mathcal{Q}$) and whose domain contains $C_c(X)$. The intersection over the domains of all such forms will be a closed subspace of $D(Q^{(N)})$. Hence, the restriction of $\mathcal{Q}$ to this domain will yield a positive closed form. We denote this form by $Q^{(D)} := Q^{(D)}_{b,c,m}$ and its domain by $D(Q^{(D)}) := D(Q^{(D)}_{b,c,m})$.

By construction $Q^{(D)}$ is the smallest closed form extending the restriction of $\mathcal{Q}$ to $C_c(X) \times C_c(X)$. Thus, we can also obtain $D(Q^{(D)}_{b,c,m})$ by taking the closure of $C_c(X)$ with respect to the norm $\|\cdot\|_{\mathcal{Q}}$, that is,

$$D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}.$$

We think of $Q^{(D)}$ as arising from some sort of Dirichlet boundary conditions and this is the reason for the superscript (D).

We furthermore provide a structural characterization of the domain of $Q^{(D)}$. This structural characterization will involve the space $\mathcal{D}_0$. This space comes about as some form of closure of $C_c(X)$ in $\mathcal{D}$ (see Definition 2.7 and its corollary). More specifically, it is the subspace of $f \in \mathcal{D}$ for which there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$.

Theorem 4.1 (Domain of $D(Q^{(D)})$). *Let (b,c) be a graph over (X,m) with associated energy form $\mathcal{Q}_{b,c}$. Then, $Q^{(D)}$ is the restriction of $\mathcal{Q}_{b,c}$ to*

$$D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X,m).$$

Remark 4.2. To put this result into perspective, we compare it with the corresponding statement for the Neumann form $Q^{(N)}$. By definition, $Q^{(N)}$ arises as the restriction of $\mathcal{Q}_{b,c}$ to

$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m).$$

So, we see that the difference between the Dirichlet and Neumann boundary conditions comes from a corresponding difference between $\mathcal{D}$ and $\mathcal{D}_0$.

Proof. To show

$$D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X,m)$$

we will prove two inclusions.

$D(Q^{(D)}) \subseteq \mathcal{D}_0 \cap \ell^2(X,m)$: By definition, $D(Q^{(D)})$ is the closure of $C_c(X)$ with respect to $\|\cdot\|_{\mathcal{Q}}$. This immediately gives the statement as $\ell^2(X,m)$ convergence implies pointwise convergence.

$\mathcal{D}_0 \cap \ell^2(X,m) \subseteq D(Q^{(D)})$: Let $f \in \mathcal{D}_0 \cap \ell^2(X,m)$. As $Q^{(D)}$ is a closed form, the restriction of $Q^{(D)}$ to the diagonal is lower semi-continuous, it suffices to find a sequence (χ_n) in $C_c(X)$ with $\chi_n \to f$ in $\ell^2(X,m)$ as $n \to \infty$ and $(Q^{(D)}(\chi_n))$ bounded.

Since $f \in \mathcal{D}_0$ we can find a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f-\varphi_n) \to 0$ as $n \to \infty$. This implies, in particular, that the sequence $(Q^{(D)}(\varphi_n)) = (\mathcal{Q}(\varphi_n))$ is bounded. We will modify the sequence (φ_n) in order to obtain a sequence (χ_n) converging to f in $\ell^2(X,m)$. Consider

$$\psi_n := \varphi_n \wedge |f|, \quad n \in \mathbb{N}.$$

Claim. We have:

- $\psi_n \in C_c(X)$ for all $n \in \mathbb{N}$.
- $\psi_n \to f$ pointwise as $n \to \infty$.
- The sequence $(Q^{(D)}(\psi_n))_n$ is bounded.

