26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 02

1.5. Markov Resolvents and Semigroups

In Sections 1.3 and 1.4 we have seen that any graph comes with a form and an operator and we have characterized the form (via compatibility with normal contractions) and the operator (via a maximum principle). Here, we introduce two further objects coming with a graph. These are families of operators. Specifically, these are the semigroup and the resolvent associated to the Laplacian of the graph. We characterize them intrinsically by the Markov property.

Let L be a self-adjoint operator on $\ell^2(X,m)$ for a finite set measure space (X,m). We call $(e^{-tL})_{t\geq 0}$ the *semigroup associated to the operator* L. We say that the semigroup is *positivity preserving* if $f \in \ell^2(X,m)$, $f \geq 0$ implies

$$e^{-tL} f \geq 0$$

for all $t \geq 0$. Recall that a function f satisfying $f \geq 0$ is called *positive*. Therefore, the semigroup $(e^{-tL})_{t\geq 0}$ is positivity preserving if it maps positive functions to positive functions.

We say that the semigroup has the *Markov property* if $f \in \ell^2(X,m)$, $0 \leq f \leq 1$ implies

$$0 \leq e^{-tL} f \leq 1$$

for all $t \geq 0$. A semigroup with the Markov property is positivity preserving. Indeed, whenever $f \geq 0$ is given then sf with a suitable $s > 0$ will satisfy $0 \leq sf \leq 1$ and $e^{-tL} f = \frac{1}{s} e^{-tL}(sf) \geq 0$ follows.

In passing we note that a semigroup $(e^{-tL})_{t\geq 0}$ is Markov if and only if $e^{-tL} f \leq 1$ holds for all $f \leq 1$ and $t \geq 0$. (Indeed, if (e^{-tL}) is Markov then

$$e^{-tL} f = e^{-tL} f_+ - e^{-tL} f_- \leq e^{-tL} f_+ \leq 1$$

holds for any $f \leq 1$ and $t \geq 0$. Conversely, $e^{-tL} f \leq 1$ for $f \leq 1$ directly implies $e^{-tL} f \leq s$ for $f \leq s$ with some $s > 0$. This in turn gives $e^{-tL} f \leq 0$ for $f \leq 0$ as any such f satisfies $f \leq s$ for all $s > 0$. Hence, the semigroup is positivity preserving.)

We will characterize in terms of L and the associated form Q when a semigroup is positivity preserving and Markov. We will need an auxiliary lemma which does not involve graphs.

Lemma 1.17 (Lie–Trotter product formula on finite set measure spaces). *Let (X,m) be a finite set measure space. If A and B are operators on $\ell^2(X,m)$, then*

$$e^{A+B} = \lim_{n\to\infty} (e^{\frac{1}{n}A} e^{\frac{1}{n}B})^n.$$

Proof. Set $S_n := e^{\frac{1}{n}(A+B)}$ and $T_n := e^{\frac{1}{n}A} e^{\frac{1}{n}B}$ for $n \in \mathbb{N}$. We want to show that $\|S_n^n - T_n^n\| \to 0$ as $n \to \infty$.

We first note that for any operator L on $\ell^2(X,m)$ we have $\|e^L\| \leq e^{\|L\|}$. Consequently, it follows that

$$\|T_n\| \leq \|e^{\frac{1}{n}A}\| \|e^{\frac{1}{n}B}\| \leq e^{\frac{1}{n}\|A\|} e^{\frac{1}{n}\|B\|} = e^{\frac{1}{n}(\|A\|+\|B\|)}$$

and

$$\|S_n\| \leq e^{\frac{1}{n}\|A+B\|} \leq e^{\frac{1}{n}(\|A\|+\|B\|)}.$$

A telescoping argument gives

$$S_n^n - T_n^n = \sum_{j=0}^{n-1} S_n^j (S_n - T_n) T_n^{n-1-j}.$$

Therefore,

$$\|S_n^n - T_n^n\| \leq C_1 n \|S_n - T_n\|,$$

where $C_1 = e^{(\|A\|+\|B\|)}$. Moreover,

$$\begin{aligned}
\|S_n - T_n\| &= \left\| \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{A+B}{n}\right)^j - \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{A}{n}\right)^k \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{B}{n}\right)^l \right\| \\
&= \left\| \sum_{j=2}^{\infty} \frac{1}{j!} \left(\frac{A+B}{n}\right)^j - \sum_{k+l\geq 2} \frac{1}{k!l!} \left(\frac{A}{n}\right)^k \left(\frac{B}{n}\right)^l \right\| \\
&\leq C \frac{1}{n^2}
\end{aligned}$$

for some constant C. Therefore,

$$\|S_n^n - T_n^n\| \leq \frac{C_1 C}{n},$$

which yields the desired statement. □

We now characterize when a semigroup is positivity preserving in terms of the matrix and the form associated to a self-adjoint operator.

