26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

Matthias Keller, Daniel Lenz, Marcel Schmidt and Christian Seifert

Lecture 01

Preface

Most of the material presented here follows closely the monograph

"Graphs and Discrete Dirichlet Spaces"

by Keller, Lenz and Wojciechowski. This monograph appeared within the Springer series *Grundlehren der mathematischen Wissenschaften* and a version can be found at Matthias Keller's webpage

`https://www.math.uni-potsdam.de/professuren/graphentheorie/team/prof-dr-matthias-keller/`.

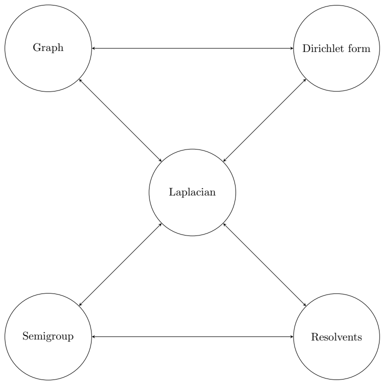
Specifically, each chapter of this monograph ends with a section called "Notes". Within these sections historical notes and references to the literature can be found there. For the lectures with the ISem 26 we therefore refer to these notes sections and provide a tabular below referencing the corresponding chapters in the lecture to the chapters in the book.

ISem 26	Monograph
Chapter 1	Chapter 0
Chapter 2	Chapter 1
Chapter 3	Appendix A,B,E
Chapter 4	Chapter 1
Chapter 5	Chapter 1, Appendix C
Chapter 6	Chapter 5
Chapter 7	Chapter 11 and 13
Chapter 8	Chapter 4
Chapter 9	Chapter 6
Chapter 10	Chapter 7
Chapter 11	Chapter 9

CHAPTER 1

Finite Graphs – The Theory in a Sandbox

Our topic deals with graphs, Dirichlet forms, Laplacians and Markovian semigroups and Markovian resolvents. It turns out that these five types of objects are in one-to-one correspondence to each other. This is the core of the theory.



In this chapter we present the theory in the situation of finite gaphs, i.e. graphs with finitely many vertices. The relevant vector spaces then become finite dimensional and the necessary operator theory is provided by linear algebra. This makes the considerations particularly accessible. The purpose of this chapter is twofold:

- The chapter introduces and discusses key topics of the course in a particularly simple situation.
- Results of this chapter are of direct use later in the context of approximation of the infinite dimensional situation by finite dimensional situations.

We also note in passing that finite graphs and their Laplacians, which are discussed in this chapter, are a topic of interest in itself.

1.1. Linear Algebra or Forms, Matrices Operators, Resolvents and Semigroups

In this section we present the necessary background from linear algebra for our considerations. At heart our theory is concerned with real valued functions (as these model diffusion processes and the like) and therefore all our vector spaces will be over the reals and all functions are real valued.

We consider a finite set X. We think of X as equipped with the discrete topology. Hence all functions on X are continuous. We denote by $C(X)$ the vector space of all (continuous) functions $f\colon X \longrightarrow \mathbb{R}$. For $x \in X$ we denote by 1_x the characteristic function of $\{x\}$. Thus, 1_x takes the value 1 at x and is 0 otherwise.

A *form* on $C(X)$ is a map

$$\mathcal{Q}\colon C(X) \times C(X) \longrightarrow \mathbb{R}$$

which is bilinear, i.e., satisfies

$$\mathcal{Q}(\alpha f + g, h) = \alpha\mathcal{Q}(f,h) + \mathcal{Q}(g,h)$$

and

$$\mathcal{Q}(f, \alpha g + h) = \alpha\mathcal{Q}(f,g) + \mathcal{Q}(f,h)$$

for all $f, g, h \in C(X)$ and all $\alpha \in \mathbb{R}$. A form $\mathcal{Q}$ is called *symmetric* if $\mathcal{Q}$ satisfies $\mathcal{Q}(f,g) = \mathcal{Q}(g,f)$ for all $f, g \in C(X)$. For the values of $\mathcal{Q}$ on the diagonal $\{(f,f) \mid f \in C(X)\}$ of $C(X) \times C(X)$ we will use the notation

$$\mathcal{Q}(f) := \mathcal{Q}(f,f).$$

In particular, when $\mathcal{Q}$ is symmetric, we get

$$\mathcal{Q}(f+g) = \mathcal{Q}(f) + 2\mathcal{Q}(f,g) + \mathcal{Q}(g).$$

To each form $\mathcal{Q}$ there exists a unique function $l\colon X \times X \longrightarrow \mathbb{R}$ with

$$\mathcal{Q}(f,g) = \sum_{x,y\in X} l(x,y)f(x)g(y)$$

for all $f, g \in C(X)$. We call $\mathcal{Q}$ the form *induced by* the matrix l and l the *matrix associated* to $\mathcal{Q}$. In particular, $\mathcal{Q}$ is symmetric if and only if the associated matrix l is symmetric. We note

$$\mathcal{Q}(1_x, 1_y) = l(x,y) \text{ and } \mathcal{Q}(1_x, 1) = \sum_{z\in X} l(x,z)$$

for all $x, y \in X$. Here, 1 denotes the function which is constant equal to 1 on X.