Proof of the claim. The first two statements are straightforward. The last statement follows from

$$\begin{aligned} |\psi_n(x) - \psi_n(y)| &\le |\varphi_n(x) - \varphi_n(y)| + |\,|f(x)| - |f(y)|\,| \\ &\le |\varphi_n(x) - \varphi_n(y)| + |f(x) - f(y)|. \end{aligned}$$

Consider now $\chi_n := \psi_n \vee -|f|$ for $n \in \mathbb{N}$. Then, we clearly have

$$\chi_n = -(-\psi_n \wedge |f|), \quad n \in \mathbb{N}.$$

Thus, we can apply the reasoning of the previous claim to obtain:

- $\chi_n \in C_c(X)$ for all $n \in \mathbb{N}$.
- $\chi_n \to f$ pointwise as $n \to \infty$.
- The sequence $(Q^{(D)}(\chi_n))_n$ is bounded.

Moreover, by construction the sequence (χ_n) satisfies

$$-|f| \le \chi_n \le |f|$$

for $n \in \mathbb{N}$. Thus, by Lebesgue's dominated convergence theorem, the sequence (χ_n) converges to f in $\ell^2(X,m)$. Hence, the sequence (χ_n) has all of the desired properties. This finishes the proof. □

4.2. Graphs and (Regular) Dirichlet Forms

Let X be a countable set and let m be a measure on X with full support. We will be concerned with forms on $\ell^2(X,m)$. Indeed, we have already seen that any graph (b,c) over (X,m) gives rise to two forms. For short we will refer to forms on $\ell^2(X,m)$ also as forms over (X,m). Throughout we will freely use the theory and notation developed in Section 3.6.

Let $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a normal contraction, i.e., a map with $C(0) = 0$ and $|C(s) - C(t)| \leq |s - t|$ for $s,t \in \mathbb{R}$. If a closed form Q over (X,m) has the property that $C \circ f$ belongs to $D(Q)$ with

$$Q(C \circ f) \leq Q(f)$$

for all $f \in D(Q)$ it is said to be *compatible* with the normal contraction C.

Definition 4.3. A closed form Q on $\ell^2(X,m)$ is called a *Dirichlet form* if it is compatible with all normal contractions.

For a graph (b,c) over (X,m), we show next that $Q^{(N)} = Q^{(N)}_{b,c,m}$ is a Dirichlet form. This form was introduced in Section 4.1 as the restriction of $\mathcal{Q} = \mathcal{Q}_{b,c}$ to $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$.

Proposition 4.4 ($Q^{(N)}$ is a Dirichlet form). *Let (b,c) be graph over (X,m). Then, $Q^{(N)}_{b,c,m}$ is a Dirichlet form.*

Proof. As $Q^{(N)}$ is a restriction of $\mathcal{Q}$, it is lower semi-continuous by Proposition 2.4. By Theorem 3.33 this implies that $Q^{(N)}$ is closed. Clearly, for all normal contractions C and $f \in \ell^2(X,m)$, it follows that $C \circ f \in \ell^2(X,m)$. Furthermore, for $f \in D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$, we find from the compatibility of $\mathcal{Q}_{b,c}$ with normal contractions

$$Q^{(N)}(C \circ f) = \mathcal{Q}_{b,c}(C \circ f) \leq \mathcal{Q}_{b,c}(f) = Q^{(N)}(f).$$

Thus, $Q^{(N)}$ is closed and compatible with normal contractions. Therefore, $Q^{(N)}$ is a Dirichlet form. □

Let $\|\cdot\|_\infty$ denote the supremum norm on $C_c(X)$. A Dirichlet form Q over (X,m) is called *regular* if $D(Q) \cap C_c(X)$ is dense in both $C_c(X)$ with respect to $\|\cdot\|_\infty$ and in $D(Q)$ with respect to the form norm $\|\cdot\|_Q$.

It turns out that a Dirichlet form Q on (X,m) is regular if and only if Q is the closure of the restriction of Q to the subspace $C_c(X)$. The "if" direction is immediate from the definition of a regular Dirichlet form. The "only if" direction is shown next.

Lemma 4.5. *Let Q be a regular Dirichlet form over (X,m). Then, $C_c(X)$ is contained in $D(Q)$. In particular, Q is the closure of the restriction of Q to $C_c(X) \times C_c(X)$.*

Proof. Let $x \in X$ be arbitrary and let $\varphi := 2 \cdot 1_x$ so that $\varphi \in C_c(X)$. We will show that $\varphi \in D(Q)$. As x is chosen arbitrarily, this will imply the first statement.