Theorem 1.18 (First Beurling–Deny criterion). *Let (X, m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with associated matrix l and form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *The matrix l of L satisfies, for all $x, y \in X$ with $x \neq y$,*

(*"Matrix"*) $$l(x, y) \leq 0.$$

(ii) *The form satisfies, for all $f \in \ell^2(X, m)$,*

(*"Form"*) $$Q(|f|) \leq Q(f).$$

(iii) *The semigroup satisfies, for all $f \in \ell^2(X, m)$, $f \geq 0$ and $t \geq 0$,*

(*"Semigroup"*) $$e^{-tL} f \geq 0.$$

Proof. (i) $\Longrightarrow$ (iii): We first decompose L into a diagonal and an off-diagonal part. More specifically, we write

$$L = \widetilde{L} + \widetilde{D},$$

where $\widetilde{L}$ has matrix elements equal to those of L on the off-diagonal and matrix elements equal to zero on the diagonal and $\widetilde{D}$ has matrix elements equal to those of L on the diagonal and matrix elements equal to zero on the off-diagonal. The Lie–Trotter formula, Lemma 1.17, then gives

$$e^{-tL} = \lim_{n\to\infty} \left(e^{-\frac{t}{n}\widetilde{L}} e^{-\frac{t}{n}\widetilde{D}}\right)^n.$$

Now, by assumption, $-\widetilde{L}$ has only non-negative entries. This is then also true for $e^{-\frac{t}{n}\widetilde{L}}$. Also, $e^{-\frac{t}{n}\widetilde{D}}$ has only non-negative entries as it is a diagonal matrix with exponential functions on the diagonal. Putting this together, we infer that e^{-tL} has only non-negative matrix entries. This gives (iii).

(iii) $\Longrightarrow$ (ii): From (iii) we easily obtain

$$|e^{-tL}f| \leq e^{-tL}|f|.$$

Indeed, write $f = f_+ - f_-$ with $f_+ = f \vee 0$ and $f_- = -f \vee 0$. Note that $f_+ \geq 0$, $f_- \geq 0$ and $|f| = f_+ + f_-$. Now, a direct computation gives

$$\begin{aligned} |e^{-tL}f| &= |e^{-tL}f_+ - e^{-tL}f_-| \\ &\leq |e^{-tL}f_+| + |e^{-tL}f_-| \\ &= e^{-tL}f_+ + e^{-tL}f_- \\ &= e^{-tL}|f|. \end{aligned}$$

Here, we used assumption (iii) in the next to last step. From this preliminary consideration we infer

$$\langle e^{-tL}f, f\rangle \leq |\langle e^{-tL}f, f\rangle| \leq \langle e^{-tL}|f|, |f|\rangle.$$

Moreover, $\langle |f|, |f|\rangle = \langle f, f\rangle$. This gives

$$\langle (e^{-tL} - I)|f|, |f|\rangle \geq \langle (e^{-tL} - I)f, f\rangle.$$

Dividing by $t > 0$ we infer

$$\langle \frac{1}{t}(e^{-tL} - I)|f|, |f|\rangle \geq \langle \frac{1}{t}(e^{-tL} - I)f, f\rangle.$$

By the discussion of semigroups in Section 1.1 we know $\partial_t e^{-tL}f = -Le^{-tL}f$ so that $\partial_t e^{-tL}|_{t=0}f = -Lf$. Letting $t \to 0^+$ in the inequality above we then find

$$-Q(|f|) = \langle -L|f|, |f|\rangle \geq \langle -Lf, f\rangle = -Q(f).$$

This gives (ii).

(ii) $\Longrightarrow$ (i): This has already been shown in Lemma 1.10 (a). $\square$

Having dealt with the positivity preserving part of the Markov property, we are now going to characterize the full Markov property.

Theorem 1.19 (Second Beurling–Deny criterion). *Let (X, m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with associated matrix l and form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *The matrix elements of the operator L satisfy, for all $x, y \in X$ with $x \neq y$,*

(*"Matrix"*) $$l(x, y) \leq 0 \quad \textit{and} \quad \sum_{z \in X} l(x, z) \geq 0.$$

(ii) *The form satisfies, for all $f \in \ell^2(X, m)$,*

(*"Form"*) $$Q(C_{[0,1]} \circ f) \leq Q(f).$$

(iii) *The semigroup satisfies, for all $t \geq 0$ and $f \in \ell^2(X, m)$, $0 \leq f \leq 1$,*

(*"Semigroup"*) $$0 \leq e^{-tL}f \leq 1.$$

PROOF. (i) $\Longleftrightarrow$ (ii): This was already shown in Theorem 1.11.

(i) $\Longleftrightarrow$ (iii): The equivalence of $l(x,y) \le 0$ for $x \neq y$ and the semigroup being positivity preserving was already shown in Theorem 1.18. For the remaining part, we start with a preliminary consideration. Set $f := L1$ so that the second inequality of (i) is equivalent to $f \ge 0$. Consider now the function u definied by $u_t := e^{-tL}1$ for $t \ge 0$. This function satisfies $u_0 = 1$ and

$$\partial_t u_t = -Le^{-tL}1 = -e^{-tL}L1 = -e^{-tL}f$$

for all $t \ge 0$. In particular,

$$\lim_{t\to 0^+} \frac{1}{t}(u_t - u_0) = \partial_t u_t|_{t=0} = -f.$$

We now turn to proving the desired equivalence. If (i) holds, then u satisfies $u_0 = 1$ and $\partial_t u_t = -e^{-tL}f \le 0$, where the last inequality follows as $(e^{-tL})_{t\ge 0}$ is positivity preserving and $f \ge 0$ due to (i). This shows that $t \mapsto u_t$ is non-increasing and gives

$$e^{-tL}1 \le 1 \quad \text{for all} \quad t \ge 0.$$

Now, let $f \in \ell^2(X,m)$, $0 \le f \le 1$. Then the inequality above implies

$$0 \le e^{-tL}f \le e^{-tL}1 \le 1$$

for all $t \ge 0$, as (e^{-tL}) is positivity preserving. This shows (iii).

