26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 03

CHAPTER 2

Infinite Graphs I – The Formal Objects

In this chapter we start the investigation of the general situation. Thus, we do not assume that the underlying set X is finite. We rather deal with (countably) infinite sets X. We will introduce the core objects and present some general theory.

Throughout we let X be a countable set. We think of X as being equipped with the discrete topology. We denote by $C(X)$ the set of all functions on X (which are automatically continuous) and by $C_c(X)$ the space of compactly, i.e., finitely, supported functions. For $x \in X$ we let 1_x denote the characteristic function of $\{x\}$. So, $1_x(y) = 1$ for $y = x$ and $1_x(y) = 0$ otherwise. Observe that the characteristic functions 1_x, $x \in X$, form a basis of $C_c(X)$. A function $m\colon X \longrightarrow (0, \infty)$ gives rise to a measure on (the σ-algebra consisting of all subsets of) X by

$$m(A) := \sum_{x \in A} m(x).$$

We will not distinguish between the measure and the function m in notation. Note that the measure has full support, i.e. $m(A) > 0$ holds for all $A \neq \emptyset$. The measure m gives naturally rise to the Hilbert space $\ell^2(X, m)$. By definition, the underlying vector space is given as

$$\ell^2(X, m) := \{f\colon X \longrightarrow \mathbb{R} \mid \sum_{x \in X} |f(x)|^2 m(x) < \infty\}.$$

The inner product is given by

$$\langle f, g\rangle := \sum_{x \in X} f(x)g(x)m(x).$$

Note that the sum in question is finite for $f, g \in \ell^2(X, m)$ since we have $|f(x)g(x)| \leq \frac{1}{2}(|f(x)|^2 + |g(x)|^2)$ for all $x \in X$. The inner product induces the norm $\|\cdot\|$ with

$$\|f\| := \langle f, f\rangle^{1/2} = \left(\sum_{x \in X} |f(x)|^2 m(x)\right)^{1/2}.$$

As is well-known from a basic course in functional analysis (and is also not hard to see) the space $\ell^2(X, m)$ is complete with respect to $\|\cdot\|$ and $C_c(X)$ is dense in $\ell^2(X, m)$.

2.1. Graphs

In this section we introduce graphs over countable X.

Definition 2.1 (Graph over X). A *graph over* X is a pair (b, c) consisting of a function $b \colon X \times X \longrightarrow [0, \infty)$ satisfying

- $b(x, y) = b(y, x)$ for all $x, y \in X$
- $b(x, x) = 0$ for all $x \in X$
- $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$

and a function $c \colon X \longrightarrow [0, \infty)$.

Whenever (b, c) is a graph over X we use the following pieces of notation (compare the Section 1.2): The elements of X are called *vertices*. A pair (x, y) with $b(x, y) > 0$ is called an *edge* with *weight* $b(x, y)$. Elements $x, y \in X$ forming an edge are also called *neighbors* and we write $x \sim y$ in this case.

The *degree* is the function

$$\deg \colon X \longrightarrow [0, \infty),\ \deg(x) := \sum_{y \in X} b(x, y) + c(x).$$

A tuple $(x_0, \ldots, x_n)$ of vertices is called a *path* from x_0 to x_n if $x_i \sim x_{i+1}$ holds for $i = 0, \ldots, n-1$. As b is symmetric there exists a path from x to y if and only if there exists a path from y to x. Whenever $x \in X$ is given the set of vertices y such that there exists a path from x to y is called the *connected component of* x. If there exists a path between any two vertices the graph is called *connected*. Clearly, the graph is connected if and only if the connected component of one (each) $x \in X$ agrees with X.

We say that a graph (b, c) is *locally finite* if for every $x \in X$ the number of neighbors of x is finite, i.e.,

$$\#\{y \in X \mid y \sim x\} < \infty$$

for all $x \in X$. Here, as above, # denote the number of elements of a set. In general, we will not assume that graphs are locally finite.

Remark 2.2 (Uncountable graphs). In our discussion we have assumed that X is countable. This is not necessary in order to set up the theory. Indeed, all of the preceding definitions make sense also for uncountable X. However, the summability condition $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$ implies that any $x \in X$ can have at most countably many neighbors. Hence, the connected component of any x must be countable (even if X were uncountable at the beginning). So any graph over an uncountable X could be considered as an (uncountable) union of connected graphs with countably many vertices. So, all of the theory developed below will apply to each connected component of such a graph. For this reason we just assume countability of X from the very beginning.

2.2. The Energy Form

Any graph comes with a bilinear map on (a subspace of) $C(X)$. This map underlies all our subsequent considerations.

To a graph (b, c) over X, we associate the subspace $\mathcal{D} := \mathcal{D}_{b,c}$ of $C(X)$ given by

$$\mathcal{D} = \{f \in C(X) \mid \frac{1}{2} \sum_{x, y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f^2(x) < \infty\}.$$

Note that $\mathcal{D}$ is indeed a subspace of $C(X)$. To (b, c) we furthermore associate the map

$$\mathcal{Q} := \mathcal{Q}_{b,c} \colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{Q}(f, g) := \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x)f(x)g(x).$$

Clearly, $\mathcal{Q}$ is bilinear, i.e. linear in each argument. We call $\mathcal{Q}$ the *energy form* and refer to elements of $\mathcal{D}$ as *functions of finite energy.*

Clearly, $\mathcal{Q}$ is *symmetric*, i.e., satisfies

$$\mathcal{Q}(f, g) = \mathcal{Q}(g, f)$$

for all $f, g \in \mathcal{D}$. The form $\mathcal{Q}$ is also *positive*, i.e., satisfies

$$\mathcal{Q}(f, f) \geq 0$$

for all $f \in \mathcal{D}$.

We will often be interested in the values of $\mathcal{Q}$ on the diagonal only. In this case, we will use the notation

$$\mathcal{Q}(f) := \mathcal{Q}(f, f)$$

for $f \in \mathcal{D}$. We can then extend this restriction of $\mathcal{Q}$ to the diagonal to a map on the whole $C(X)$, again denoted by $\mathcal{Q}$, defined by $\mathcal{Q}\colon C(X) \longrightarrow [0, \infty]$ via

$$\mathcal{Q}(f) = \begin{cases} \mathcal{Q}(f, f) & \text{if } f \in \mathcal{D} \\ \infty & \text{else.} \end{cases}$$

Recall that at map $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a normal contraction if $C(0) = 0$ and $|C(s) - C(t)| \leq |t - s|$ for all $s, t \in \mathbb{R}$. The form $\mathcal{Q}$ is compatible with normal contractions in the following sense.