As Q is regular, $C_c(X) \cap D(Q)$ is dense in $C_c(X)$ with respect to the supremum norm, so there exists a $\psi \in D(Q)$ with $1 < \psi(x) < 3$ and $|\psi(y)| < 1$ for all $y \neq x$, i.e.,

$$\|\varphi - \psi\|_\infty < 1.$$

As Q is a Dirichlet form, $D(Q)$ is invariant under taking the modulus and we can assume $\psi \geq 0$. Furthermore, as taking the minimum with 1 is also a normal contraction, $\psi \wedge 1 \in D(Q)$. As $D(Q)$ is a vector space it contains $\psi - \psi \wedge 1$ and this is a nonzero multiple of φ by construction. Thus $\varphi \in D(Q)$ and as $x \in X$ is arbitrary, the first statement follows.

As Q was assumed to be regular, the space $C_c(X) = C_c(X) \cap D(Q)$ is dense in $D(Q)$ with respect to the form norm and the "in particular" statement follows. □

Next, we will show that the domain of $Q^{(D)} = Q^{(D)}_{b,c,m}$ is preserved by normal contractions.

Lemma 4.6 ($Q^{(D)}$ is a regular Dirichlet form). *Let (b, c) be a graph over (X, m). Then, $Q^{(D)}$ is a regular Dirichlet form.*

Proof. We first show that $Q^{(D)} = Q^{(D)}_{b,c,m}$ is a Dirichlet form. We denote the restriction of $Q^{(D)}$ to $C_c(X) \times C_c(X)$ by $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$. Note that $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$ is a restriction of $\mathcal{Q}_{b,c}$ by the very definition of $Q^{(D)}$. Note also that $C \circ \varphi$ belongs to $C_c(X)$ whenever φ belongs to $C_c(X)$ and C is a normal contraction. From the compatibility of $\mathcal{Q}_{b,c}$ with normal contractions we then infer for a normal contraction C that

$$\begin{aligned} Q^{(D)}(C \circ \varphi) &= \mathcal{Q}^{(\mathrm{comp})}_{b,c}(C \circ \varphi) = \mathcal{Q}_{b,c}(C \circ \varphi) \\ &\leq \mathcal{Q}_{b,c}(\varphi) = \mathcal{Q}^{(\mathrm{comp})}_{b,c}(\varphi) = Q^{(D)}(\varphi) \end{aligned}$$

for all $\varphi \in C_c(X)$.

We will next extend this inequality to the whole $D(Q^{(D)})$. In particular, we will show that $C \circ f \in D(Q^{(D)})$ for $f \in D(Q^{(D)})$.

As $Q^{(D)}$ is the closure of its restriction $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$ to $C_c(X) \times C_c(X)$, there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ with respect to $\|\cdot\|_Q$. In particular, $\varphi_n \to f$ in $\ell^2(X, m)$. Then, clearly, the sequence $(C \circ \varphi_n)$ belongs to $C_c(X)$ and converges to $C \circ f$ in $\ell^2(X, m)$. Moreover, the sequence $(Q^{(D)}(C \circ \varphi_n))$ is bounded as

$$Q^{(D)}(C \circ \varphi_n) = \mathcal{Q}^{(\mathrm{comp})}_{b,c}(C \circ \varphi_n) \leq \mathcal{Q}^{(\mathrm{comp})}_{b,c}(\varphi_n) = Q^{(D)}(\varphi_n) \to Q^{(D)}(f)$$

as $n \to \infty$. From Proposition 3.35, we then infer $C \circ f \in D(Q^{(D)})$ and

$$Q^{(D)}(C \circ f) \leq Q^{(D)}(f).$$

Therefore, $Q^{(D)}$ is a Dirichlet form.