Conversely, if (iii) holds, then we infer

$$-L1 = \partial_t e^{-tL}1|_{t=0} = \lim_{t\to 0^+} \frac{1}{t}\left(e^{-tL} - 1\right) \le 0$$

from which $\sum_{z\in X} l(x,z) \ge 0$ follows. $\square$

We now conclude this section with a characterization of the validity of the Markov property via graphs.

THEOREM 1.20 (Characterization of the Markov property). *Let (X,m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with associated form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *There exists a graph (b,c) over (X,m) with*

(*"Graph"*) $$Q = Q_{b,c} \quad \text{and} \quad L = L_{b,c,m}.$$

(ii) *The semigroup $(e^{-tL})_{t\ge 0}$, satisfies the Markov property, i.e.,*

(*"Semigroup"*) $$0 \le e^{-tL}f \le 1 \quad \text{for all} \quad 0 \le f \le 1,\ t \ge 0.$$

PROOF. The statement follows by combining the second Beurling–Deny criterion in Theorem 1.19, with Lemma 1.6. $\square$

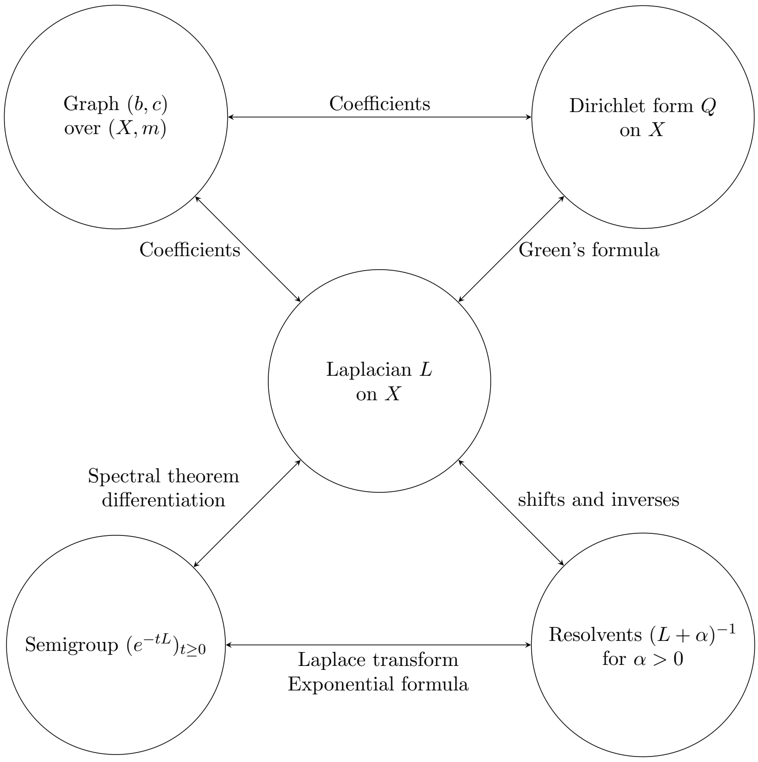
As discussed in Section 1.1, semigroups and resolvents associated to self-adjoint operators share many features. One of these turns out to be validity of the Markov property. The resolvent has the Markov property if and only if the semigroup has the Markov property.

COROLLARY 1.21. *Let (X,m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with non-negative eigenvalues. Then, the following statements are equivalent:*

(i) *For all* $t \geq 0$ *and all* $f \in \ell^2(X,m)$ *with* $0 \leq f \leq 1$,
$$0 \leq e^{-tL} f \leq 1.$$
(ii) *For all* $\alpha > 0$ *and all* $f \in \ell^2(X,m)$ *with* $0 \leq f \leq 1$,
$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1.$$
(iii) *There exists a graph* (b,c) *over* (X,m) *with* $L = L_{b,c,m}$.

Proof. The equivalence between (i) and (ii) follows easily from the formulae given in Lemma 1.3. The equivalence between (i) and (iii) was shown in Theorem 1.20. □

We can now visualize all the relations between the relevant objects in the following figure.



Remark 1.22 (Stochastic completeness). Let (b,c) be a graph over (X,m) and L the associated Laplacian. Then the Markov property implies $e^{-tL}1 \leq 1$ for all $t \geq 0$. The question whether actually $e^{-tL}1 = 1$ holds for all t is an interesting one. The graph is called *stochastically complete* if $e^{-tL}1 = 1$ holds for all $t \geq 0$. We will have a much closer look on this phenomenon in the case of infinite X. In the case considered here, where X is finite, it turns out that $e^{-tL}1 = 1$ holds if and only if $c = 0$. Indeed, $t \mapsto e^{-tL}1$ is the unique solution of the heat equation
$$\partial_t u_t = -Lu_t, \quad u_0 = 1.$$
Now, for $c = 0$ we have $L1 = 0$ and $u = 1$ is clearly a solution of that equation and from uniqueness $u_t = e^{-tL}1 = 1$ follows for all $t \geq 0$. Conversely, for $c \neq 0$ we have $L1 = c \neq 0$. In this case $u = 1$ is not a solution of the heat equation. Hence, $e^{-tL}1$ can not be identically to 1 for all t. A further

analysis then shows that $e^{-tL}1$ is strictly less than 1 for all $t > 0$ on each connected component on which c does not vanish.