Proposition 2.3 (Compatibility with normal contractions). *Let (b, c) be a graph over X and let $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a normal contraction. Then,*

$$\mathcal{Q}(C \circ f) \leq \mathcal{Q}(f)$$

holds for all $f \in C(X)$.

Proof. This is immediate from the definitions. □

The energy form has the following semi-continuity property.

Proposition 2.4 (Lower semi-continuity of $\mathcal{Q}$ on $C(X)$). *Let (b, c) be a graph over X. If a sequence (f_n) in $C(X)$ converges pointwise to $f \in C(X)$, i.e., $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$, then*

$$\mathcal{Q}(f) \leq \liminf_{n\to\infty} \mathcal{Q}(f_n).$$

Proof. This is a consequence of Fatou's lemma. Indeed, consider the measure space $X \times X$ with the measure B and X with the measure C given by

$$B(M) := \frac{1}{2} \sum_{(x,y) \in M} b(x, y) \quad \text{and} \quad C(N) := \sum_{x \in N} c(x)$$

for $M \subseteq X \times X$, $N \subseteq X$, and for $n \in \mathbb{N}$ the functions $F_n, F\colon X \times X \longrightarrow [0,\infty)$ defined by

$$F_n(x,y) := (f_n(x) - f_n(y))^2 \quad \text{and} \quad F(x,y) := (f(x) - f(y))^2.$$

Then, clearly $F_n(x,y) \to F(x,y)$ for all $x, y \in X$, $f_n(x)^2 \to f(x)^2$ for all $x \in X$ as $n \to \infty$ and

$$\int_{X\times X} F \mathrm{d}B + \int_X f^2 \mathrm{d}C = \mathcal{Q}(f), \qquad \int_{X\times X} F_n \mathrm{d}B + \int_X f_n^2 \mathrm{d}C = \mathcal{Q}(f_n).$$

Now, Fatou's lemma gives the desired statement. □

For $o \in X$, we define the map

$$\langle\cdot,\cdot\rangle_o\colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

by

$$\langle f, g\rangle_o := \mathcal{Q}(f,g) + f(o)g(o)$$

for $f, g \in \mathcal{D}$. Clearly, $\langle\cdot,\cdot\rangle_o$ is linear in each argument, symmetric and satisfies $\langle f, f\rangle_o \geq 0$ for all $f \in \mathcal{D}$. Hence, the map is a semi-scalar product. We let $\|\cdot\|_o$ be the corresponding semi-norm, i.e.

$$\|f\|_o := \langle f, f\rangle_o^{1/2}$$

for $f \in \mathcal{D}$.

For connected graphs the map $\langle\cdot,\cdot\rangle_o$ is a scalar-product and $\|\cdot\|_o$ is a norm on $\mathcal{D}$ and $\mathcal{D}$ becomes a Hilbert space. This (and more) is the content of the next lemma.

Lemma 2.5 (The Hilbert space $(\mathcal{D}, \langle\cdot,\cdot\rangle_o)$). *Let (b,c) be a connected graph over (X,m) and $o \in X$. Then, the following holds:*

(a) *The map $\langle\cdot,\cdot\rangle_o$ is a scalar-product and $\|\cdot\|_o$ is a norm.*
(b) *For any $x \in X$ the norms $\|\cdot\|_o$ and $\|\cdot\|_x$ are equivalent. In particular, for any $x \in X$ the pointwise evaluation map $\mathcal{D} \longrightarrow \mathbb{R}, f \mapsto f(x)$, is continuous.*
(c) *$(\mathcal{D}, \|\cdot\|_o)$ is a Hilbert space.*
(d) *A sequence (f_n) in $\mathcal{D}$ converges to f w.r.t. $\langle\cdot,\cdot\rangle_o$ if and only if $f_n \to f$ pointwise and*

$$\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f).$$

Proof. (a) We already know that $\langle\cdot,\cdot\rangle_o$ is a semi-inner product. Thus, it suffices to show that it is non-degenerate i.e. that $\langle f, f\rangle_o = 0$ is only possible for $f = 0$. Now, writing out the definition gives

$$\begin{aligned} 0 &= \langle f, f\rangle_o = \mathcal{Q}(f,f) + f(o)^2 \\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x) - f(y))^2 + \sum_{x\in X} c(x) f(x)^2 + f(o)^2. \end{aligned}$$

As b, c and all squares appearing are non-negative, we infer $f(o) = 0$ and $f(x) = f(y)$ for all $x \sim y$. As the graph is connected this implies $f = 0$.

(b) Since the graph is connected for any $x \in X$ there exists a path $(x_0, \ldots, x_n)$ with $x_0 = o$ and $x_n = x$. Without loss of generality we assume that the x_j are pairwise different. Then, for any $f \in \mathcal{D}$ the following estimate holds

$$\begin{aligned}
&|f(x) - f(o)| \\
&\leq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \\
&\leq \left(\sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})} \right)^{1/2} \left(\sum_{i=0}^{n-1} b(x_i, x_{i+1}) |f(x_i) - f(x_{i+1})|^2 \right)^{1/2} \\
&\leq C_{o,x} \mathcal{Q}(f)^{1/2}
\end{aligned}$$

with $C_{o,x} := \left(\sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})} \right)^{1/2}$. From this the desired equivalence follows rather easily. Here are the details: From $|f(x)| \leq |f(o)| + |f(x) - f(o)|$ we obtain

$$f(x)^2 \leq 2f(o)^2 + 2|f(x) - f(o)|^2 \leq 2f(o)^2 + 2C_{o,x}^2 \mathcal{Q}(f).$$

This then implies

$$\|f\|_x^2 = \mathcal{Q}(f) + f(x)^2 \leq \max\{2, 2C_{o,x} + 1\} \|f\|_o^2$$

for all $f \in \mathcal{D}$. Reversing the roles of x and o we also find $C \geq 0$ such for all $f \in \mathcal{D}$ the inequality

$$\|f\|_o^2 \leq C \|f\|_x^2$$

holds. (Actually, $C = \max\{2, 2C_{o,x} + 1\}$ will do as well as we could just use the path above in the reverse order). This shows that $\| \cdot \|_o$ and $\| \cdot \|_x$ are equivalent.

Now, clearly the point evaluation $\mathcal{D} \longrightarrow \mathbb{R}, f \mapsto f(x)$ is continuous with respect to $\| \cdot \|_x$. As $\| \cdot \|_o$ is equivalent to $\| \cdot \|_x$ the point evaluation is continuous with respect to $\| \cdot \|_o$ as well.