By construction, $Q^{(D)}$ is the closure of $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$. Hence, $Q^{(D)}$ is regular. This finishes the proof. □

It turns out that the converse to the previous lemma holds as well.

LEMMA 4.7 (Regular Dirichlet forms arise from graphs). *Let Q be a regular Dirichlet form over (X,m). Then, there exists a graph (b,c) over (X,m) with $Q = Q^{(D)}_{b,c,m}$.*

PROOF. By Lemma 4.5, $C_c(X)$ is contained in $D(Q)$. Define $b\colon X \times X \longrightarrow \mathbb{R}$ by

$$b(x,y) := -Q(1_x, 1_y)$$

for $x \neq y$ and $b(x,x) := 0$ for $x \in X$, and define $c\colon X \longrightarrow \mathbb{R}$ by

$$c(x) := Q(1_x) - \sum_{y\in X} b(x,y).$$

We will show that (b,c) is a graph with $Q^{(D)}_{b,c,m} = Q$. This will also show that the sum appearing in the definition of c is absolutely convergent.

For a finite subset K of X we let

$$i_K\colon C(K) \longrightarrow C_c(X)$$

be the canonical inclusion (i.e. $i_K(f)(x) = f(x)$ for $x \in K$ and $i_K(f)(x) = 0$ for $x \notin K$). Note that — due to the finiteness of K — the map i_K maps indeed into $C_c(X)$. Due to regularity $C_c(X)$ belongs to the domain of Q (see Lemma 4.5). Thus, Q induces a form Q_K on $C(K)$ defined by

$$Q_K(f,g) := Q(i_K(f), i_K(g)).$$

Clearly, $C \circ i_K(f) = i_K(C \circ f)$ holds for all normal contractions C and all $f \in C(K)$. As Q is a Dirichlet form we therefore find

$$Q_K(Cf) = Q(i_K(Cf)) = Q(Ci_K(f)) \leq Q(i_K(f)) = Q_K(f).$$

Hence, Q_K is a Dirichlet form. By the results of the first chapter we then find

(a) For any $x, y \in X$ with $x \neq y$, we have

$$b(x,y) = -Q(1_x, 1_y) = -Q_{\{x,y\}}(1_x, 1_y) \leq 0.$$

(b) For any finite $K \subseteq X$ and $x \in K$, we have

$$Q(1_K, 1_x) = Q_K(1, 1_x) \geq 0.$$

From (a) the function b is positive. Moreover, for any $K \subseteq X$ finite and any $x \in K$ we compute

$$\begin{aligned} Q(1_x) &= Q(1_K, 1_x) - \sum_{y\in K, y\neq x} Q(1_y, 1_x) = Q(1_K, 1_x) + \sum_{y \in K, y\neq x} b(x,y) \\ &= Q(1_K, 1_x) + \sum_{y\in K} b(x,y). \end{aligned}$$

As, by (a) and (b) both $Q(1_K, 1_x)$ and b are positive, we can now conclude

$$\sum_{y\in K} b(x,y) \leq Q(1_x)$$

for any $K \subseteq X$ finite and this gives

$$\sum_{y\in X} b(x,y) \leq Q(1_x) < \infty.$$

From this we infer

$$Q(1_x) - \sum_{y \in X} b(x,y) \geq 0$$

for all $x \in X$. Thus, c defined at the beginning of the proof exists and is non-negative. Hence, (b,c) is indeed a graph.

Moreover, from the very definitions of b and c we conclude for $x, y \in X$ with $x \neq y$

$$Q(1_x, 1_y) = -b(x,y) = Q^{(D)}_{b,c,m}(1_x, 1_y)$$

and for $x \in X$

$$Q(1_x) = c(x) + \sum_{y \in X} b(x,y) = Q^{(D)}_{b,c,m}(1_x).$$

By bilinearity, Q and $Q^{(D)}_{b,c,m}$ agree on $C_c(X)$. As both are regular Dirichlet forms, they must then be equal. □

Combining the preceding lemmas we infer that regular Dirichlet forms on discrete sets are exactly the forms arising from graphs with Dirichlet boundary conditions. Specifically, the following holds.