1.6. Connectedness and Large Time Behaviour

In this section we consider the behaviour of e^{-tL} for large t. This is intimately related to the behaviour of L at the bottom of the spectrum. Let (X, m) be a finite set measure space.

For a function $f \colon X \longrightarrow \mathbb{R}$ we write $f > 0$ provided $f(x) > 0$ for all $x \in X$.

Definition 1.23 (Positivity improving). Let $A \colon \ell^2(X, m) \longrightarrow \ell^2(X, m)$ be an operator on $\ell^2(X, m)$, i.e. A is linear. Then A is called *positivity improving* if $Af > 0$ holds for all $f \geq 0$ with $f \neq 0$.

Proposition 1.24 (Characterization of positivity improving semigroups and resolvents). *Let (b, c) be a graph over (X, m) with associated Laplacian $L = L_{b,c,m}$. Then, the following statements are equivalent:*

(i) *The semigroup operator e^{-tL} is positivity improving for one (all) $t > 0$.*
(ii) *The resolvent $(L + \alpha)^{-1}$ is positivity improving for one (all) $\alpha > 0$.*
(iii) *The graph (b, c) is connected.*

Proof. (i) $\Longrightarrow$ (ii): This follows immediately from the Laplace transform, i.e. from the fact that $(L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} \mathrm{d}t$, which is shown in Lemma 1.3 (b).

(ii) $\Longrightarrow$ (iii): Suppose that (b, c) is not connected so that there exists a non-empty connected component U of X with $U \neq X$. Write m_A for the restriction of m to the set $A \subseteq X$. We identify $\ell^2(X, m)$ with

$$\ell^2(U, m_U) \oplus \ell^2(X \setminus U, m_{X \setminus U})$$

and write elements of the latter space as (f, g).

As U is not connected to $X \setminus U$, the operator L can be decomposed as

$$L = L_U \oplus L_{X \setminus U},$$

where L_U is the restriction of L to $\ell^2(U, m_U)$ and $L_{X\setminus U}$ the restriction of L to $\ell^2(X \setminus U, m_{X \setminus U})$. It follows that

$$(L+\alpha)^{-1} = (L_U + \alpha)^{-1} \oplus (L_{X \setminus U} + \alpha)^{-1}.$$

Let $f \in \ell^2(U, m_U)$ be positive and non-trivial. Then we clearly have that $(f, 0) \in \ell^2(U, m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$ is positive and non-trivial but

$$(L+\alpha)^{-1}(f, 0) = ((L_U + \alpha)^{-1} f, (L_{X\setminus U} + \alpha)^{-1} 0) = ((L_U + \alpha)^{-1} f, 0)$$

is not strictly positive. Hence $(L + \alpha)^{-1}$ is not positivity improving.

(iii) $\Longrightarrow$ (i): Let $f \geq 0$ with $f \neq 0$. Let $u \colon [0, \infty) \times X \longrightarrow [0, \infty)$ be defined by

$$u(t, x) := u_t(x) := e^{-tL} f(x).$$

By Corollary 1.21 we have $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in X$. We wish to show that $u_t(x) > 0$ for all $t > 0$ and $x \in X$.

Assume that $u_{t_0}(x_0) = 0$ for some $t_0 > 0$ and some $x_0 \in X$. Then, $t \mapsto u_t(x_0)$ has a minimum at t_0. Thus,

$$\partial_t u_{t_0}(x_0) = 0.$$

As u_t solves $\partial_t u_t = -Lu_t$, this implies

$$\begin{aligned}
0 &= Lu_{t_0}(x_0) \\
&= \frac{1}{m(x_0)} \sum_{y \in X} b(x_0, y)(u_{t_0}(x_0) - u_{t_0}(y)) + \frac{c(x_0)}{m(x_0)} u_{t_0}(x_0) \\
&= -\frac{1}{m(x_0)} \sum_{y \in X} b(x_0, y) u_{t_0}(y).
\end{aligned}$$

By $u \geq 0$ we conclude $u_{t_0}(y) = 0$ for all $y \sim x_0$. By connectedness of the graph, we obtain inductively that $u_{t_0} = 0$. This gives the contradiction $f = e^{t_0 L} u_{t_0} = 0$. □

Lemma 1.25 (Speed of convergence). *Let L be a self-adjoint operator on $\ell^2(X, m)$. Let λ_0, λ_1 be the smallest and second smallest eigenvalues of L, respectively, and let $\alpha := \lambda_1 - \lambda_0 > 0$. If E_0 is the orthogonal projection onto the eigenspace of λ_0, then*

$$\|e^{\lambda_0 t} e^{-tL} - E_0\| \leq e^{-\alpha t} \quad \textit{for all} \quad t \geq 0.$$

In particular,

$$\|e^{-tL} - E_0\| \leq e^{-\lambda_1 t} \quad \textit{for all} \quad t \geq 0$$

if $\lambda_0 = 0$.