(c) We have to show that $(\mathcal{D}, \langle \cdot, \cdot \rangle_o)$ is complete. Let (f_n) be a Cauchy sequence with respect to $\| \cdot \|_o$. This implies that

$$\mathcal{Q}(f_n - f_m) + |f_n(o) - f_m(o)|^2$$

becomes arbitrarily small for n, m sufficiently large. In particular, $(f_n(o))$ must be a Cauchy sequence. Now, by (b) the (f_n) must be a Cauchy sequence with respect to $\| \cdot \|_x$ for any $x \in X$. Hence, $(f_n(x))$ must be a Cauchy sequence for each x (by the same reasoning that we had applied just now for $x = o$). Altogether, we infer that (f_n) converges pointwise to some f. This f is now our candidate for a limit of (f_n) and we have to show that f is indeed the limit of (f_n) with respect to $\| \cdot \|_o$.

Since (f_n) is a Cauchy sequence, $(\|f_n\|_o)$ is bounded by some $C \geq 0$. Thus, by Fatou's lemma

$$\mathcal{Q}(f) \leq \liminf_{n \to \infty} \mathcal{Q}(f_n) \leq \liminf_{n \to \infty} \|f_n\|_o^2 \leq C^2.$$

Thus, f belongs to $\mathcal{D}$ and, again by Fatou's lemma,

$$\mathcal{Q}(f - f_n) \leq \liminf_{k \to \infty} \mathcal{Q}(f_k - f_n) \leq \liminf_{k \to \infty} \|f_k - f_n\|_o^2,$$

which becomes arbitrarily small since (f_n) is a Cauchy sequence. Hence, (f_n) converges indeed to f with respect to $\| \cdot \|_o$.

(d) Let (f_n) be a sequence in $\mathcal{D}$ that converges with respect to $\|\cdot\|_o$ to f. Then, the continuity of the point evaluation from (b) gives that (f_n) converges pointwise to f. Continuity of $\|\cdot\|_o$ then implies

$$\mathcal{Q}(f_n) = \|f_n\|_o^2 - f_n(o)^2 \to \|f\|_o^2 - f(o)^2 = \mathcal{Q}(f).$$

This shows one implication.

Conversely, assume pointwise convergence of $f_n \to f$ and the bound given in (d). This then implies

$$\limsup_{n\to\infty} \|f_n\|_o^2 = \limsup_{n\to\infty} \left(\mathcal{Q}(f_n) + f_n(o)^2\right) \leq \mathcal{Q}(f) + f(o)^2 = \|f\|_o^2.$$

As the sequence $((f_n)$ is bounded in the Hilbert space $(\mathcal{D}, \|\cdot\|_o)$, the sequence as well as any of its subsequences must have weakly converging subsequences. By pointwise convergence the limit of any weakly converging subsequence coincides with f. Hence, (f_n) converges weakly to f itself, i.e., $\langle f_n, g\rangle_o \to \langle f, g\rangle_o$ for all $g \in \mathcal{D}$. This gives in particular

$$\langle f_n, f\rangle_o \to \|f\|_o^2, \quad n \to \infty.$$

From the preceding convergence statement and

$$0 \leq \|f - f_n\|_o^2 = \|f\|_o^2 + \|f_n\|_o^2 - 2\langle f, f_n\rangle_o$$

we obtain

$$0 \leq \limsup_{n\to\infty} \|f - f_n\|_o^2 \leq 0.$$

This gives the desired convergence and finishes the proof. □

Remark 2.6. Let (b, c) be a connected graph over (X, m) and $o \in X$. The inner product $\langle\cdot,\cdot\rangle_o$ can be rewritten as $\langle\cdot,\cdot\rangle_o = \mathcal{Q}_{b,\tilde{c}}$ with $\tilde{c} := c + 1_o$. In this sense, up to a scaling the previous lemma can be understood as a statement about connected graphs with nonvanishing c. In particular, if $c \neq 0$, then $(\mathcal{D}, \mathcal{Q})$ is a Hilbert space.

The preceding results make $\mathcal{D}$ a Hilbert space, whenever the underlying graph is connected. In particular, in this case a sequence of functions (f_n) converges to f if and only if (f_n) converges pointwise to f and $\mathcal{Q}(f - f_n) \to 0$ holds. The latter can be used to define convergence of sequences in $\mathcal{D}$ in the general case even if the graph is not connected. We will be interested in the 'closure' of $C_c(X)$ with respect to this notion of convergence.

Definition 2.7. Let (b, c) be a graph over X. We define $\mathcal{D}_0(X)$ to be the subspace of $\mathcal{D}$ consisting of those $f \in \mathcal{D}$ for which there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$.

Corollary 2.8. *Let (b, c) be a connected graph over X and $o \in X$. Then,*

$$\mathcal{D}_0(X) = \overline{C_c(X)}^{\|\cdot\|_o}$$

holds.

Proof. By (d) of Lemma 2.5, convergence of (φ_n) to f in $\mathcal{D}$ implies pointwise convergence of (φ_n) to f. Given this the equality is rather immediate. □

2.3. The Laplacian

Besides the energy form $\mathcal{Q}$ associated to a graph we will also consider the *formal Laplacian*. Details are discussed in this section.

Let m be a measure on X of full support and let (b, c) be a graph over X. We let $\mathcal{L} = \mathcal{L}_{b,c,m}$ be the operator acting on

$$\mathcal{F} = \mathcal{F}_b := \{f \in C(X) \mid \sum_{y \in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\}$$

by

$$\mathcal{L}f(x) := \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).$$

We call $\mathcal{L}$ the *formal Laplacian* associated to (b, c) over (X, m). The word 'formal' appears as this is not an operator in an ℓ^2 space.

We note that the formal Laplacian $\mathcal{L}$ depends on b as well as c and m while the domain $\mathcal{F}$ depends only on b.

The operator $\mathcal{L}$ has a certain symmetry property and the form $\mathcal{Q}$ and operator $\mathcal{L}$ are related by an integration by parts formula which we refer to as Green's formula. This is the content of the next proposition.

Proposition 2.9 (Green's formula). *Let (b, c) be a graph over (X, m).*

(a) *Every $\varphi \in C_c(X)$ belongs to $\mathcal{F}$ and for all $f \in \mathcal{F}$ and $\varphi \in C_c(X)$*

$$\sum_{x \in X} \varphi(x)\mathcal{L}f(x)m(x) = \sum_{x \in X} \mathcal{L}\varphi(x)f(x)m(x)$$
$$= \frac{1}{2} \sum_{x,y \in X} b(x,y)(\varphi(x) - \varphi(y))(f(x) - f(y)) + \sum_{x \in X} c(x)\varphi(x)f(x)$$

holds, where all of the sums are absolutely convergent.

