Solution Exercise 13

Johannes Dietzschold, Tobias Habacker, Pablo Lummerzheim (all Leipzig University)

Exercise 1

Let an antitree with $s(r) = \lfloor r^\gamma \rfloor, r \geq 1, \gamma > 0$ be given and $m = 1$. Let ρ be the degree path metric. Show that $s \mapsto \# B_s^\rho(o)$ grows exponentially for $\gamma = 2$ and polynomially for $\gamma < 2$, and thee graph is bounded for $\gamma > 2$.

Proof

First we note that Deg is spherically symmetric and $Deg(r) = \lfloor (r-1)^\gamma \rfloor + \lfloor (r+1)^\gamma \rfloor$ for $r \geq 1$. It's also easy to see that for $x \in S_r$

$$\rho(o,x) = \inf_{o=x_0\sim\ldots\sim x_n=x} \sum_{i=1}^{n} (Deg(i-1) \vee Deg(i))^{-\frac{1}{2}} = \sum_{i=1}^{r} Deg(i)^{-\frac{1}{2}}$$

especially $\rho(o,\cdot)$ is spherical symmetric.

For the case $\gamma > 2$ one calculates for $r \geq 2$

$$\begin{aligned}
\rho(o,r) &= \sum_{i=1}^{r} Deg(i)^{-\frac{1}{2}} \\
&= \sum_{i=1}^{r} (\lfloor (i-1)^\gamma \rfloor + \lfloor (i+1)^\gamma)^{-\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}} \sum_{i=1}^{r} \lfloor (i-1)^\gamma \rfloor^{-\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}} \sum_{i=0}^{r-1} i^{-\frac{\gamma}{2}} \qquad \leq \qquad \frac{1}{\sqrt{2}} \sum_{i=0}^{\infty} i^{-\frac{\gamma}{2}}
\end{aligned}$$

were the last expression is independent of r and finite since $\frac{\gamma}{2} > 1$, so the graph is bounded.

For the case $\gamma < 2$, fix $s > 0, r \in B_s^\rho(o)$, then

$$\begin{aligned}
s &> \rho(o,r) \\
&= \sum_{i=1}^{r} Deg(i)^{-\frac{1}{2}} \\
&= \sum_{i=1}^{r} (\lfloor (i-1)^\gamma \rfloor + \lfloor (i+1)^\gamma \rfloor)^{-\frac{1}{2}} \\
&\geq \frac{1}{\sqrt{2}} \sum_{i=1}^{r} \lfloor (i+1)^\gamma \rfloor^{-\frac{1}{2}} \\
&\geq \frac{1}{\sqrt{2}} \sum_{i=1}^{r} (i+2)^{-\frac{\gamma}{2}} \\
&= \frac{1}{\sqrt{2}} \sum_{i=3}^{r+2} i^{-\frac{\gamma}{2}} \\
&\geq \frac{1}{\sqrt{2}} \int_3^{r+4} t^{-\frac{\gamma}{2}} \, dt \quad = \quad C(r+4)^{1-\frac{\gamma}{2}} - D
\end{aligned}$$

for some constants C, D. This implies

$$r < \left(\frac{s+D}{C}\right)^{\frac{\gamma}{2}-1} - 4 < \left(\frac{s+D}{C}\right)^{\frac{\gamma}{2}-1} - 1 =: f(s)$$

so $B_s^\rho(o) \subseteq B_{\lfloor f(s) \rfloor}$ thus

$$\begin{aligned}
\# B_s^\rho(o) &\leq \# B_{\lfloor f(s) \rfloor} \\
&= 1 + \sum_{n=1}^{\lfloor f(s) \rfloor} \lfloor n^\gamma \rfloor \\
&\leq 1 + \sum_{n=1}^{\lfloor f(s) \rfloor} n^\gamma \\
&\leq 1 + \int_1^{\lfloor f(s) \rfloor + 1} t^\gamma \, dt \\
&\leq 1 + \int_1^{f(s)+1} t^\gamma \, dt \\
&= 1 + \frac{1}{\gamma+1}(f(s)+1)^{\gamma+1} - \frac{1}{\gamma+1} \\
&= 1 + \frac{1}{\gamma+1}\left(\left(\frac{s+D}{C}\right)^{\frac{\gamma}{2}-1}\right)^{\gamma+1} - \frac{1}{\gamma+1} \quad = \quad 1 - \frac{1}{\gamma+1} + \frac{1}{\gamma+1}\left(\frac{s+D}{C}\right)^{\frac{\gamma}{2}}
\end{aligned}$$

which grows polynomially in s.