Proof. We write $L = \sum_{j=0}^n \lambda_j E_j$ with pairwise different eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ of L and E_j the associated pairwise orthogonal spectral projections onto the eigenspaces. Then,

$$e^{\lambda_0 t} e^{-tL} = E_0 + \sum_{j=1}^n e^{-t(\lambda_j - \lambda_0)} E_j.$$

From this we derive

$$\|e^{\lambda_0 t} e^{-tL} - E_0\| \leq e^{-(\lambda_1 - \lambda_0)t}$$

as follows: Let $f \in \ell^2(X, m)$. We use the fact that the E_j are pairwise orthogonal twice to get

$$\begin{aligned}
\|(e^{\lambda_0 t} e^{-tL} - E_0) f\|^2 &= \sum_{j,k=1}^n e^{-t(\lambda_j - \lambda_0)} e^{-t(\lambda_k - \lambda_0)} \langle E_j f, E_k f \rangle \\
&\overset{E_j \text{ pw. orth.}}{=} \sum_{j=1}^n e^{-2t(\lambda_j - \lambda_0)} \|E_j f\|^2 \\
&\leq e^{-2\alpha t} \sum_{j=0}^n \|E_j f\|^2 \\
&\overset{E_j \text{ pw. orth.}}{=} e^{-2\alpha t} \|\sum_{j=0}^n E_j f\|^2 \\
&= e^{-2\alpha t} \|f\|^2.
\end{aligned}$$

Since this holds for all $f \in \ell^2(X,m)$, taking square roots yields the conclusion. $\square$

The result above shows that $(e^{\lambda_0 t}e^{-tL})_{t\geq 0}$ converges exponentially to E_0, the orthogonal projection onto the eigenspace of λ_0. In particular, if $\lambda_0 = 0$, we get that the semigroup $(e^{-tL})_{t\geq 0}$ converges exponentially to E_0.

We will now investigate the properties of E_0 in the case when the graph is connected. The following result is known as the Perron–Frobenius theorem. It states that the eigenspace of λ_0 is of dimension one and consists of functions of a fixed sign.

We recall that by the variational characterization of the bottom of the spectrum we have

$$\lambda_0 = \inf Q(f),$$

where the infimum is taken over all $f \in \ell^2(X,m)$ with $\|f\| = 1$.

THEOREM 1.26 (Perron–Frobenius). *Let (b,c) be a connected graph over a finite set measure space (X,m). Let $L = L_{b,c,m}$ be the associated Laplacian with form $Q = Q_{b,c}$ and let λ_0 be the smallest eigenvalue of L with E_0 the associated orthogonal projection. Then, the eigenspace of λ_0 is one-dimensional and there exists a unique normalized strictly positive eigenfunction u corresponding to λ_0 with*

$$E_0 f = \langle u, f\rangle u$$

for all $f \in \ell^2(X,m)$.

PROOF. We first note the following general fact.

Claim. A normalized function u is an eigenfunction corresponding to λ_0 if and only if $Q(u) = \lambda_0$.

Proof of the claim. If $Lu = \lambda_0 u$ with $\|u\| = 1$, then $Q(u) = \langle Lu, u\rangle = \lambda_0 \|u\|^2 = \lambda_0$.

Conversely, let u be normalized with $Q(u) = \lambda_0$. Let $\lambda_0 < \ldots < \lambda_n$ denote the eigenvalues of L. Writing $L = \sum_{j=0}^n \lambda_j E_j$, we note that

$$\lambda_0 = Q(u) = \langle u, Lu\rangle = \langle u, \sum_{j=0}^n \lambda_j E_j u\rangle = \sum_{j=0}^n \lambda_j \|E_j u\|^2$$

with $\sum_{j=0}^n \|E_j u\|^2 = \|u\|^2 = 1$. This shows $E_j u = 0$ for $j \geq 1$ and $E_0 u = u$, so that $Lu = \lambda_0 u$.

We now show that any eigenfunction corresponding to λ_0 is either strictly positive or strictly negative:

Let u be a normalized eigenfunction corresponding to λ_0. Then,

$$\lambda_0 \leq Q(|u|) \leq Q(u) = \lambda_0.$$

Here, we used the variational characterization of λ_0 in the first inequality and that Q is a Dirichlet form in the second inequality. Therefore,

$$\lambda_0 = Q(|u|).$$

As $|u|$ is normalized as well, we infer that $|u|$ is also an eigenfunction corresponding to λ_0 by the claim.

We now write $u = u_+ - u_-$, where $u_+ = u \vee 0$ and $u_- = -u \vee 0$, so that $|u| = u_+ + u_-$. Then

$$u_+ = \frac{1}{2}(|u| + u) \quad \text{and} \quad u_- = \frac{1}{2}(|u| - u)$$

are also eigenfunctions corresponding to λ_0 (or vanish identically). Assume, without loss of generality, that $u_+ \neq 0$. As e^{-tL} is positivity improving for all $t > 0$ by Proposition 1.24, we infer

$$0 < e^{-L}u_+ = e^{-\lambda_0}u_+.$$

This implies

$$u_+ > 0 \quad \text{and} \quad u_- = 0.$$

These considerations show that any eigenfunction corresponding to λ_0 has a strict sign. We conclude that the eigenspace of λ_0 is one-dimensional as eigenfunctions with a strict sign cannot be orthogonal to one another.

Now, as the eigenspace of λ_0 is one-dimensional, we then obtain

$$E_0 f = \langle u, f \rangle u$$

for any normalized eigenfunction u and $f \in \ell^2(X, m)$. Hence, any normalized strictly positive u has the desired properties and is uniquely determined by these properties. $\square$

Definition 1.27 (Ground state and ground state energy). Let (b, c) be a connected graph over (X, m) with associated Laplacian $L = L_{b,c,m}$. The smallest eigenvalue λ_0 of L is called the *ground state energy* and the normalized positive eigenfunction u corresponding to λ_0 is called the *ground state*.