In order to make use of inner products and the powerful methods coming with them we will make $C(X)$ into an ℓ^2 space. So, we introduce next measures on discrete sets X. If $m\colon X \longrightarrow (0,\infty)$ is a strictly positive function on X, then we can extend m to a measure on X via

$$m(A) := \sum_{x\in A} m(x)$$

for all subsets $A \subseteq X$. Therefore, the pair (X,m) can be seen as a measure space. Clearly, m has *full support*, i.e. $m(A) > 0$ for all $A \neq \emptyset$. We refer to (X,m) consisting of a finite set X together with a measure of full support as a *finite set measure space*.

Example 1.1 (Counting measure). Let $m = 1$. Then m is called the *counting measure* on X. In this case, the measure of a set $A \subseteq X$ is the number of elements in the set, i.e.,

$$m(A) = \sum_{x\in A} 1.$$

The vector space $C(X)$ equipped with the inner product

$$\langle f, g\rangle := \sum_{x\in X} f(x)g(x)m(x)$$

and the norm

$$\|f\| := \langle f, f\rangle^{1/2}$$

is denoted by $\ell^2(X,m)$. It is finite dimensional and, hence, automatically complete and, therefore, a Hilbert space.

A linear map $L\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ is called an *operator* on $\ell^2(X,m)$. The set of all operators is a vector space. It becomes a normed space with norm given by

$$\|L\| := \sup\{\|Lf\| \mid \|f\| \leq 1\}.$$

The characteristic feature of $\|L\|$ is that any $f \in \ell^2(X,m)$ satisfies

$$\|Lf\| \leq \|L\|\|f\|$$

and $\|L\|$ is the smallest number with this property. Convergence on the space of operators is always understood to refer to convergence with respect to this norm.

Any operator L on $\ell^2(X,m)$ comes with a form Q_L defined by

$$Q_L(f,g) := \langle f, Lg\rangle.$$

We call Q_L the form associated to L.

Remark 1.2. Note that we write Q whenever we will interpret a form on $\ell^2(X,m)$ instead of $\mathcal{Q}$, the form on $C(X)$. In case X is finite we have $Q = \mathcal{Q}$ since $C(X) = \ell^2(X,m)$ as vector spaces. This will become different for infinite X, which we will see in the upcoming chapters.

An operator L on $\ell^2(X,m)$ is called *self-adjoint* if L satisfies

$$\langle Lf, g\rangle = \langle f, Lg\rangle$$

for all $f, g \in \ell^2(X,m)$. Clearly, L is self-adjoint if and only if its form is symmetric.

Whenever L is an operator and Q_L is the associated form we call the matrix l induced by Q_L the *matrix of the operator*. Thus,

$$\langle f, Lg\rangle = \sum_{x,y\in X} f(x)l(x,y)g(y)$$

holds for all $f, g \in \ell^2(X,m)$. In particular, l can be recovered from L by

$$l(x,y) = \langle 1_x, L1_y\rangle$$

and a direct calculation gives

$$(Lg)(x) = \frac{1}{m(x)} \sum_{y\in X} l(x,y)g(y)$$

for all $g \in \ell^2(X,m)$. So, L is uniquely determind by l and we call L the *operator induced by the matrix l*. Clearly, L is self-adjoint on $\ell^2(X,m)$ if and only if the matrix l associated to L is symmetric.

From these considerations we infer that there is a one-to-one correspondence between self-adjoint operators, symmetric forms and symmetric matrices.

Let now a self-adjoint L on $\ell^2(X,m)$ be given. An $e \in \ell^2(X,m)$ with $e \neq 0$ is called an *eigenvector* of L if

$$Le = \lambda e$$

holds for some $\lambda \in \mathbb{R}$ (note that self-adjointness implies that λ can only be real). The number λ is then called an *eigenvalue*. Eigenvectors to a fixed eigenvalue form a subspace of $\ell^2(X,m)$ called the *eigenspace* of this eigenvalue.

The set of all eigenvalues of L is called the *spectrum* of L and denoted by $\sigma(L)$. By basic results in linear algebra the operator L can be diagonalized, i.e. there exist pairwise orthogonal normalized eigenvectors $e_0, \ldots, e_{\#X-1}$ to eigenvalues $\lambda_0, \ldots, \lambda_{\#X-1}$ of L. Here, $\#X$ denotes the number of elements of X. The pairwise orthogonal normalized eigenvectors $e_0, \ldots, e_{\#X-1}$ form an orthonormal basis of $\ell^2(X,m)$, i.e. any $f \in \ell^2(X,m)$ can be represented as

$$f = \sum_{j=0}^{\#X-1} \langle e_j, f \rangle e_j,$$

and

$$Lf = \sum_{j=0}^{\#X-1} \lambda_j \langle e_j, f \rangle e_j$$

holds (as each e_j is an eigenvector to λ_j). If we take together all the $j \in \{0, \ldots, \#X - 1\}$ whose λ_j agree we can exhibit these formulae in a more succinct way as follows: For $\lambda \in \sigma(A)$ let

$$E_\lambda := \sum_{j:\lambda_j = \lambda} \langle e_j, \cdot \rangle e_j$$

be the orthogonal projection onto the eigenspace of λ. Then, the mutual orthogonality of the e_j, the preceding formula for f and the preceding formula for Lf easily give

- $E_\lambda E_\mu = 0$ for $\lambda \neq \mu$.
- $I = \sum_{\lambda \in \sigma(L)} E_\lambda$.
- $L = \sum_{\lambda \in \sigma(L)} \lambda E_\lambda$.

