Lecture 12 - Exercises

Exercise 1 (**Subgraphs I**). Let b be a stochastically incomplete graph over (X, m). Show that there exist X' with $X \subseteq X'$, b' and m' which extend b and m to X' such that b' over (X', m') is stochastically complete.

Solution. Without loss of generality we can assume $X = \mathbb{N}$. We define

$$X' := \mathbb{N} \times \{1, 2\},$$
$$b'((x, i), (y, j)) := \begin{cases} b(x, y) & \text{if } i = j = 1, \\ \omega_x & \text{if } y = x \text{ and } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we have $m'(x, 1) = m(x)$ for all $x \in \mathbb{N}$. We will use Theorem 10.25 (v.b) and verify stochastic completeness by showing that for all $\alpha > 0$ every $0 \leq f \in \ell^\infty(X')$ satisfying $(\mathcal{L} + \alpha)f = 0$ is trivial. For $x \in X$ the equation $(\mathcal{L} + \alpha)f(x, 2) = \frac{1}{m'(x,2)}\omega_x(f(x, 2) - f(x, 1)) + \alpha f(x, 2) = 0$ implies

$$f(x, 2) = \frac{1}{1 + \alpha m'(x, 2)/\omega_x} f(x, 1). \tag{1}$$

Inserting gives

$$\begin{aligned}(\mathcal{L} + \alpha)f(x, 1) &= \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x, 1) - f(y, 1)) + \frac{\omega_x}{m(x)}(f(x, 1) - f(x, 2)) + \alpha f(x, 1) \\ &= (\mathcal{L}_X + \alpha)(f(\cdot, 1))(x) + \frac{\omega_x}{m(x)(1 + \alpha m'(x, 2)/\omega_x)} f(x, 1) = 0\end{aligned}$$

and implies that

$$\frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x, 1) - f(y, 1)) + \alpha f(x, 1) = -\frac{\omega_x}{m(x)(1 + \alpha m'(x, 2)/\omega_x)} f(x, 1). \tag{2}$$

Setting $M := \|f\|_\infty$ and bounding the absolute values gives

$$(\alpha + \mathrm{Deg}_X(x))f(x, 1) + \mathrm{Deg}_X(x)M \geq \frac{\omega_x}{m(x)(1 + \alpha m'(x, 2)/\omega_x)} f(x, 1).$$

With the choice $m'(x, 2) = 1$ and $\omega_x = (2^x - 1)m(x) \sup(\mathrm{Deg}_X(x), 1)$ this inequality can only be true if $f(x, 1) = 0$ for large $x \in \mathbb{N}$. Since X is connected equation (2) and the assumption $f \geq 0$ gives $f(x, 1) = 0$ for $x \in \mathbb{N}$. Equation (1) finishes the proof.

Exercise 2 (**Subgraphs II**). Let b be a graph over (X, m). Let $Y \subseteq X$ and suppose that the associated subgraph b_Y over (Y, m_Y) is stochastically incomplete. Let

$$\mathrm{Deg}_{X \setminus Y}(x) = \frac{1}{m(x)} \sum_{y \in X \setminus Y} b(x, y)$$

for $x \in Y$. Suppose that $\mathrm{Deg}_{X \setminus Y}(x)$ is bounded on the set

$$\{x \in Y | \text{ there exists a } y \sim x, y \notin Y\}.$$

Show that b over (X, m) is stochastically incomplete.

Solution. We will make use of the Omori-Yau maximum principle which, according to Theorem 10.25, is equivalent to stochastic completeness:

If $u \in \mathcal{F}$ satisfies $\sup u \in (0, \infty)$ and $b \in (0, \sup u)$, then

$$\sup_{x \in X_\beta} \mathcal{L}u(x) \geq 0,$$

where $X_\beta := \{x \in X | u(x) > \sup u - \beta\}$.