We also introduce the heat kernel, which arises from the heat semigroup $(e^{-tL})_{t \geq 0}$.

Definition 1.28 (Heat kernel). Let (b, c) be a graph over (X, m) with associated Laplacian $L = L_{b,c,m}$. The map

$$p \colon [0, \infty) \times X \times X \longrightarrow [0, \infty)$$

defined by

$$e^{-tL} f(x) = \sum_{y \in X} p_t(x, y) f(y) m(y)$$

for all $t \geq 0$, $f \in \ell^2(X, m)$ and $x \in X$ is called the *heat kernel*.

Theorem 1.29 (Convergence to the ground state and ground state energy). *Let (b, c) be a connected graph over (X, m). Let $L = L_{b,c,m}$ be the associated Laplacian with ground state energy λ_0, ground state u and heat kernel p. Let $\lambda_1 > \lambda_0$ be the second smallest eigenvalue of L and let $\alpha := \lambda_1 - \lambda_0$.*

(a) *For all $x, y \in X$,*

$$|e^{\lambda_0 t} p_t(x, y) - u(x)u(y)| \leq \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}.$$

(*"Theorem of Chavel–Karp for finite graphs"*)

(b) *For all $x, y \in X$,*

$$\lim_{t\to\infty} \frac{1}{t} \log p_t(x, y) = -\lambda_0.$$

(“Theorem of Li for finite graphs”)

PROOF. To prove (a), first observe that for any $f \in \ell^2(X, m)$ we have $|f(x)| \leq \|f\|/\sqrt{m(x)}$ for all $x \in X$. Now, the formula for E_0 in Theorem 1.26 gives, for all $x, y \in X$, that $E_0 1_y(x)/m(y) = u(x)u(y)$ while $p_t(x, y) = e^{-tL} 1_y(x)/m(y)$ by definition. From Lemma 1.25 we then obtain

$$\begin{aligned}
|e^{\lambda_0 t} p_t(x, y) - u(x)u(y)| &= \frac{|e^{\lambda_0 t} e^{-tL} 1_y(x) - E_0 1_y(x)|}{m(y)} \\
&\leq \frac{\|e^{\lambda_0 t} e^{-tL} - E_0\| \|1_y\|}{m(y)\sqrt{m(x)}} \\
&\leq \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}.
\end{aligned}$$

This gives (a).

To prove (b), note from the above that

$$u(x)u(y) - \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}} \leq e^{\lambda_0 t} p_t(x, y) \leq u(x)u(y) + \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}$$

for all $x, y \in X$. As u is strictly positive by Theorem 1.26, (b) follows after taking logarithms for large t, dividing by t and letting $t \to \infty$. □

1.7. The Dirichlet Problem

In this section we discuss some further aspects of the theory. Throughout we assume that X is a finite set and the measure m is just the counting measure and we remove it from notation. We leave it as an exercise to include a measure in the considerations.

LEMMA 1.30 (Non-vanishing c characterizes the bijectivity of $L_{b,c}$). *Let (b, c) be a graph over a finite set X and let $L_{b,c}$ be the associated Laplacian on $\ell^2(X)$. The operator $L_{b,c}$ is bijective if and only if c does not vanish identically on any connected component of (b, c).*

PROOF. As $L_{b,c}$ is a linear operator on a finite dimensional vector space, bijectivity is equivalent to injectivity. Furthermore, we can assume without loss of generality that the graph is connected.

If $c = 0$, then clearly $L_{b,c} 1 = 0$. Therefore, $L_{b,c}$ is not injective in this case.

Now, suppose that c does not vanish at all $x \in X$. Let $u \in \ell^2(X)$ satisfy $L_{b,c} u = 0$. Green’s formula, Proposition 1.14, gives

$$\begin{aligned}
0 &= \sum_{x\in X} u(x) L_{b,c} u(x) = Q_{b,c}(u) \\
&= \frac{1}{2} \sum_{x,y\in X} b(x, y)(u(x) - u(y))^2 + \sum_{x\in X} c(x) u^2(x).
\end{aligned}$$

As all terms appearing in the sums are non-negative, we infer $u(x) = u(y)$ whenever $b(x, y) > 0$ and $u(x) = 0$ whenever $c(x) \neq 0$. As the graph is connected, the first set of conditions implies u is constant and the second set of conditions implies $u = 0$ as c does not vanish identically. Therefore, $L_{b,c}$ is injective. $\square$

THEOREM 1.31 (The Dirichlet problem). *Let (b, c) be a connected graph over a finite set X. Let $B \subseteq X$ with $B \neq \emptyset$, $A := X \setminus B$ and $g\colon B \longrightarrow \mathbb{R}$. Then, the Dirichlet problem* (DP)*:*

- $L_{b,c}u = 0$ *on* A
- $u = g$ *on* B

has a unique solution. Moreover, for the set

$$\mathcal{A}_g := \{h \in C(X) \mid h = g \text{ on } B\}$$

and $f \in \mathcal{A}_g$ the following statements are equivalent:

(i) $Q_{b,c}(f) = \min\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}$.
(ii) *The function f solves the Dirichlet problem* (DP)*.*

In particular, there exists a unique minimizer in (i)*. Moreover, if $0 \leq g \leq 1$, then $0 \leq f \leq 1$.*

PROOF. We will show a series of claims which will prove the theorem (and a bit more).

Claim 1. The solution of (DP) exists and is unique.