We refer to these formulae as "spectral theorem" in the finite dimensional case. The formulae allow us to define for any function

$$\Phi : \sigma(L) \longrightarrow \mathbb{R}$$

the operator $\Phi(L)$ by

$$\Phi(L) := \sum_{\lambda \in \sigma(L)} \Phi(\lambda) E_\lambda.$$

The map

$$\text{functions on } \sigma(L) \longrightarrow \text{linear operators on } \ell^2(X,m), \quad \Phi \mapsto \Phi(L),$$

is called *spectral calculus*. It has the following features:

- It is linear, i.e. $(\Phi + \lambda\Psi)(L) = \Phi(L) + \lambda\Psi(L)$ holds for all Φ, Ψ and $\lambda \in \mathbb{R}$.
- It is multiplicative, i.e. $(\Phi\Psi)(L) = \Phi(L)\Psi(L)$ holds for all Φ, Ψ.

- It is bounded, i.e. $\|\Phi(L)\| \leq \|\Phi\|_\infty$ holds (where we have $\|\Phi\|_\infty = \max\{|\Phi(\lambda)| \mid \lambda \in \sigma(L)\}$).

Indeed, these features follows easily from the definition and the mutual orthogonality of the E_λ, $\lambda \in \sigma(L)$. From these features, we easily infer that $\Phi_n(L) \to \Phi(L)$ whenever (Φ_n) is a sequence of functions on $\sigma(L)$ converging pointwise to Φ. Indeed, we have

$$\|\Phi(L) - \Phi_n(L)\| = \|(\Phi - \Phi_n)(L)\| \leq \|\Phi - \Phi_n\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Our definition of $\Phi(L)$ is consistent with natural definitions of $\Phi(L)$ for special Φ. We will discuss this in two special instances viz for the resolvents and the semigroup. Resolvents and semigroup are core objects of our further study.

We start by dealing with resolvents. Consider the linear equation — for specific L also known as *Poisson problem* — given by

$$(L + \alpha)u = f$$

for given $f \in \ell^2(X, m)$ and $\alpha \in \mathbb{R}$. By the very definition of $\sigma(L)$, for $\alpha \notin \sigma(L)$ the operator $(L + \alpha)$ is bijective. Its inverse is a linear operator known as the *resolvent* (of L at $-\alpha$). The linear equation above is uniquely solved by

$$u = (L + \alpha)^{-1} f.$$

The resolvent can be computed by spectral calculus as follows. Define $\Phi_{(\alpha)}$ on $\sigma(L)$ by $\Phi_{(\alpha)}(s) := \frac{1}{s+\alpha}$ for $\alpha \notin \sigma(L)$. A short computation gives that

$$\Phi_{(\alpha)}(L) = \sum_{\lambda \in \sigma(L)} \frac{1}{\lambda + \alpha} E_\lambda$$

satisfies

$$\Phi_\alpha(L)(L + \alpha) = I = (L + \alpha)\Phi_{(\alpha)}(L).$$

This gives that $\Phi_{(\alpha)}(L)$ is just the inverse $(L + \alpha)^{-1}$ of $(L + \alpha)$.

Invoking

$$\alpha(1 - \frac{\alpha}{s + \alpha}) \to s \quad \text{as} \quad \alpha \to \infty,$$

for all $s \in \mathbb{R}$ we find from spectral calculus that we can recover L from the resolvents by

$$L = \lim_{\alpha \to \infty} \alpha(I - \alpha(L + \alpha)^{-1}).$$

We now turn to the semigroup. Consider the *Cauchy problem* — for specific L also referred to as *heat equation* — stated as:

$$-Lu = \partial_t u, \qquad u_0 = f.$$

Here, $u \colon [0, \infty) \longrightarrow \ell^2(X, m)$, $t \mapsto u_t$ is called a solution to the Cauchy problem if u is continuous with $u_0 = f$ and

$$\partial_t u_t := \lim_{s \to t} \frac{1}{t - s}(u(t) - u(s))$$

exists for all $t \in (0, \infty)$ and $-Lu_t = \partial_t u_t$ holds for all $t > 0$. Now, for $\Phi^{(t)}$ defined on $\sigma(L)$ by $\Phi^{(t)}(s) := e^{-ts}$ we can use spectral calculus to define

$$\Phi^{(t)}(L) = \sum_{\lambda \in \sigma(L)} e^{-t\lambda} E_\lambda.$$

We set

$$e^{-tL} := \Phi^{(t)}(L).$$

Then, a short computation shows that

$$u_t = e^{-tL} f$$

is a solution to the Cauchy problem (and by standard theory of ordinary differential equations this solution is unique).

A direct computation also gives that

$$e^{-tL} = \sum_{k=0}^{\infty} \frac{(-tL)^k}{k!}$$

is valid. Spectral calculus again gives that we can recover

$$L = \lim_{t \to 0} \frac{1}{t}(I - e^{-tL}).$$

Clearly, $e^{-(t+s)L} = e^{-tL}e^{-sL}$ holds for all $s, t \geq 0$ as well as $e^{-0L} = I$ and for this reason we refer to the family $(e^{-tL})_{t\geq 0}$, as the *semigroup* of L.

In the preceding discussion we have seen two families of operators associated to a self-adjoint L viz the semigroup and the resolvent. These families are — in some sense — equivalent objects. For example, we have already seen that the operator L can be obtained from either family by a limiting procedure. Moreover, one can actually obtain each of these families from the other as shown in the next lemma. The lemma can be understood to give a precise sense in which resolvent and semigroup are equivalent.