THEOREM 4.8 (Regular Dirichlet forms and graphs). *The map*

$$(b,c) \mapsto Q^{(D)}_{b,c,m}$$

is a bijective correspondence between graphs (b,c) over (X,m) and regular Dirichlet forms over (X,m).

PROOF. This is a direct consequence of Lemmas 4.6 and 4.7. In particular, injectivity of the map follows directly from the first lines of the proof of Lemma 4.7. □

4.3. Laplacians on Hilbert Spaces

In this section we investigate the Laplacian associated to $Q^{(D)}_{b,c,m}$.

Let (b,c) be a graph over (X,m) and $Q^{(D)} := Q^{(D)}_{b,c,m}$ the associated regular form. By the theory of closed forms, see Lemma 3.30 and Corollary 3.37, there exists a unique self-adjoint operator $L^{(D)} := L^{(D)}_{b,c,m}$ on $\ell^2(X,m)$ whose domain $D(L^{(D)})$ is contained in $D(Q^{(D)})$ and which satisfies

$$\langle g, L^{(D)} f \rangle = Q^{(D)}(g,f)$$

for all $f \in D(L^{(D)})$ and $g \in D(Q^{(D)})$. We call $L^{(D)}$ the *Dirichlet Laplacian* or just *the Laplacian* associated to a graph. We denote the *spectrum* of $L^{(D)}$ by $\sigma(L^{(D)})$ and the bottom of the spectrum of $L^{(D)}$ by $\lambda_0(L^{(D)}) := \inf \sigma(L^{(D)})$. We note that $L^{(D)}$ is positive and thus $\sigma(L^{(D)}) \subseteq [0,\infty)$ and $\lambda_0(L^{(D)}) \geq 0$.

In general, it is rather hard to describe explicitly the domain of $L^{(D)}$. Still, the action of this operator is easy to describe.

To do so, we recall the definition of the formal operator $\mathcal{L} := \mathcal{L}_{b,c,m}$ associated to a graph (b,c) over the measure space (X,m). This operator has domain $\mathcal{F}$ and acts via

$$\mathcal{L}f(x) = \frac{1}{m(x)}\mathcal{L}_{b,c}f(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(f(x)-f(y)) + c(x)f(x)\right).$$

This operator is intimately linked to the form $\mathcal{Q} := \mathcal{Q}_{b,c}$. Indeed, Green's formula in Proposition 2.9 gives that any $f \in \mathcal{D}$ satisfies $f \in \mathcal{F}$ and

$$\mathcal{Q}(\varphi, f) = \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x)$$

holds. This is the essential step in the proof of the following result.

Theorem 4.9 (Action of the Dirichlet Laplacian). *Let (b,c) be a graph over (X,m) and let $L^{(D)}$ be the Dirichlet Laplacian. Then,*

$$L^{(D)}f(x) = \mathcal{L}f(x)$$

for all $f \in D(L^{(D)})$ and $x \in X$.

Proof. By definition, $L^{(D)}$ is the unique self-adjoint operator with $D(L^{(D)}) \subseteq D(Q^{(D)})$ which satisfies $\langle g, L^{(D)}f\rangle = Q^{(D)}(g,f)$ for all $f \in D(L^{(D)})$ and $g \in D(Q^{(D)})$. Furthermore, as $Q^{(D)}$ is a restriction of $\mathcal{Q}$ and $C_c(X) \subseteq D(Q^{(D)}) \subseteq \mathcal{D} \subseteq \mathcal{F}$, from Green's formula, Proposition 2.9, we have

$$\langle \varphi, L^{(D)}f\rangle = Q^{(D)}(\varphi, f) = \mathcal{Q}(\varphi, f) = \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x)$$

for all $\varphi \in C_c(X)$ and $f \in D(Q^{(D)})$. The conclusion follows by choosing $\varphi := 1_x/m$ for arbitrary x. □

4.4. Semigroups and Resolvents

In Section 3.5 we have discussed how positive operators come with a semigroup and a resolvent and how semigroups and resolvents give rise to solutions of the heat equation and the Poisson equation, respectively. In this section we apply this to the graph Laplacians.