Proof of Claim 1. We transform the problem to an equivalent problem for which we will establish existence and uniqueness. Let f be a solution of $L_{b,c}f = 0$ on A with $f = g$ on B, that is, let f solve (DP). For any $x \in A$, we then have

$$\begin{aligned}
0 &= L_{b,c}f(x)\\
&= \sum_{y\in X} b(x,y)(f(x) - f(y)) + c(x)f(x)\\
&= \sum_{y\in A} b(x,y)(f(x) - f(y)) + \sum_{y\in B} b(x,y)(f(x) - f(y)) + c(x)f(x)\\
&= \sum_{y\in A} b(x,y)(f(x) - f(y)) + \Big(c(x) + \sum_{y\in B} b(x,y)\Big)f(x) - \sum_{y\in B} b(x,y)g(y)\\
&= \sum_{y\in A} b(x,y)(f(x) - f(y)) + d(x)f(x) - h(x)
\end{aligned}$$

with

$$d(x) := c(x) + \sum_{y\in B} b(x,y) \quad \text{and} \quad h(x) := \sum_{y\in B} b(x,y)g(y).$$

Note that both d and h do not depend on f.

We let $L_A^{(D)} := L_{b_A,d}$, which we call the *Dirichlet Laplacian* associated to the graph (b_A, d) over A, given by $b_A(x, y) := b(x, y)$ for $x, y \in A$, d as above and the restriction f_A of f to A, we obtain from the above that

(P) $L_A^{(D)} f_A = h$ on A.

Now, if f is a solution of (DP), then f_A solves (P), as shown by the above calculation. Conversely, any solution $\widetilde{f}$ of (P) becomes a solution f to (DP) after extending $\widetilde{f}$ by g on B. This gives:

$$f \text{ solves (DP)} \quad \Longleftrightarrow \quad f_A \text{ solves (P)}.$$

Therefore, it suffices to show that (P) has a unique solution, that is, $L_A^{(D)}$ is bijective. By construction, $L_A^{(D)}$ is the Laplacian associated to the graph (b_A, d) over A. Thus, by Lemma 1.30, it suffices to show that d does not vanish on any connected component of A, where the connected components are defined with respect to b_A. Let Z be such a connected component. Invoking the definition of d, it suffices to find $x \in Z$ and $y \in B$ with $b(x, y) > 0$. First, we choose an arbitrary $y' \in B$ and $o \in Z$. As the graph is connected there exists a path $(x_0, x_1, \ldots, x_n)$ in (X, b) with $x_0 = o$ and $x_n = y'$. Let j be the smallest index such that x_j does not belong to Z. Then, letting $y := x_j$, y belongs to B as otherwise it would belong to Z since Z is a connected component. Thus, $x := x_{j-1} \in Z$ and $y = x_j \in B$ satisfy $b(x, y) > 0$. This finishes the proof of Claim 1.

Claim 2. Any minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ solves (DP).

Proof of Claim 2. Suppose that there exists an $f \in \mathcal{A}_g$ with

$$Q_{b,c}(f) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

Let φ be an arbitrary function supported on A. Then, $f + \lambda\varphi$ belongs to $\mathcal{A}_g$ for all $\lambda \in \mathbb{R}$. Thus, the function

$$\lambda \mapsto Q_{b,c}(f + \lambda\varphi) = Q_{b,c}(f) + 2\lambda Q_{b,c}(f, \varphi) + \lambda^2 Q_{b,c}(\varphi)$$

has a minimum at $\lambda = 0$. Taking the derivative at $\lambda = 0$ yields

$$0 = Q_{b,c}(f, \varphi) = \sum_{x \in X} L_{b,c} f(x)\varphi(x)$$

by Green's formula, Proposition 1.14. As φ supported in A was arbitrary, we conclude that $L_{b,c} f = 0$ on A.

Claim 3. There exists a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$.

Proof of Claim 3. Let (f_n) be a sequence in $\mathcal{A}_g$ with

$$\lim_{n\to\infty} Q_{b,c}(f_n) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

It follows that $(Q_{b,c}(f_n))$ is a bounded sequence. Let o be an arbitrary point in B. Then, $f_n(o) = g(o)$ for all $n \in \mathbb{N}$ as $f_n \in \mathcal{A}_g$. As we will show below, the boundedness of $(Q_{b,c}(f_n))$ together with the boundedness of $(f_n(o))$ implies that $(f_n(x))$ is bounded for any $x \in X$. By choosing a suitable subsequence we can, without of loss of generality, assume that (f_n) converges pointwise to a function f. Obviously, $f \in \mathcal{A}_g$ and

$$Q_{b,c}(f) = Q_{b,c}\left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} Q_{b,c}(f_n) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

Thus, f is a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$.