The lemma features the integral $\int_0^\infty e^{-t\alpha}e^{-tL}\mathrm{d}t$ over the operator valued function $t \mapsto e^{-t\alpha}e^{-tL} =: A(t)$ (for $\alpha > 0$). There are various ways to make sense out of this integral and they all lead to the same result. One way is to think about the $A(t)$ as matrices (after taking a basis) and then the integral is just the matrix whose entries are the improper Riemann integrals over the corresponding entries of A. Another way is to first define $\int_0^N A(t)\mathrm{d}t$ by taking the limit of Riemann sums and then take the limit $N \to \infty$.

LEMMA 1.3 (Laplace transform, exponential formula). *Let (X, m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with non-negative eigenvalues.*

(a) *For all $\alpha > 0$,*

$$(L + \alpha)^{-1} = \int_0^\infty e^{-t\alpha}e^{-tL}\mathrm{d}t.$$

(*"Laplace transform"*)

(b) *For all $t > 0$,*

$$e^{-tL} = \lim_{n\to\infty} \left(\frac{n}{t}\left(L + \frac{n}{t}\right)^{-1}\right)^n.$$

(*"Exponential formula"*)

PROOF. (a) Spectral calculus gives

$$e^{-t\alpha}e^{-tL} = \sum_{\lambda\in\sigma(L)} e^{-t(\alpha+\lambda)}E_\lambda \quad \text{and} \quad (L+\alpha)^{-1} = \sum_{\lambda\in\sigma(L)} \frac{1}{\lambda+\alpha}E_\lambda.$$

Now, the desired statement follows easily by integration.

(b) As follows from spectral calculus, for all natural numbers n we have

$$e^{-tL} = \sum_{\lambda\in\sigma(L)} e^{-t\lambda}E_\lambda \quad \text{and} \quad \left(\frac{n}{t}\left(\frac{n}{t}+L\right)^{-1}\right)^n = \sum_{\lambda\in\sigma(L)} \left(\frac{\frac{n}{t}}{\frac{n}{t}+\lambda}\right)^n E_\lambda.$$

Now, the desired statement follows easily from

$$\lim_{n\to\infty}\left(\frac{\frac{n}{t}}{\frac{n}{t}+\lambda}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{t\lambda}{n}}\right)^n = e^{-t\lambda}.$$

This completes the proof. □

Finally, we turn to the smallest eigenvalue of L, which can also be thought of as the minimum of the spectrum of L. We denote it by λ_0. One has

$$\lambda_0 = \inf_{\|f\|=1} Q_L(f).$$

Indeed, from $f = \sum_{\lambda\in\sigma(L)} E_\lambda f$ and $Lf = \sum_{\lambda\in\sigma(L)} \lambda E_\lambda$ and $E_\lambda E_\mu = 0$ for $\lambda \neq \mu$ we easily find

$$Q_L(f) = \langle f, Lf\rangle = \sum_{\lambda\in\sigma(L)} \lambda\|E_\lambda f\|^2$$

as well as

$$\sum_{\lambda\in\sigma(L)} \|E_\lambda f\|^2 = \|f\|^2$$

for all $f \in \ell^2(X,m)$. So, for f with $\|f\| = 1$ we find that $Q_L(f)$ is a linear combination of the eigenvalues $\lambda \in \sigma(L)$ with coefficients $\|E_\lambda f\|^2$ adding up to 1. This shows the claim on λ_0.

1.2. Graphs and Matrices

In this section we introduce (finite) graphs and discuss some background. We then show that each such graph naturally comes with a matrix. This matrix in turn gives a form and a linear operator. The forms and linear operators associated to graphs share distinct features and are the topic of subsequent sections.

DEFINITION 1.4 (Graph over finite X). Let X be a finite set. A *graph over X* or a *finite graph* is a pair (b,c) consisting of a function $b\colon X\times X \longrightarrow [0,\infty)$ satisfying

- $b(x,y) = b(y,x)$ for all $x,y\in X$
- $b(x,x) = 0$ for all $x \in X$

and a function $c\colon X \longrightarrow [0,\infty)$. If $c(x) = 0$ for all $x \in X$, then we speak of b as a graph over X (instead of $(b,0)$).

In the context of graphs we use the following notation: The elements of X are called the *vertices* of the graph. The map b is called the *edge weight*. The map c is called the *killing term*. Moreover, a pair (x, y) with $b(x, y) > 0$ is called an *edge* with *weight* $b(x, y)$ connecting x to y. The vertices x and y are called *neighbors* if they form an edge. We write $x \sim y$ in this case. The graph (b, c) is called *connected* if for any $x, y \in X$ there exists a finite number of vertices $x_0, \ldots, x_n$ with $x = x_0$, $x_n = y$ and $x_j \sim x_{j+1}$ for $j = 0, \ldots, n-1$. The vertices $x_0, \ldots, x_n$ are then called a *path* from x to y. For any vertex x we define the *connected component* of x to be the set of $y \in X$ such that there exists a path from x to y.

Any graph comes with a matrix.

Definition 1.5 (Matrix associated to a graph). Let (b, c) be a graph over a finite set X. The matrix $l_{b,c}$ given by

$$l_{b,c}(x, y) = \begin{cases} -b(x, y) & \text{if} \quad x \neq y \\ \sum_{z \in X} b(x, z) + c(x) & \text{if} \quad x = y \end{cases}$$

is called the *matrix associated to the graph* (b, c). We say that (b, c) *induces* the matrix $l_{b,c}$.