Consider a graph (b,c) over (X,m) and let $\mathcal{L}$ be the associated formal Laplacian. A *solution of the heat equation* (for $\mathcal{L}$)) is a function $u\colon [0,\infty) \times X \longrightarrow \mathbb{R}$, $(t,x) \longmapsto u_t(x)$ such that $t \mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ for all x and satisfies $u_t \in \mathcal{F}$ for all $t > 0$ such that

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

holds for all $x \in X$ and $t > 0$. We note that if $f \in \ell^2(X,m)$, then the function u given by

$$u_t(x) := e^{-tL^{(D)}}f(x), \quad t \geq 0, x \in X$$

is a solution of the heat equation with initial condition f, see Theorem 3.24. Similarly, for $f \in \ell^2(X,m)$ and $\alpha > 0$ the Poisson equation (associated to $\mathcal{L} + \alpha$)

$$(\mathcal{L} + \alpha)u = f$$

has the solution

$$u := (L^{(D)} + \alpha)^{-1} f.$$

By Theorem 3.26 the Laplace transform gives

$$(L^{(D)} + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL^{(D)}} \mathrm{d}t.$$

and the exponential formula gives

$$e^{-tL^{(D)}} = \lim_{n\to\infty} \left(\frac{n}{t} \left(L^{(D)} + \frac{n}{t} \right)^{-1} \right)^n .$$

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Sheet 6

Infinite Graphs II

Exercise 1 (Densitiy of C_c) 4 points

Let (X,m) be an infinite discrete measure space and $p \in [1,\infty]$. Show that $C_c(X)$ is dense in $\ell^p(X,m)$ if and only if $p \in [1,\infty)$.

Exercise 2 (Inclusion of ℓ^p spaces) 4 points

Let (X, m) be a discrete measure space.

a) Show the equivalence of the following statements:
   (i) $\ell^1(X, m) \subseteq \ell^\infty(X)$,
   (ii) $\ell^1(X, m) \subseteq C_0(X) := \overline{C_c(X)}^{\|\cdot\|_{\ell^\infty}}$,
   (iii) There exists $\alpha > 0$ such that $m \geq \alpha$.

b) Show the equivalence of the following statements:
   (i) $\ell^1(X, m) \supseteq \ell^\infty(X, m)$,
   (ii) $m(X) < \infty$.

Exercise 3 (Boundedness) 4 points

Let (b,c) be a graph over (X,m). Show that $\mathcal{L}$ is bounded on $\ell^2(X,m)$ if and only if $\mathcal{L}$ is bounded on $\ell^p(X,m)$ for some $p \in [1,\infty]$.

Exercise 4 (Forms between $Q^{(D)}$ and $Q^{(N)}$) 4 points

Let (b,c) be a graph over (X,m) and $U \subseteq X$ and

$$\begin{aligned} D(Q^{(U)}) &= \overline{\{u \in D(Q^N) \mid U \cap \operatorname{supp} u \text{ is finite}\}}^{\|\cdot\|_{Q^{(N)}}} \\ Q^{(U)}(f,g) &= Q^{(N)}(f,g) \end{aligned}$$

(a) $Q^{(U)}$ is a Dirichlet form.

(b) $Q^{(D)} \subseteq Q^{(U)} \subseteq Q^{(N)}$, where $Q_1 \subseteq Q_2$ if $D(Q_1) \subseteq D(Q_2)$ and $Q_1 = Q_2$ on $D(Q_1)$. Furthermore, $Q^{(X \setminus F)} = Q^{(D)}$ and $Q^{(F)} = Q^{(N)}$ for finite $F \subseteq X$.

Bonus Exercise 1 (Inclusion of ℓ^p spaces) 1 point

Show that the equivalences of Exercise 2 are still true if $\ell^1(X, m)$ is replaced by $\ell^p(X, m)$ with $p \in (1, \infty)$.