For the case $\gamma = 2$, first fix $s > 0, r \in B_s^\rho(o)$ then

$$\begin{aligned}
s &> \rho(o,r) \\
&= \sum_{i=1}^{r} Deg(i)^{-\frac{1}{2}} \\
&= \sum_{i=1}^{r}((i-1)^2 + (i+1)^2)^{-\frac{1}{2}} \\
&= \sum_{i=1}^{r}(2i^2+2)^{-\frac{1}{2}} \\
&\geq \frac{1}{\sqrt{2}} \sum_{i=1}^{r} \frac{1}{i} \\
&\geq \frac{1}{\sqrt{2}} \int_1^{r+1} \frac{1}{t}\ dt = \frac{1}{\sqrt{2}} \ln(r+1)
\end{aligned}$$

which implies

$$r < \exp(\sqrt{2}s) - 1 =: f(s)$$

so $B_s^\rho(o) \subseteq B_{\lfloor f(s) \rfloor}$ thus

$$\begin{aligned}
\#B_s^\rho(o) &\leq \#B_{\lfloor f(s) \rfloor} \\
&= 1 + \sum_{n=1}^{\lfloor f(s) \rfloor} n^2 \\
&\leq 1 + \int_1^{\lfloor f(s) \rfloor + 1} t^2\ dt \\
&\leq 1 + \int_1^{f(s)+1} t^2\ dt \\
&= 1 + \frac{1}{3}(f(s)+1)^3 - \frac{1}{3} = \frac{2}{3} + \exp(3\sqrt{2}s)
\end{aligned}$$

so $\#B_s^\rho$ grows at most exponentially. On the other hand fix $r \notin B_s^\rho(o)$ and assume $s \geq 1$ then

$$\begin{aligned}
s &\leq \rho(o,r) \\
&= \sum_{i=1}^{r} Deg(i)^{-\frac{1}{2}} \\
&= \sum_{i=1}^{r}((i-1)^2 + (i+1)^2)^{-\frac{1}{2}} \\
&= \sum_{i=1}^{r}(2i^2+2)^{-\frac{1}{2}} \\
&\leq \sum_{i=1}^{r}(4i^2)^{-\frac{1}{2}} \\
&= \frac{1}{2} \sum_{i=1}^{r} \frac{1}{i} \\
&\leq \frac{1}{2} + \frac{1}{2} \int_1^{r} \frac{1}{t}\ dt = \frac{1}{2} + \frac{1}{2}\ln(r)
\end{aligned}$$

which implies

$$r > \exp(2s-1) =: g(s)$$

so $B_{\lfloor g(s)\rfloor} \subseteq B_s^\rho(o)$ thus

$$\begin{aligned}
\#B_s^\rho(o) &\geq B_{\lfloor g(s)\rfloor} \\
&= 1 + \sum_{n=1}^{\lfloor g(s)\rfloor} n^2 \\
&\geq 1 + \int_0^{\lfloor g(s)\rfloor} t^2 \, dt \\
&\geq 1 + \int_0^{g(s)-1} t^2 \, dt \\
&= 1 + \frac{1}{3}(g(s)-1)^3 \quad = \quad 1 + \frac{1}{3}(\exp(2s-1)-1)^3
\end{aligned}$$

so $\#B_s^\rho(o)$ grows exponentially. □

Exercise 2

Let an antitree be given with $s(r) = \lfloor r^\gamma \rfloor$ with $r \geq 1$.

Proof that $\lambda_0(L) > 0$ if and only if $\gamma \leq 2$.

Proof

For $\gamma > 2$ we calculate:

$$\begin{aligned}
a &:= \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{2s(r)s(r+1)} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{r=1}^{\infty} \frac{1 + \sum_{n=1}^{r} \lfloor n^\gamma \rfloor}{\lfloor r^\gamma \rfloor \lfloor (r+1)^\gamma \rfloor} \\
&\leq \frac{1}{2} + \frac{1}{4}\sum_{r=1}^{\infty} \frac{1}{r^{2\gamma}} + \frac{1}{2}\sum_{r=1}^{\infty} \frac{r\lfloor r^\gamma \rfloor}{\lfloor r^\gamma \rfloor \lfloor (r+1)^\gamma \rfloor} \\
&\leq \frac{1}{2} + \frac{1}{4}\sum_{r=1}^{\infty} \frac{1}{r^{2\gamma}} + \frac{1}{2}\sum_{r=1}^{\infty} \frac{r}{r^\gamma} \quad < \quad \infty
\end{aligned}$$

and by Example 11.15 $\lambda_0(L) \geq \frac{1}{2a} > 0$.

For $\gamma < 2$ by Exercise 1 we have $m(B_r^\rho(o)) \leq ar^n$ for some $a \geq 0, n \in \mathbb{N}$ so

$$\begin{aligned}
\mu &= \liminf_{r\to\infty} \inf_{x\in X} \frac{1}{r} \log \frac{m(B_r^\rho(x))}{m(B_1^\rho(x))} \\
&\leq \liminf_{r\to\infty} \frac{1}{r} \log \left(\frac{m(B_r^\rho(o))}{m(B_1^\rho(o))} \right) \\
&\leq \liminf_{r\to\infty} \frac{1}{r} \log \left(\frac{ar^n}{m(B_1^\rho(o))} \right) \quad = \quad 0
\end{aligned}$$

So by the Theorem of Brooks-Sturm (Theorem 7.12) $\lambda_0(L) \leq \frac{\mu^2}{8} \leq 0$ □

Exercise 3

(Antitrees and trees - Recurrence)