Lemma 1.6 (Characterizing matrices arising from graphs). *Let X be a finite set. Let $l \colon X \times X \longrightarrow \mathbb{R}$ be a symmetric matrix. Then, the following statements are equivalent:*

(i) *There exists a graph (b, c) such that $l = l_{b,c}$.* (*“Graph”*)

(ii) *The matrix l satisfies*

$$l(x, y) \leq 0 \text{ and } \sum_{z \in X} l(x, z) \geq 0$$

for all $x, y \in X$ with $x \neq y$. (*“Matrix”*)

Moreover, if (i) *and* (ii) *hold, then the graph (b, c) and the matrix l are related by the equations*

$$l(x, y) = -b(x, y) \text{ and } c(x) = \sum_{z \in X} l(x, z)$$

for all $x, y \in X$ with $x \neq y$.

Proof. (i) $\Longrightarrow$ (ii): Let $l = l_{b,c}$ be the matrix associated to a graph (b, c). By the definition of $l_{b,c}$ we have $l(x, y) = -b(x, y) \leq 0$ for all $x \neq y$ as $b(x, y) \geq 0$. Furthermore,

$$\begin{aligned}\sum_{z \in X} l(x, z) &= l(x, x) + \sum_{z \neq x} l(x, z) = \sum_{z \in X} b(x, z) + c(x) - \sum_{z \neq x} b(x, z) \\ &= c(x) \geq 0\end{aligned}$$

for all $x \in X$ as $b(x, x) = 0$. This gives (ii).

(ii) $\Longrightarrow$ (i): Define $b \colon X \times X \longrightarrow \mathbb{R}$ for $x \neq y$ by

$$b(x, y) = -l(x, y) \quad \text{and} \quad b(x, x) = 0.$$

Define $c \colon X \longrightarrow \mathbb{R}$ by

$$c(x) = \sum_{z \in X} l(x, z).$$

Then, (b, c) is a graph over X by (ii) and the symmetry of l.

Furthermore, by construction, $l_{b,c}(x, y) = -b(x, y) = l(x, y)$ for $x \neq y$ and

$$\begin{aligned} l_{b,c}(x, x) &= \sum_{z \in X} b(x, z) + c(x) = \sum_{z \neq x} b(x, z) + c(x) \\ &= -\sum_{z \neq x} l(x, z) + \sum_{z \in X} l(x, z) = l(x, x) \end{aligned}$$

for all $x \in X$. Therefore, l is the matrix associated to the graph (b, c). This gives (i).

The last statement is clear from the considerations above. $\square$

1.3. Graphs and Dirichlet Forms

Any graph gives rise to a form. This form has specific features. It is a Dirichlet form. Details are discussed in this section.

Definition 1.7 (Form associated to a graph). Let (b, c) be a graph over a finite set X. The form $\mathcal{Q}_{b,c}$ acting on $C(X) \times C(X)$ by

$$\mathcal{Q}_{b,c}(f, g) := \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x) f(x) g(x)$$

is called the *form associated to the graph* (b, c) or the *energy form.*

Forms associated to graphs are particularly compatible with contractions. Specifically, let (b, c) be a graph over the finite set X and $\mathcal{Q}_{b,c}$ the associated form. Let $f, g \in C(X)$ be given and assume that g is a contraction of f in the sense that $|g(x)| \leq |f(x)|$ holds for all $x \in X$ and $|g(x) - g(y)| \leq |f(x) - f(y)|$ holds for all $x, y \in X$. Then, we immediately obtain from the definition that

$$\mathcal{Q}_{b,c}(g) \leq \mathcal{Q}_{b,c}(f).$$

To explore this more systematically, we make the following definition. A map $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a *normal contraction* if

$$C(0) = 0 \quad \text{and} \quad |C(s) - C(t)| \leq |s - t|$$

for all $s, t \in \mathbb{R}$. In particular, we note that $|C(s)| \leq |s|$ for all $s \in \mathbb{R}$ when C is a normal contraction.

In the context of normal contractions it is convenient to define

$$s \wedge t := \min\{s, t\} \quad \text{and} \quad s \vee t := \max\{s, t\}$$

for real numbers or for real-valued functions s and t. Examples of normal contractions include $C_+ : \mathbb{R} \longrightarrow \mathbb{R}, C_+(t) := t \vee 0$ and $C_- : \mathbb{R} \longrightarrow \mathbb{R}, C_-(t) := -(t \wedge 0)$ as well as

$$C_{[0,1]} : \mathbb{R} \longrightarrow \mathbb{R}, \quad C_{[0,1]}(t) := 0 \vee (t \wedge 1)$$

and the modulus $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}, t \mapsto |t|$.

PROPOSITION 1.8 (Compatibility of graph forms with normal contractions). *Let (b,c) be a graph over a finite set X and let $\mathcal{Q}_{b,c}$ be the form associated to (b,c). If $f \in C(X)$ is given and C is a normal contraction, then*

$$\mathcal{Q}_{b,c}(C \circ f) \leq \mathcal{Q}_{b,c}(f)$$

holds.

PROOF. Clearly, $C \circ f$ satisfies $|C \circ f(x)| \leq |f(x)|$ and

$$|C \circ f(x) - C \circ f(y)| \leq |f(x) - f(y)|$$

for all $x, y \in X$. Thus, the desired inequality follows directly from the definition of $\mathcal{Q}_{b,c}$. □

We are heading towards proving a converse to the proposition. We need the following general result.