(b) *We have*

$$\mathcal{D} \subseteq \mathcal{F}$$

and thus for all $f \in \mathcal{D}$ and $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi, f) = \sum_{x \in X} \varphi(x)\mathcal{L}f(x)m(x) = \sum_{x \in X} \mathcal{L}\varphi(x)f(x)m(x).$$

Proof. (a) By the assumptions on f, φ and b we have

$$\sum_{x,y \in X} |b(x,y)f(y)\varphi(x)| = \sum_{x \in X} |\varphi(x)| \sum_{y \in X} b(x,y)|f(y)| < \infty$$

and

$$\sum_{x,y \in X} |b(x,y)f(x)\varphi(x)| = \sum_{x \in X} |f(x)\varphi(x)| \sum_{y \in X} b(x,y) < \infty.$$

Given this finiteness, the desired equalities follow easily by direct computations.

(b) Given (a), it suffices to show that every $f \in \mathcal{D}$ belongs to $\mathcal{F}$. To see this, we calculate

$$\sum_{y \in X} b(x,y)|f(y)| \leq \sum_{y \in X} b(x,y)|f(x) - f(y)| + \sum_{y \in X} b(x,y)|f(x)|.$$

Now, the first term can be seen to be finite via the Cauchy–Schwarz inequality as

$$\left(\sum_{y\in X} b(x,y)\right)^{1/2} \left(\sum_{y\in X} b(x,y)(f(x)-f(y))^2\right)^{1/2} \leq \deg(x)^{1/2}\mathcal{Q}(f)^{1/2}$$

and the second term is finite by the assumption on b. This gives the desired statement. □

For $\alpha \in \mathbb{R}$ we say that a function u is *α-subharmonic* if $u \in \mathcal{F}$ and

$$(\mathcal{L}+\alpha)u \leq 0.$$

We say that u is *α-superharmonic* if $-u$ is α-subharmonic. We say that u is *α-harmonic* if u is both α-subharmonic and α-superharmonic, i.e., $u \in \mathcal{F}$ satisfies

$$(\mathcal{L}+\alpha)u = 0.$$

When $\alpha = 0$, we say that u is *(sub/super)harmonic*. We will see that various features of such functions are intimately related to the geometric, spectral and stochastic properties of graphs.

We next present three basic results concerning solutions of the equation

$$(\mathcal{L}+\alpha)u = f$$

which will be used in various later considerations. We refer to this equation as the *Poisson equation*.

As above, we will use the notation $u \wedge v := \min\{u,v\}$ and $u \vee v := \max\{u,v\}$ for the minimum and maximum of two functions u and v, respectively.

We start with a minimum principle for certain supersolutions of the Poisson equation.

THEOREM 2.10 (Minimum principle). *Let (b,c) be a graph over (X,m). Let $U \subseteq X$. Assume that the function $u \in \mathcal{F}$ satisfies*

- *$(\mathcal{L}+\alpha)u \geq 0$ on U for some $\alpha \geq 0$*
- *$u \wedge 0$ attains a minimum on every connected component of U*
- *$u \geq 0$ on $X \setminus U$.*

If $\alpha > 0$ or if every connected component of U is connected to $X \setminus U$, then $u \geq 0$ and, in fact, on each connected component of U either $u = 0$ or $u > 0$.

PROOF. Without loss of generality we can assume that U is connected. If $u > 0$ there is nothing to show. Therefore, assume there exists a vertex $x \in U$ with $u(x) \leq 0$. As $u \wedge 0$ attains a minimum on U, there exists a vertex $x_0 \in U$ with $u(x_0) \leq 0$ and $u(x_0) \leq u(y)$ for all $y \in U$. As $u(y) \geq 0$ for $y \in X \setminus U$, we obtain $u(x_0) - u(y) \leq 0$ for all $y \in X$. By the supersolution assumption we then find

$$\begin{aligned} 0 &\leq (\mathcal{L}+\alpha)u(x_0) \\ &= \frac{1}{m(x_0)}\left(\sum_{y\in X} b(x_0,y)(u(x_0)-u(y)) + c(x_0)u(x_0)\right) + \alpha u(x_0) \leq 0. \end{aligned}$$

Therefore, if $\alpha > 0$, then $0 = u(x_0)$ and $u(y) = u(x_0) = 0$ for all $y \sim x_0$. As U is connected, iteration of this argument shows that $u = 0$ on U.

On the other hand, for $\alpha = 0$, we obtain by the same argument that u is constant on U. As U is connected to $X \setminus U$, namely there exist $x \in U$ and $z \in X \setminus U$ such that $x \sim z$, we conclude from the formula above for x that

$$0 = \frac{1}{m(x)} b(x,z)(u(x) - u(z)) + \frac{1}{m(x)} \left(\sum_{y \neq z} b(x,y)(u(x) - u(y)) + c(x)u(x) \right).$$

Since $u(x) = u(x_0)$, the second term is clearly smaller or equal to 0. Hence, the first term must be greater or equal to zero. This implies $0 \geq u(x_0) = u(x) \geq u(z) \geq 0$ and we conclude $u = 0$ on U. $\square$

For the following lemma, given a sequence of functions (u_n) and a function u we write

$$u_n \nearrow u \quad \text{as} \quad n \to \infty$$

if $u_n(x) \leq u_{n+1}(x)$ for all $x \in X$, $n \in \mathbb{N}$ and $u_n \to u$ pointwise as $n \to \infty$.

Lemma 2.11 (Monotone convergence of solutions). *Let (b, c) be a graph over (X, m). Let $\alpha \in \mathbb{R}$ and let $u, f \in C(X)$. Let (u_n) be a sequence of functions in $\mathcal{F}$ with $u_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $u_n \nearrow u$ and $(\mathcal{L} + \alpha)u_n(x) \to f(x)$ for all $x \in X$ as $n \to \infty$. Then, $u \in \mathcal{F}$ and*

$$(\mathcal{L} + \alpha)u = f.$$

Proof. Without loss of generality, we assume that $m = 1$. By assumption

$$\sum_{y \in X} b(x,y)(u_n(x) - u_n(y)) + (c(x) + \alpha)u_n(x) = (\mathcal{L} + \alpha)u_n(x) \to f(x)$$

as $n \to \infty$ for any $x \in X$. As $\left(\sum_{y \in X} b(x,y)u_n(x)\right)_{n \in \mathbb{N}}$ converges increasingly to $u(x)\sum_{y \in X} b(x,y) < \infty$, the assumptions on (u_n) show that $\left(\sum_{y \in X} b(x,y)u_n(y)\right)_{n \in \mathbb{N}}$ must converge as well and, in fact, must converge to $\sum_{y \in X} b(x,y)u(y)$ by the monotone convergence theorem. From this, we easily obtain the conclusion. $\square$

We let

$$u_+ := u \vee 0 \qquad \text{and} \qquad u_- := -u \vee 0$$

denote the positive and negative part of u so that $u = u_+ - u_-$ and $|u| = u_+ + u_-$. The next lemma then shows that the positive and negative part of an α-harmonic function are α-subharmonic.