(a) Let b be a spherically symmetric tree with branching number k. Then b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\prod_{n=0}^{r} k(n)} = \infty.$$

(b) Let b be an antitree with sphere size s. Then, b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{s(r)s(r+1)} = \infty.$$

Proof

(a) From Theorem 11.17, we know that a locally finite weakly spherically symmetric graph is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} = \infty.$$

As b is a tree, $O = \{o\}$, $k_+(x) = k(r)$ if $x \in S_r(o)$ and $k_-(x) = 1$ for each $x \neq o$. Furthermore, we can compute the number of elements in $S_r(o)$ inductively: Clearly $m(S_0(o)) = \#S_0(o) = \#\{o\} = 1$, and $\#S_1(o) = k(0)$, as o connects to $k_+(o) = k(0)$ elements in $S_1(o)$. Now assume, that $\#S_r(o) = \prod_{n=0}^{r-1} k(n)$. Then each of the elements of $S_r(o)$ connects to exactly $k(r)$ elements: For $x \in S_r(o)$, we have $k_+(x) = k(r)$ connections to $S_{r+1}(o)$, and any $y \in S_{r+1}(o)$ that is connected to x is not connected to any other element of $S_r(o)$, since $k_-(y) = 1$. Therefore $\#S_{r+1}(o) = \#S_r(o) \cdot k(r) = \prod_{n=0}^{r} k(n)$.

Now, we compute $b(\partial B_r(o))$. From the lecture, we have for locally finite weakly spherically symmetric graphs

$$b(\partial B_r(o)) = 2k_+(r)m(S_r(o)) = 2k(r)\prod_{n=0}^{r-1} k(n) = 2\prod_{n=0}^{r} k(n).$$

Now we have

$$b \text{ is recurrent} \iff \left(\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{\prod_{n=0}^{r} k(n)} =\right)\sum_{r=0}^{\infty}\frac{1}{b(\partial B_r(O))} = \infty \iff \sum_{r=0}^{\infty}\frac{1}{\prod_{n=0}^{r} k(n)} = \infty,$$

which proves the assertion.

(b) Again, we use theorem 11.17. Since b is an antitree, we have $O = \{o\}$, $k_+(x) = s(r+1)$ whenever $x \in S_r(o)$ and $m(S_r(o)) = s(r)$. Therefore

$$b(\partial B_r(o)) = 2k_+(r)m(S_r(o)) = 2s(r+1)s(r),$$

so we have the equivalences

$$b \text{ is recurrent } \iff \left(\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{s(r+1)s(r)} =\right)\sum_{r=0}^{\infty}\frac{1}{b(\partial B_r(O))} = \infty \iff \sum_{r=0}^{\infty}\frac{1}{s(r+1)s(r)} = \infty,$$

which prove the statement.

□

Exercise 4

(Antitrees and trees - Stochastic completeness)

(a) Let b be a spherically symmetric tree with branching number k. Then b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty}\frac{1+\sum_{n=1}^{r}\prod_{j=0}^{n-1}k(j)}{\prod_{l=0}^{r}k(l)} = \infty.$$

(b) Let b be an anti-tree with sphere size s. Then, b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty}\frac{\sum_{n=0}^{r}s(n)}{s(r)s(r+1)} = \infty.$$

Proof

(a) For this proof, we use theorem 11.21 which states that a locally finite weakly spherically symmetric graph is stochatically complete at infinity if and only if

$$\sum_{r=0}^{\infty}\frac{m(B_r(O))}{b(\partial B_r(O))} = \infty.$$

As b is a tree, we have (again) $O = \{o\}$, $k_+(x) = k(r)$ if $x \in S_r(o)$ and $k_-(x) = 1$ for each $x \neq o$. Moreover, we use the computation from exercise 3:

$$b(\partial B_r(o)) = 2\prod_{n=0}^{r}k(n),$$

$$m(B_r(o)) = m(S_0(o)) + \sum_{n=1}^{r}m(S_n(o)) = 1 + \sum_{n=1}^{r}\prod_{j=0}^{n-1}k(j).$$

Thus, we obtain

$$b \text{ is stochastically complete } \iff \infty = \sum_{r=0}^{\infty}\frac{m(B_r(O))}{b(\partial B_r(O))}\left(=\frac{1}{2}\sum_{r=0}^{\infty}\frac{1+\sum_{n=1}^{r}\prod_{j=0}^{n-1}k(j)}{\prod_{l=0}^{r}k(l)}\right)$$

$$\iff \sum_{r=0}^{\infty}\frac{1+\sum_{n=1}^{r}\prod_{j=0}^{n-1}k(j)}{\prod_{l=0}^{r}k(l)} = \infty.$$

(b) Again by 11.21 with the previous computations

$$b(\partial B_r(o)) = 2s(r+1)s(r),$$
$$m(B_r(o)) = \sum_{n=o}^{r} m(S_n(o)) = \sum_{n=0}^{r} s(n),$$

we get

$$b \text{ is stochastically complete} \iff \infty = \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} \left(= \frac{1}{2} \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r+1)s(r)} \right)$$
$$\iff \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r+1)s(r)} = \infty.$$

□