PROPOSITION 1.9 (Representing forms via differences). *Let X be a finite set. Let $\mathcal{Q}$ be a symmetric form over X with associated matrix $l\colon X \times X \longrightarrow \mathbb{R}$. Define $b_{\mathcal{Q}}\colon X \times X \longrightarrow \mathbb{R}$ and $c_{\mathcal{Q}}\colon X \longrightarrow \mathbb{R}$ by*

$$b_{\mathcal{Q}}(x,y) := \begin{cases} -l(x,y) & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

and

$$c_{\mathcal{Q}}(x) := \sum_{y \in X} l(x,y).$$

Then, the form $\mathcal{Q}$ satisfies

$$\mathcal{Q}(f,g) = \frac{1}{2} \sum_{x,y \in X} b_{Q}(x,y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c_{Q}(x) f(x) g(x)$$

for all $f, g \in C(X)$.

PROOF. This follows by a direct computation. Here are the details: By definition,

$$\mathcal{Q}(f,g) = \sum_{x,y \in X} \mathcal{Q}(1_x, 1_y) f(x) g(y).$$

Furthermore, by using the definitions of $c_{\mathcal{Q}}$ and $b_{\mathcal{Q}}$, we get

$$(*) \quad \mathcal{Q}(1_x, 1_x) = \mathcal{Q}(1, 1_x) - \sum_{y \neq x} \mathcal{Q}(1_y, 1_x) = c_{\mathcal{Q}}(x) + \sum_{y \in X} b_{\mathcal{Q}}(x,y).$$

Therefore,

$$
\begin{aligned}
&\mathcal{Q}(f,g)\\
&= \sum_{x,y\in X} \mathcal{Q}(1_x,1_y)f(x)g(y)\\
&= \sum_{x\in X} \mathcal{Q}(1_x,1_x)f(x)g(x) + \sum_{x\in X}\sum_{y\neq x} \mathcal{Q}(1_x,1_y)f(x)g(y)\\
&\overset{(*)}{=} \sum_{x\in X}\left(c_{\mathcal{Q}}(x) + \sum_{y\in X} b_{\mathcal{Q}}(x,y)\right) f(x)g(x) - \sum_{x,y\in X} b_{\mathcal{Q}}(x,y)f(x)g(y)\\
&= \sum_{x,y\in X} b_{\mathcal{Q}}(x,y)f(x)(g(x)-g(y)) + \sum_{x\in X} c_{\mathcal{Q}}(x)f(x)g(x)\\
&= \frac{1}{2}\sum_{x,y\in X} b_{\mathcal{Q}}(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c_{\mathcal{Q}}(x)f(x)g(x),
\end{aligned}
$$

where in the last equality we use the symmetry of $b_{\mathcal{Q}}$ which follows from the symmetry of $\mathcal{Q}$. □

Lemma 1.10 (Characterization of compatibility with normal contractions). *Let X be a finite set. Let $\mathcal{Q}$ be a symmetric form over X with associated matrix $l\colon X\times X \longrightarrow \mathbb{R}$.*

(a) *The following statements are equivalent:*
  (i) *The form $\mathcal{Q}$ satisfies, for all $f\in C(X)$,*
$$\mathcal{Q}(|f|) \le \mathcal{Q}(f).$$
  (ii) *The matrix l satisfies, for all $x\neq y$,*
$$l(x,y)\le 0.$$

(b) *The following statements are equivalent:*
  (i) *The form $\mathcal{Q}$ satisfies, for all $f\in C(X)$,*
$$\mathcal{Q}(C_{[0,1]}\circ f) \le \mathcal{Q}(f).$$
  (ii) *The matrix l satisfies, for all $x\in X$ and $y\in X$ with $x\neq y$,*
$$l(x,y)\le 0 \quad \textit{and} \quad \sum_{z\in X} l(x,z)\ge 0.$$

Proof. As shown in Proposition 1.9, we have

$$\mathcal{Q}(f) = \frac{1}{2}\sum_{x\neq y} b_{\mathcal{Q}}(x,y)(f(x)-f(y))^2 + \sum_{x\in X} c_{\mathcal{Q}}(x)f^2(x)$$

with

$$b_{\mathcal{Q}}(x,y) = -l(x,y) \text{ for } x\neq y$$

and

$$c_{\mathcal{Q}}(x) = \sum_{z\in X} l(x,z).$$

This easily shows the implication (ii) $\Longrightarrow$ (i) in both (a) and (b).

(i) $\Longrightarrow$ (ii) in (a): Assume that $\mathcal{Q}$ satisfies $\mathcal{Q}(|f|) \leq \mathcal{Q}(f)$ for all $f \in C(X)$. Let $x, y \in X$ with $x \neq y$ and consider $f := 1_x - 1_y$. Then, $|f| = 1_x + 1_y$. Hence, the assumption on $\mathcal{Q}$ gives

$$\mathcal{Q}(1_x + 1_y) \leq \mathcal{Q}(1_x - 1_y).$$

Invoking the bilinearity and symmetry of $\mathcal{Q}$, we can easily infer

$$4\mathcal{Q}(1_x, 1_y) \leq 0.$$

Since $l(x, y) = \mathcal{Q}(1_x, 1_y)$, the desired statement follows.

(i) $\Longrightarrow$ (ii) in (b): Assume that $\mathcal{Q}$ satisfies $\mathcal{Q}(C_{[0,1]} \circ f) \leq \mathcal{Q}(f)$ for all $f \in C(X)$.

We start by showing $l(x, y) \leq 0$ for all $x \neq y$. By part (a), which has already been proven, it suffices to show that $\mathcal{Q}(|f|) \leq \mathcal{Q}(f)$ holds for all $f \in C(X)$.