Lemma 2.12 (α-subharmonic and α-superharmonic functions). *Let (b, c) be a graph over (X, m). Let $\alpha \in \mathbb{R}$. If $u, v \in \mathcal{F}$ are α-subharmonic (α-superharmonic, respectively), then $u \vee v$ is α-subharmonic ($u \wedge v$ is α-superharmonic, respectively). In particular, if u is α-harmonic, then u_+, u_- and $|u|$ are all α-subharmonic.*

Proof. Let u, v be α-subharmonic for some $\alpha \in \mathbb{R}$ and let $w = u \vee v$. Let $x \in X$ and assume without loss of generality that $w(x) = u(x) \geq v(x)$.

Then,

$$w(x) - w(y) = \begin{cases} u(x) - u(y) & \text{if } u(y) \geq v(y) \\ u(x) - v(y) & \text{else} \end{cases}$$
$$\leq u(x) - u(y).$$

Thus,

$$(\mathcal{L} + \alpha)w(x) \leq (\mathcal{L} + \alpha)u(x) \leq 0$$

holds. As $x \in X$ was arbitrary, we infer that w is α-subharmonic.

Now, let u, v be α-superharmonic. We first observe that $u \wedge v = -((-u) \vee (-v))$. Hence, by what we have shown above, $(-u) \vee (-v)$ is α-subharmonic as $-u$ and $-v$ are α-subharmonic. Therefore, $u \wedge v$ is α-superharmonic. The "in particular" statement follows as $u_\pm = (\pm u) \vee 0$ and $|u| = u_+ + u_-$. □

We now introduce the *heat equation*

$$(\mathcal{L} + \partial_t)u = 0.$$

More specifically, a function $u\colon [0, \infty) \times X \longrightarrow \mathbb{R}$ is called a *solution of the heat equation* if, for every $x \in X$, the mapping $t \mapsto u_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $u_t \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t > 0$. If u has all of the properties above but instead of equality in the heat equation satisfies $(\mathcal{L} + \partial_t)u_t \geq 0$ for all $t > 0$, then we call u a *supersolution* of the heat equation. If u is a solution of the heat equation and $u_0 = f$ for $f \in C(X)$, then f is called the *initial condition* for u. We will say that u satisfies the heat equation with initial condition f in this case. We think of x as a space variable and t as time.

We now prove a minimum principle for the heat equation. In particular, for supersolutions of the heat equation on certain subsets, positivity on the boundary propagates to positivity on the subset. This will be used later to establish the minimality of certain solutions.

Theorem 2.13 (Minimum principle for the heat equation). *Let (b, c) be a graph over (X, m). Let $U \subseteq X$ be a connected subset and suppose that U contains a vertex which is connected to a vertex outside of U. Let $T \geq 0$ and let $u\colon [0, T] \times X \longrightarrow \mathbb{R}$ be such that $t \mapsto u_t(x)$ is continuously differentiable on $(0, T)$ for every $x \in U$ and $u_t \in \mathcal{F}$ for all $t \in (0, T]$. Assume u satisfies*

- *$(\mathcal{L} + \partial_t)u_t \geq 0$ on U for $t \in (0, T)$,*
- *$u \wedge 0$ attains a minimum on $U \times [0, T]$,*
- *$u \geq 0$ on $((0, T] \times (X \setminus U)) \cup (\{0\} \times U)$.*

Then, $u \geq 0$ on $[0, T] \times U$.

Proof. By definition we have $u \wedge 0 \leq 0$. Let (t, x) be a point where $u \wedge 0$ attains a minimum on $[0, T] \times U$. If $u_t(x) \geq 0$, the conclusion follows so we assume $u_t(x) < 0$. Since u is positive on $\{0\} \times U$ we have $t > 0$. Furthermore, since u attains a minimum at (t, x) with respect to t we obtain

$$\partial_t u_t(x) = 0 \text{ if } t < T \text{ and } \partial_t u_t(x) \leq 0 \text{ if } t = T.$$

Since u also attains a negative minimum at (t,x) with respect to x, we infer from a direct computation

$$\mathcal{L}u_t(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y) \underbrace{(u_t(x) - u_t(y))}_{\leq 0} + \frac{c(x)}{m(x)} \underbrace{u_t(x)}_{<0} \leq 0.$$

Put together this gives $(\mathcal{L} + \partial_t)\, u_t(x) \leq 0$. As u also satisfies $(\mathcal{L} + \partial_t) u_t \geq 0$ we obtain

$$(\mathcal{L} + \partial_t)\, u_t(x) = 0$$

and hence $\mathcal{L}u_t(x) = 0$. Looking at the formula for $\mathcal{L}u_t(x)$ again, we find

$$u_t(y) = u_t(x) < 0$$

for all $y \sim x$. Iterating this argument and using the assumption that U is connected we find that u_t is a negative constant on U. At the vertex $z \in U$ which has a neighbor not in U, the equation $\mathcal{L}u_t(z) = 0$ then contradicts the assumption $u \geq 0$ on $(0,T] \times (X \setminus U)$. $\square$

2.4. Boundedness of Forms and Operators

We have associated to each graph a form and an operator. Now, in general form and operator will not be bounded and this will pose a major challenge right at the beginning of the investigation. We will deal with this challenge by using the theory of closed forms and unbounded self-adjoint operators. Basic elements of this theory will be discussed in the next chapter and this will allow us to proceed in the investigation of the general case. There is, however, a situation which can already be investigated right now without the theory of closed forms and unbounded operators. This is the situation that form and operator are bounded. Details will be discussed in this section.

For a graph (b,c) over (X,m) we define the *weighted degree*

$$\mathrm{Deg}\colon X \longrightarrow [0,\infty)$$

by

$$\mathrm{Deg}(x) := \frac{1}{m(x)} \left(\sum_{y \in X} b(x,y) + c(x) \right)$$

We will characterize boundedness of the operator and the form in terms of the weighted degree. We first investigate the issue of whether the space of functions of compact support $C_c(X)$ is mapped into $\ell^2(X,m)$ by $\mathcal{L}$. We start by examples.