It remains to show the desired boundedness of $(f_n(x))$ for $x \in X$. Let $x \in X$ and let $\gamma := (x_0, \ldots, x_m)$ with $x_0 = o$ and $x_m = x$ be a path from o to x. Then, for any function u, we have by the Cauchy–Schwarz inequality

$$|u(x) - u(o)|$$

$$\begin{aligned}
&\leq \sum_{j=0}^{m-1} |u(x_j) - u(x_{j+1})| \\
&= \sum_{j=0}^{m-1} |u(x_j) - u(x_{j+1})| b(x_j, x_{j+1})^{1/2} \cdot \frac{1}{b(x_j, x_{j+1})^{1/2}} \\
&\leq \left(\sum_{j=0}^{m-1} (u(x_j) - u(x_{j+1}))^2 b(x_j, x_{j+1}) \right)^{1/2} \left(\sum_{j=0}^{m-1} b(x_j, x_{j+1})^{-1} \right)^{1/2} \\
&\leq Q_{b,c}(u)^{1/2} C(\gamma)
\end{aligned}$$

with $C(\gamma) := \left(\sum_{j=1}^{m} b(x_j, x_{j+1})^{-1} \right)^{1/2}$. Applying this to f_n and noting that $f_n(o) = g(o)$ for all n since $o \in B$, we get

$$|f_n(x) - g(o)| \leq C(\gamma) Q_{b,c}(f_n)^{1/2}.$$

As $(Q_{b,c}(f_n))_n$ is bounded and $C(\gamma)$ does not depend on n, it follows that $(f_n(x))_n$ is bounded.

Claim 4. If $0 \leq g \leq 1$, then $0 \leq f \leq 1$.

Proof of Claim 4. Recall that $C_{[0,1]} \circ f = 0 \vee f \wedge 1$. If $f \in \mathcal{A}_g$, then $C_{[0,1]} \circ f \in \mathcal{A}_g$ since $C_{[0,1]} \circ g = g$. Therefore, $C_{[0,1]} \circ f$ is also a minimizer of $Q_{b,c}$ as $Q_{b,c}$ is a Dirichlet form and thus $Q_{b,c}(C_{[0,1]} \circ f) \leq Q_{b,c}(f)$. The already proven uniqueness then gives $f = C_{[0,1]} \circ f$, which is equivalent to $0 \leq f \leq 1$.

By combining the preceding statements we now prove the theorem: Claim 1 yields the existence and uniqueness of solutions to (DP). Claim 2 shows the implication (i) $\Longrightarrow$ (ii). Furthermore, in Claim 3, we have shown the existence of a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$. We next turn to (ii) $\Longrightarrow$ (i): The solution of (DP) and the minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ both exist and are unique by the considerations above. As the minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ solves (DP) by Claim 2, it coincides with the unique solution of (DP). Thus, this unique solution minimizes $Q_{b,c}$ on $\mathcal{A}_g$. Finally, the last statement of the theorem follows from Claim 4. $\square$

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Sheet 2

Finite Graphs II

Exercise 1 (Positivity improvement of the inverse operator) 4 points

Let X be a finite set and let L be an injective operator on $C(X)$. Show that the following assertions are equivalent:

(i) The inverse operator L^{-1} is positivity improving, i. e. for all $f \in C(X)$ such that $f \geq 0$ and $f \neq 0$, we have $L^{-1}f > 0$.

(ii) For each function $u \in C(X)$ satisfying the inequalities $\max_{x\in X} u(x) \geq 0$ and $Lu \leq 0$, we have $u \equiv 0$.

Exercise 2 (Cauchy problem / Heat equation) 4 points

Let (X, m) be a finite measure space and let L be a self-adjoint operator on $\ell^2(X, m)$ and for $t \geq 0$ let e^{-tL} be defined via spectral calculus.

a) Show that for all $t \geq 0$,

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{1}{n!}(-tL)^n$$

In particular, show that the sum absolutely convergent with respect to the operator norm.

b) Show that $\{e^{-tL} \mid t \geq 0\}$, equipped with the composition of operators, is an operator semigroup, i.e., $e^{0L} = I$ and $e^{(t+s)L} = e^{tL}e^{sL}$ for all $t,s \geq 0$ and $t \mapsto e^{-tL}f$ is continuously differentiable at $t = 0$ for all $f \in \ell^2(X,m)$. Moreover, show that (in this finite dimensional case)

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-tL} = -Le^{-tL} = -e^{-tL}L.$$

c) Show that for all $f \in \ell^2(X, m)$, the function $t \mapsto \varphi_t := e^{-tL}f$ is the unique solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = -L\varphi_t, \quad \varphi_0 = f,$$

for all $t \geq 0$.

Exercise 3 (Stochastic incompleteness) 4 points

Let (b,c) be a connected graph over (X,m) and let $L = L_{b,c,m}$ denote the associated Laplacian.

(a) Show that $e^{-tL}1 < 1$ for all $t > 0$ if and only if $c \neq 0$.

(b) Show that if $e^{-tL}1 < 1$ for some $t > 0$, then $e^{-tL}1 < 1$ for all $t > 0$.

Exercise 4 (Effective resistance) 4 points

Let b be a graph over a finite set X, let $Q = Q_b$ be the associated form and let

$$W_{\text{eff}}(x,y) := \sup\left\{\frac{1}{Q(h)} \mid h \in C(X), h(x) - h(y) = 1\right\}$$

be the *effective resistance* for $x \neq y$ and $W_{\text{eff}}(x,x) = 0$.

a) Prove the following equation

$$W_{\text{eff}}(x,y) = \max\{|f(x) - f(y)|^2 \mid Q(f) \leq 1\}.$$

b) Show that

$$\varrho : X \times X \to [0,\infty), \quad (x,y) \mapsto W_{\text{eff}}(x,y)^{1/2}$$

defines a metric on the graph.

Bonus Exercise 1 (Effective resistance II) 1 point

Given the assumptions of Exercise 4, show that

$$\varrho : X \times X \to [0,\infty), \quad (x,y) \mapsto W_{\text{eff}}(x,y)$$

defines a metric on the graph, as well.