Let $f \in C(X)$. After replacing f by αf with a suitable $\alpha > 0$, we can assume without loss of generality that $f \leq 1$. Now, consider the decomposition of f into positive and negative parts $f = f_+ - f_-$ where $f_+(x) := f(x) \vee 0$ and $f_-(x) := -f(x) \vee 0$ for $x \in X$. Clearly, $|f| = f_+ + f_-$. For $s > 0$ set

$$f_s := f_+ - sf_-.$$

Then, $C_{[0,1]} \circ f_s = f_+$ for all $s > 0$. Thus, our assumption gives

$$\mathcal{Q}(f_+) = \mathcal{Q}(C_{[0,1]} \circ f_s) \leq \mathcal{Q}(f_s) = \mathcal{Q}(f_+ - sf_-).$$

Invoking the bilinearity of $\mathcal{Q}$ and dividing by $s > 0$, we can then easily infer

$$0 \leq -2\mathcal{Q}(f_+, f_-) + s\mathcal{Q}(f_-)$$

for all $s > 0$. Letting $s \to 0$, we obtain

$$0 \leq -\mathcal{Q}(f_+, f_-).$$

Given this inequality, it follows that

$$\begin{aligned}\mathcal{Q}(|f|) &= \mathcal{Q}(f_+ + f_-) = \mathcal{Q}(f_+) + 2\mathcal{Q}(f_+, f_-) + \mathcal{Q}(f_-) \\ &\leq \mathcal{Q}(f_+) - 2\mathcal{Q}(f_+, f_-) + \mathcal{Q}(f_-) = \mathcal{Q}(f).\end{aligned}$$

This gives the desired compatibility of $\mathcal{Q}$ with $|\cdot|$.

We now turn to proving $\sum_{z \in X} l(x, z) \geq 0$ for all $x \in X$. Let $x \in X$ and consider $f := 1 + s1_x$ with $s > 0$. Then, $C_{[0,1]} \circ f = 1$ for all $s > 0$ and we obtain by assumption

$$\mathcal{Q}(1) = \mathcal{Q}(C_{[0,1]} \circ f) \leq \mathcal{Q}(f) = Q(1 + s1_x).$$

By the bilinearity of $\mathcal{Q}$ and after dividing by s, we find

$$0 \leq 2\mathcal{Q}(1, 1_x) + s\mathcal{Q}(1_x).$$

Letting $s \to 0$, we obtain

$$0 \leq \mathcal{Q}(1, 1_x) = \sum_{z \in X} l(x, z).$$

This gives the desired inequality for every $x \in X$. □

Theorem 1.11 (Characterization of forms associated to graphs). *Let $\mathcal{Q}$ be a symmetric form over a finite set X. Then, the following statements are equivalent:*

(i) *There exists a graph* (b, c) *over* X *with*

$$\text{(“Graph”)} \qquad \mathcal{Q} = \mathcal{Q}_{b,c}.$$

(ii) *The matrix* l *associated to* $\mathcal{Q}$ *satisfies, for* $x, y \in X$ *with* $x \neq y$,

$$\text{(“Matrix”)} \qquad l(x,y) \leq 0 \qquad \text{and} \qquad \sum_{z \in X} l(x,z) \geq 0.$$

(iii) *For all* $f \in C(X)$,

$$\mathcal{Q}(C_{[0,1]} \circ f) \leq \mathcal{Q}(f).$$

(*“Form compatible with one normal contraction”*)

(iv) *For all normal contractions* C *and* $f \in C(X)$,

$$\mathcal{Q}(C \circ f) \leq \mathcal{Q}(f).$$

(*“Form compatible with normal contractions”*)

(v) *If* $f, g \in C(X)$ *satisfy, for all* $x, y \in X$,

$$|f| \leq |g| \qquad \text{and} \qquad |f(x) - f(y)| \leq |g(x) - g(y)|,$$

then

$$\mathcal{Q}(f) \leq \mathcal{Q}(g).$$

Proof. This follows from the preceding considerations. Indeed, by Lemma 1.6, the equivalence between (i) and (ii) follows. The equivalence between (ii) and (iii) is the content of Lemma 1.10 (b). The implication (i) $\Longrightarrow$ (v) can be directly read off from the definition of $\mathcal{Q}_{b,c}$. The implication (v) $\Longrightarrow$ (iv) is clear from the definition of a normal contraction. Finally, (iv) $\Longrightarrow$ (iii) is obvious as $C_{[0,1]}$ is a normal contraction. $\square$

Definition 1.12 (Dirichlet form). A form $\mathcal{Q}$ on $C(X)$ is called a *Dirichlet form* if

$$\mathcal{Q}(C_{[0,1]} \circ f) \leq \mathcal{Q}(f)$$

holds for all $f \in C(X)$.

With this definition the preceding theorem can be seen as a characterization of Dirichlet forms.

1.4. Graphs and Laplacians

From Section 1.3 we know that any graph comes with a form. Here, we discuss how it comes with an operator.

Definition 1.13 (Laplacian on $\ell^2(X, m)$). Let (b, c) be a graph over a finite set measure space (X, m). The operator $L_{b,c,m}$ acting on $\ell^2(X, m)$ via

$$L_{b,c,m} f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x)$$

is called the *Laplacian on* $\ell^2(X, m)$ *associated to the graph* (b, c).