Example 2.14 ($\mathcal{L}C_c(X)$ in $\ell^2(X,m)$). Let (b,c) be a locally finite graph over (X,m). Then,

$$\mathcal{L}C_c(X) \subseteq C_c(X) \subseteq \ell^2(X,m).$$

Indeed, for any $x \in X$, the function $\mathcal{L}1_x$ vanishes outside of $\{y \mid y \sim x\}$ and this set is finite by the local finiteness assumption. Hence, $\mathcal{L}1_x$ belongs to $C_c(X)$ for any $x \in X$. As the 1_x, $x \in X$, form a basis of $C_c(X)$ the inclusion $\mathcal{L}C_c(X) \subseteq C_c(X)$ follows and the inclusion $C_c(X) \subseteq \ell^2(X,m)$ is clear anyway.

EXAMPLE 2.15 ($\mathcal{L}C_c(X)$ not contained in $\ell^2(X,m)$). We consider a star shaped graph. Specifically, let $X := \mathbb{N}_0$ and $b(0,k) = b(k,0) := k^{-2}$ for $k \geq 1$ and $b(k,l) := 0$ for $k,l \geq 1$ with $c := 0$. Furthermore, let m be given by $m(k) := k^{-3}$ for $k \geq 1$ with $m(0) := 1$. Then,

$$\sum_{k=0}^{\infty} (\mathcal{L}1_0(k))^2 m(k) \geq \sum_{k=1}^{\infty} \frac{b^2(0,k)}{m(k)} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Thus, $\mathcal{L}1_0$ does not belong to $\ell^2(X,m)$. Note that the set X has finite measure. So, finiteness of the measure is not relevant in this context.

We can now characterize when $C_c(X)$ is mapped into $\ell^2(X,m)$.

THEOREM 2.16. *Let (b,c) be a graph over (X,m). Then, the following statements are equivalent:*

(i) $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$.
(ii) *The functions $X \longrightarrow [0,\infty)$, $y \mapsto b(x,y)/m(y)$, belong to $\ell^2(X,m)$ for all $x \in X$.*
(iii) $\ell^2(X,m) \subseteq \mathcal{F}$.

Furthermore, the equivalent conditions above are satisfied if

$$\inf_{y \sim x} m(y) > 0$$

for all $x \in X$ which holds, in particular, if the graph is locally finite.

PROOF. For any $x \in X$ we define φ_x on X by $\varphi_x(y) := b(x,y)/m(y)$. Clearly, φ_x vanishes outside of

$$N_x := \{y \in X \mid y \sim x\}.$$

(i) $\Longleftrightarrow$ (ii): Let $x \in X$ be arbitrary. We first observe

$$\mathcal{L}1_x = \mathrm{Deg}(x)1_x - \varphi_x.$$

Thus, $\mathcal{L}1_x$ belongs to $\ell^2(X,m)$ if and only if $\varphi_x \in \ell^2(X,m)$. As $x \in X$ is arbitrary this gives the desired equivalence.

(ii) $\Longleftrightarrow$ (iii): Assume that φ_x belongs to $\ell^2(X,m)$ for all $x \in X$. Then, for $f \in \ell^2(X,m)$, we get by the Cauchy–Schwarz inequality

$$\sum_{y \in X} b(x,y)|f(y)| = \sum_{y \in X} \varphi_x(y)|f(y)|m(y) \leq \|\varphi_x\|\|f\|.$$

Hence, f belongs to $\mathcal{F}$.

Conversely, assume $\ell^2(X,m) \subseteq \mathcal{F}$. Let $x \in X$. Then, $\ell^1(N_x, b(x,\cdot)) = \{f \in \mathcal{F} \mid \mathrm{supp} f \subseteq N_x\}$ and we find

$$\ell^2(N_x, m_{N_x}) \subseteq \ell^1(N_x, b(x,\cdot)).$$

Thus, the map

$$j\colon \ell^2(N_x, m_{N_x}) \longrightarrow \ell^1(N_x, b(x,\cdot)), \quad f \mapsto f$$

is well-defined and linear. As convergence in an ℓ^p-space always implies pointwise convergence, we easily infer that the map j is closed (i.e. $f_n \to f$ and $j(f_n) \to g$ implies $j(f) = g$). Hence, by the closed graph theorem we infer the existence of a constant $C \geq 0$ such that for all $f \in \ell^2(X,m)$ and all $x \in X$

$$\sum_{y \in X} b(x,y)|f(y)| = \|f1_{N_x}\|_{\ell^1(N_x, b(x,\cdot))} \leq C\|f1_{N_x}\| \leq C\|f\|.$$

Therefore,

$$\sum_{y \in X} \varphi_x(y)|f(y)|m(y) = \sum_{y \in X} b(x,y)|f(y)| \leq C\|f\|.$$

Hence, $\varphi_x \in \ell^2(X,m)$ by the Riesz representation theorem.

Finally, the condition $\inf_{y \sim x} m(y) = C_x > 0$ implies that $\varphi_x \in \ell^2(X,m)$ for $x \in X$, since

$$\|\varphi_x\|^2 = \sum_{y \in X} \frac{b^2(x,y)}{m(y)} \leq \frac{1}{C_x} \sum_{y \in X} b^2(x,y) < \infty.$$

This shows the "in particular" statement. □

We now turn to characterizing boundedness of $\mathcal{L}$ and $\mathcal{Q}$ on $\ell^2(X,m)$. The crucial feature will be a bound on the Deg. We start with a general lemma.

Lemma 2.17 (The bound lemma). *Let X be a countable set and m a measure on X of full support. Let $a\colon X \times X \longrightarrow [0,\infty)$ be symmetric and assume that for some $\kappa > 0$ we have*

$$\sum_{y \in X} a(x,y)m(y) \leq \kappa$$

for all $x \in X$. Then:

(a) *The map $A\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ given by*

$$Af(x) := \sum_{y \in X} a(x,y)f(y)m(y)$$

for $f \in \ell^2(X,m)$ and $x \in X$ is well-defined and linear with $\|Af\| \leq \kappa\|f\|$.

(b) *The map $q\colon \ell^2(X,m) \times \ell^2(X,m) \longrightarrow \mathbb{R}$ with*

$$q(f,g) = \frac{1}{2} \sum_{x,y \in X} a(x,y)(f(x)-f(y))(g(x)-g(y))m(x)m(y)$$
$$+ \sum_{x \in X} a(x,x)f(x)g(x)m(x)m(x).$$

is well-defined and bilinear with $|q(f,g)| \leq 2\kappa\|f\|\|g\|$ for all $f,g \in \ell^2(X,m)$.

Proof. The proofs of (a) and (b) is very similar.

(a) We have to show that $\sum_{y \in X} a(x,y)f(y)m(y)$ is defined for all $x \in X$ and Af belongs to $\ell^2(X,m)$ with the given bound. It suffices to show

$$\sum_{x \in X} \left(\sum_{y \in X} a(x,y)|f(y)|m(y) \right)^2 m(x) \leq \kappa^2\|f\|^2.$$

This in turn follows by a direct computation using the Cauchy–Schwarz inequality (CSI):

$$\begin{aligned}
&\sum_{x\in X}\left(\sum_{y\in X}\underbrace{a(x,y)|f(y)|m(y)}_{=a(x,\cdot)^{1/2}m^{1/2}\cdot a(x,\cdot)^{1/2}fm^{1/2}}\right)^2 m(x)\\
&\overset{\text{CSI}}{\leq}\sum_{x\in X}\left(\underbrace{\sum_{z\in X}a(x,z)m(z)}_{\leq\kappa}\right)\left(\sum_{z\in X}a(x,z)f(z)^2m(z)\right)m(x)\\
&\leq\kappa\sum_{x\in X}\left(\sum_{z\in X}a(x,z)f(z)^2m(z)\right)m(x)\\
&\overset{\text{Fubini}}{=}\kappa\sum_{z\in X}f(z)^2m(z)\left(\underbrace{\sum_{x\in X}a(x,z)m(x)}_{\leq\kappa}\right)\\
&\leq\kappa^2\sum_{z\in X}f(z)^2m(z)=\kappa^2\|f\|^2.
\end{aligned}$$

Here, we used that a is symmetric to obtain

$$\sum_{x\in X}a(x,z)m(x)=\sum_{x\in X}a(z,x)m(x)\leq\kappa.$$

(b) We first consider the case $f=g$. Clearly, $(f(x)-f(y))^2\leq 2f(x)^2+2f(y)^2$ holds. Now, a direct computation invoking Fubini's theorem and the symmetry of a gives

$$\begin{aligned}
&\frac{1}{2}\sum_{x,y\in X,\,x\neq y}a(x,y)(2f(x)^2+2f(y)^2)m(x)m(y)+\sum_{x\in X}a(x,x)f(x)^2m(x)m(x)\\
&\leq\frac{1}{2}\sum_{x,y\in X,\,x\neq y}a(x,y)(2f(x)^2+2f(y)^2)m(x)m(y)\\
&\quad+\sum_{x\in X}a(x,x)f(x)^2m(x)m(x)+\sum_{y\in X}a(y,y)f(y)^2m(y)\\
&=\sum_{x\in X}f(x)^2m(x)\sum_{y\in X}a(x,y)m(y)+\sum_{y\in X}f(y)^2m(y)\sum_{x\in X}a(x,y)m(x)\\
&\leq\kappa\|f\|^2+\kappa\|f\|^2=2\kappa\|f\|^2.
\end{aligned}$$

This easily gives the claim for $f=g$. The general case now follows from $q(f,f)\geq 0$ and Cauchy–Schwarz inequality. □

THEOREM 2.18 (Characterization of boundedness). *Let (b,c) be a graph over (X,m). Then, the following statements are equivalent:*

(i) *The weighted degree* Deg *is a bounded function on X.*

(ii) *The form* $\mathcal{Q}$ *is bounded on* $\ell^2(X,m)$ *(i.e.* $\ell^2(X,m)$ *belongs to* $\mathcal{D}$ *and there exists a* $C \geq 0$ *with* $|\mathcal{Q}(f,g)| \leq C\|f\|\|g\|$ *for all* $f,g \in \ell^2(X,m)$*).*

(iii) *The operator* $\mathcal{L}$ *is bounded on* $\ell^2(X,m)$ *(i.e.* $\ell^2(X,m)$ *belongs to* $\mathcal{F}$ *and there exists* $D \geq 0$ *such that* $\|\mathcal{L}f\| \leq D\|f\|$ *for any* $f \in \ell^2(X,m)$*).*

Moreover, if Deg *is bounded by* D*, then*

$$|\mathcal{Q}(f,g)| \leq 2D\|f\|\|g\|$$

holds as well as

$$\|\mathcal{L}f\| \leq D\|f\|$$

for all $f,g \in \ell^2(X,m)$.

Proof. The implications (i) $\Longrightarrow$ (ii) and (i)$\Longrightarrow$(iii) follow from (a) and (b) respectively of the preceding lemma with $a(x,y) := \frac{b(x,y)}{m(x)m(y)}$ for $x,y \in X$ with $x \neq y$ and $a(x,x) := \frac{c(x)}{m(x)m(x)}$ for $x \in X$. Indeed, the assumption of that lemma is satisfied with $\kappa := \sup_{x\in X} \operatorname{Deg}$. The conclusion of the lemma then gives the bound $C = 2\kappa$ for the estimate of $\mathcal{Q}$ and the bound $D = \kappa$ for the estimate of $\mathcal{L}$.

(ii)$\Longrightarrow$(i): Assume $\mathcal{Q}(f) \leq C\|f\|^2$ for all $f \in \ell^2(X,m)$. Choosing $f := 1_x$ for $x \in X$ we find

$$\sum_{y\in X} b(x,y) + c(x) = \mathcal{Q}(1_x) \leq C\|1_x\|^2 = Cm(x).$$

This implies Deg $\leq C$.

(iii)$\Longrightarrow$(i): Assume that $\|\mathcal{L}f\| \leq C\|f\|$ holds for all $f \in \ell^2(X,m)$. Choosing $f := 1_x$ for $x \in X$ we find

$$\sum_{y\in X} b(x,y) + c(x) = m(x)\mathcal{L}1_x(x) = \langle 1_x, \mathcal{L}1_x\rangle \leq C\|1_x\|\|1_x\| = Dm(x)$$

for all $x \in X$. This implies Deg $\leq D$.

The last statement of the theorem has been proven along the way in (i)$\Longrightarrow$(ii) and (i)$\Longrightarrow$(iii) respectively. □

Remark 2.19 (Relationship between the theorems in this section). The boundedness of $\mathcal{L}$ obviously implies that $\mathcal{L}$ maps $C_c(X)$ in $\ell^2(X,m)$. Thus, the condition that Deg is bounded (appearing in Theorem 2.18) is stronger than the condition $\sum_{y\in X} \frac{b(x,y)^2}{m(y)} < \infty$ for all $x \in X$ (appearing in Theorem 2.16). Indeed, this can already be seen directly as boundedness of the degree gives

$$\sum_{z\in X} (b(y,z) + c(z)) \leq Cm(y)$$

which in turn implies

$$\sum_{y\in X} \frac{b(x,y)^2}{m(y)} \leq C \sum_{y\in X} \frac{b(x,y)^2}{\sum_z (b(y,z) + c(z))} \leq C \sum_{y\in X} b(x,y) < \infty.$$

2.5. Graphs with Standard Weights

Special b and c with rather restricted range have attracted particular attention in the literature. We present these cases here under the heading of standard weights.