It is not hard to see that the Laplacian $L_{b,c,m}$ on $\ell^2(X,m)$ associated to (b,c) is self-adjoint. At this point we have associated to each graph (b,c) over (X,m) a symmetric form, viz $Q_{b,c} := \mathcal{Q}_{b,c}$ (recall Remark 1.2) and a self-adjoint operator, viz $L_{b,c,m}$. It turns out that form and operator are related. In fact, the form is exactly the form associated to the operator. This is the content of the next proposition.

Proposition 1.14 (Green's formula). *Let (b,c) be a graph over a finite set measure space (X,m). Let $Q_{b,c}$ and $L_{b,c,m}$ be the form and Laplacian associated to (b,c). Then, $Q_{b,c} = Q_{L_{b,c,m}}$ holds, i.e. the Green's formulae*

$$Q_{b,c}(f,g) = \langle L_{b,c,m} f, g\rangle = \langle f, L_{b,c,m} g\rangle$$

are valid for all $f,g \in \ell^2(X,m)$.

Proof. The proof ist left as an exercise. □

In Section 1.3 we have seen that the forms associated to graphs are characterized within the set of all symmetric forms by their compatibility with normal contractions. It turns out that the Laplacians associated to graphs can be characterized within the set of self-adjoint operators by a distinctive feature. This feature is introduced next.

Definition 1.15 (Maximum principle and Laplacians). Let (X,m) be a finite set measure space and let L be a self-adjoint operator on $\ell^2(X,m)$. The operator L is said to satisfy the *maximum principle* if

$$Lf(x) \geq 0$$

whenever $f \in \ell^2(X,m)$ has a non-negative maximum at $x \in X$. An operator satisfying the maximum principle is called *Laplacian*.

Theorem 1.16 (Maximum principle and graphs). *Let (X,m) be a finite set measure space and let L be a self-adjoint operator on $\ell^2(X,m)$. Then, the following statements are equivalent:*

(i) *The operator L satisfies the maximum principle.*
(ii) *There exists a graph (b,c) over (X,m) such that $L = L_{b,c,m}$ is the Laplacian associated to (b,c).*

Proof. (i) $\Longrightarrow$ (ii): Let l be the matrix associated to L. By Lemma 1.6 it suffices to show that $l(x,y) \leq 0$ for all $x \neq y$ and $\sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Applying the maximum principle to $f = 1$, we directly obtain $L1(x) = \sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Applying the maximum principle at $x \in X$ to $f = -1_y$ for an arbitrary $y \in X$ with $y \neq x$ we infer $-L1_y(x) = -l(x,y) \geq 0$ so that $l(x,y) \leq 0$ for all $x \neq y$.

(ii) $\Longrightarrow$ (i): As $L = L_{b,c,m}$ is the Laplacian associated to a graph (b,c) it follows that if f has a non-negative maximum at x, then

$$Lf(x) = \frac{1}{m(x)} \sum_{y\in X} b(x,y) \underbrace{(f(x) - f(y))}_{\geq 0} + c(x) \underbrace{f(x)}_{\geq 0} \geq 0,$$

which completes the proof. □

Matthias Keller
Daniel Lenz
Marcel Schmidt
Christian Seifert

Sheet 1

Finite Graphs I

Exercise 1 (Normal contractions) 4 points

(a) Show that the following maps from $\mathbb{R}$ to $\mathbb{R}$ are normal contractions:

- $C_+ : t \mapsto t \vee 0$,
- $C_- : t \mapsto (-t) \vee 0$,
- $C_{(-\infty,1]} : t \mapsto t \wedge 1$ and
- $C_{[0,1]} : t \mapsto 0 \vee (t \wedge 1)$.

For which $a \leq b$ is $C_{[a,b]} : t \mapsto a \vee (t \wedge b)$ a normal contraction?

(b) Let X be a finite set and let Q be a symmetric bilinear form on $C(X)$. Show that Q is compatible with normal contractions if it is compatible with the map $C_{(-\infty,1]}$.

Exercise 2 (First Beurling–Deny criterion) 4 points

Let X be a finite set and let Q be a symmetric bilinear form over X. For any $f \in C(X)$, let $f_+ = f \vee 0$ be the *positive part* and let $f_- = (-f) \vee 0$ be the *negative part* of f.

Show the following equivalence:

(i) $Q(|f|) \leq Q(f)$ for all $f \in C(X)$.

(ii) $Q(f_+, f_-) \leq 0$ for all $f \in C(X)$.

(iii) $Q(f \vee g) + Q(f \wedge g) \leq Q(f) + Q(g)$.

and for Q positive show that this is also equivalent to:

(iv) $Q(f_+) \leq Q(f)$ for all $f \in C(X)$.

Exercise 3 (Harmonic functions and connected components) 4 points

Let (b, c) be a graph over a finite set measure space (X,m) with associated Laplacian $L = L_{b,c,m}$ and let

$$H = \{f \in C(X) \mid Lf = 0\}$$

be the subspace of *harmonic functions*. Show that $\dim H$ is equal to the number of connected components of (b, c) on which c vanishes.

Exercise 4 (Poisson equation for $\alpha = 0$) 4 points

Let b be a graph over a finite set measure space (X,m) (that is $c = 0$) and let $L = L_{b,0,m}$ be the associated Laplacian. Furthermore, let

$$V := \{f \in C(X) \mid \sum_{x \in X} f(x)m(x) = 0\}.$$

Show that for each $f \in V$, there is a unique function $u \in V$ such that

$$Lu = f.$$

Hint: Observe that for the scalar product in $\ell^2(X,m)$, we have for all $f \in \ell^2(X,m)$

$$\sum_{x \in X} f(x)m(x) = \langle f,1 \rangle$$