DEFINITION 2.20 (Graphs with standard weights). Let (b, c) be a graph over X. If b takes values in $\{0, 1\}$ and $c = 0$, we say that b is a graph with *standard weights.*

We denote the edges of the graph by

$$E = \{(x, y) \in X \times X \mid x \sim y\}.$$

For graphs with standard weights, the degree function deg given by $\deg(x) := \sum_{y\in X} b(x, y)$ is the *combinatorial degree*, i.e., if $x \in X$, then

$$\deg(x) = \#\{y \in X \mid x \sim y\} = \# (E \cap (\{x\} \times X)).$$

The assumption $\sum_{y\in X} b(x, y) < \infty$ for all $x \in X$ clearly implies that graphs with standard weights are locally finite.

We now explicitly write out the energy form and the Laplacian in the case of standard weights.

For a graph b with standard weights, the energy form $\mathcal{Q}$ is given by

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y\in X,\, x\sim y} (f(x) - f(y))^2$$

for $f \in C(X)$. Furthermore, by local finiteness, the domain $\mathcal{F}$ of the formal Laplacian consists of all functions, i.e.,

$$\mathcal{F} = C(X).$$

2.5.1. The Counting Measure.

The counting measure $m = 1$ counts the number of vertices in a subset of X. In this case, the degree and the weighted degree satisfy

$$\deg = \mathrm{Deg}$$

and are equal to the combinatorial degree.

We denote the formal Laplacian $\mathcal{L}$ for graphs b with standard weights by Δ. This operator acts as

$$\Delta f(x) = \sum_{y\in X,\, y\sim x} (f(x) - f(y)).$$

We deduce the following corollaries from the results of the previous sections.

COROLLARY 2.21 (Characterization of boundedness). *Let b be a graph with standard weights and let $m = 1$ be the counting measure. Then, the following statements are equivalent:*

(i) *The combinatorial degree* deg *is a bounded function on X.*
(ii) *The form $\mathcal{Q}$ is bounded on $\ell^2(X)$.*
(iii) *The operator Δ is bounded on $\ell^2(X)$.*

PROOF. This follows directly from Theorem 2.18 and the equality of the combinatorial and weighted degrees, $\deg = \mathrm{Deg}$, in this case. $\square$

Furthermore, as a graph with standard weights is always locally finite we have

$$\Delta C_c(X) \subseteq C_c(X) \subseteq \ell^2(X)$$

2.5.2. The Normalizing Measure. We now introduce the normalizing measure and discuss how the resulting Laplacian is always bounded.

The normalizing measure n is given by deg which is the combinatorial degree in the case of standard weights. This measure counts the number of edges for a subset of vertices, more specifically,

$$n(A) = \#E_A + \frac{1}{2}\#\partial_E A$$

for $A \subseteq X$, where $E_A := E \cap (A \times A)$ and

$$\partial_E A := E \cap (((X \setminus A) \times A) \cup (A \times (X \setminus A)))$$

(Exercise). Letting $m := n$, the weighted degree Deg satisfies

$$\mathrm{Deg} = 1.$$

For the normalizing measure $n = \deg$, we denote the Laplacian by Δ_n refered to as the *normalized Laplacian*. We have

$$\Delta_n f(x) = \frac{1}{\deg(x)} \sum_{y \in X,\, y \sim x} (f(x) - f(y)).$$

COROLLARY 2.22 (Δ_n is bounded). *Let b be a graph with standard weights and let $n = \deg$ be the normalizing measure. Then, the normalized Laplacian Δ_n is a bounded operator on $\ell^2(X, n)$. In particular, $C_c(X) \subseteq D(\Delta_n) = \ell^2(X, n)$.*

PROOF. This follows directly from Theorem 2.18 and the equality $\mathrm{Deg} = 1$ in this case. $\square$

ISEM26 Winter Semester 2022/23
Graphs and Discrete Dirichlet Spaces

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Sheet 3

Infinite Graphs I

Exercise 1 ($\mathcal{F} = C(X)$) 4 points

Let $\mathcal{F}$ be the domain of the formal Laplacian $\mathcal{L}$ associated to a graph. Show that the following statements are equivalent:

- The graph is locally finite.
- $\mathcal{L}C_c(X) \subseteq C_c(X)$.
- $C(X) = \mathcal{F}$.

Exercise 2 (Maximum principle) 4 points

Let $\mathcal{A} : C_c(X) \longrightarrow C_c(X)$ be a symmetric linear operator, i.e., $\mathcal{A}1_x(y) = \mathcal{A}1_y(x)$ for all $x,y \in X$. Show the following equivalence:

(i) $\mathcal{A} = \mathcal{L}_{b,c}$ on $C_c(X)$ for a locally finite graph (b,c) over X.

(ii) $\mathcal{A}$ satisfies a maximum principle, i.e., if $f \in C_c(X)$ has a non-negative local maximum in $x \in X$, then $\mathcal{A}f \geq 0$.

Exercise 3 (Uncountable graphs) 4 points

Let X be an arbitrary set and assume that $b\colon X \times X \longrightarrow [0,\infty)$ satisfies $b(x,y) = b(y,x)$, $b(x,x) = 0$ and

$$\sum_{z\in X} b(x,z) = \sup_{U\subseteq X \text{ finite}} \sum_{y\in U} b(x,y) < \infty$$

for all $x \in X$. Call a subset Y of X connected if for arbitrary $x, y \in Y$ there exists $n \in \mathbb{N}$ and $x_0, \ldots, x_n \in Y$ with $x_0 = x$, $x_n = y$ and $b(x_k,x_{k+1}) > 0$ for all $k = 0, \ldots, n-1$. Show that any connected subset of X is countable.

Exercise 4 (Summability) 4 points

Let X be a countable set, $b : X \times X \longrightarrow [0,\infty)$ and $\mathcal{Q} : C(X) \longrightarrow [0,\infty]$

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y\in X} b(x,y)(f(x) - f(y))^2.$$

Show that

$$\mathcal{Q}(\varphi) < \infty$$

for all $\varphi \in C_c(X)$ if and only if

$$\sum_{y\in X} b(x,y) < \infty$$

for all $x \in X$.

Bonus Exercise 1 (Local finiteness) 1 point

Let $\mathcal{A}\colon C(X) \longrightarrow C(X)$ be a symmetric linear operator, i.e., $\mathcal{A}1_x(y) = \mathcal{A}1_y(x)$ for all $x,y \in X$. Does one have $\mathcal{A}C_c(X) \subseteq C_c(X)$?