Therefore we find $u \in \mathcal{F}_Y$ and $\beta_Y \in (0, \sup u)$ such that

$$\sup_{x \in Y_\beta} \mathcal{L}u(x) =: -M_Y < 0.$$

We set

$$\begin{aligned} C &:= 1 \vee \inf\{c > 0 | c > \mathrm{Deg}_{X \setminus Y}(x) \forall x \in Y\}, \\ \beta_X &:= \frac{\beta_Y \wedge M_Y}{2C}, \\ \gamma &:= \sup_{y \in Y} u(y), \\ u_X(x) &:= \begin{cases} u(x), & \text{if } x \in Y, \\ \gamma - \beta_X, & \text{if } x \in \setminus Y, \end{cases} \end{aligned}$$

which gives that $X_{\beta_X} \subseteq Y_{\beta_Y}$, since $\beta_X < \beta_Y$. Finally for all $x \in X_{\beta_X}$ it holds

$$\begin{aligned} \mathcal{L}u_X(x) &= \frac{1}{m(x)} \sum_{y \in Y} b(x,y)(u(x) - u(y)) + \frac{1}{m(x)} \sum_{y \in X \setminus Y} b(x,y)(u(x) - (\gamma - \beta_X)) \\ &= \mathcal{L}_Y u(x) - \frac{1}{m(x)} \sum_{y \in X \setminus Y} b(x,y)(u(x) - (\gamma - \beta_X)) \\ &\leq -M_Y + C\beta_X \leq -M_Y + \frac{M_Y}{2} < 0 \end{aligned}$$

which shows that b is stochastically incomplete over (X, m).

Exercise 3 (**Khasminskii criterion**)**.** Let $(b, 0)$ be a connected graph over (X, m). If there exists an $\alpha > 0$ and a positive function $v \in \mathcal{F}$ such that $v(x) \to \infty$ as $x \to \infty$ and $(\mathcal{L} + \alpha)v \geq 0$ then $(b, 0)$ is stochastically complete.

Solution. Suppose that $u \in \ell^\infty$ is such that $(\mathcal{L} + \alpha)u = 0$. We show that $u = 0$. Without loss of generality assume that $0 \leq u \leq 1$. The assumption $v(x) \to \infty$ as $x \to \infty$ means that for each $C > 0$ there exists a finite set $K \subseteq X$ such that $v \geq C$ on the complement of K. Set $w = v - Cu$. We have that $(\mathcal{L} + \alpha)w = (\mathcal{L} + \alpha)v \geq 0$ and that $w \geq 0$ on the complement of K. Furthermore, $w \wedge 0$ has a minimum on K since K is finite. By the minimum principle, it follows that $w \geq 0$. Thus $Cu \leq v$ for each $C > 0$. Therefore $u = 0$.

Exercise 4 (**Uniqueness of bounded solutions with** $c \neq 0$)**.** Let (b, c) be a graph over (X, m) with formal Laplacian $L = L_{b,c,m}$. Assume that b (without c) is stochastically complete. Show that the following equations only have the trivial solution.

(a) $(L + \alpha)f = 0$ with $\alpha > 0$ and $f \in \ell^\infty(X)$.

(b) $(\partial_t + L)u = 0$ with $u : [0, \infty) \times X \to \mathbb{R}$ bounded and $u(0, \mathring{u}) = 0$.

Solution. We know for $|f| \in \ell^\infty(X)$ the inequality

$$(L_{b,c} + \alpha) |f| \leq 0.$$

holds. We conclude

$$(L_{b,0} + \alpha) |f| \leq (L_{b,c} + \alpha) |f| \leq 0$$

and since $(b, 0)$ is stochastically complete $f = 0$ follows. Statement (b) follows with Laplace transformation.

Bonus Exercise 1 (**Adding potentials**). Let b be a graph over (X, m). Show that there exists

$$c : X \to [0, \infty)$$

such that the inequality $(L+\alpha)f \leq 0$ with $\alpha > 0$ and $f \in \ell^\infty(X)_+$ only has the trivial solution $f = 0$.

Solution. Without loss of generality we can assume $X = \mathbb{N}$. Let $x \in X$, $\alpha > 0$ and let $0 \leq f \in \ell^\infty(X)$ satisfy $(L+\alpha)f \leq 0$. Rewriting $(L+\alpha)f(x) \leq 0$ we derive the inequality

$$(\deg(x) + \alpha m(x))f(x) \leq \sum_{y \in X} b(x,y)f(y).$$

Setting $M := \|f\|_\infty$ yields

$$\frac{\deg(x) + \alpha m(x)}{\sum_{y\in X} b(x,y)} f(x) \leq M$$

and moreover

$$(1 + \frac{c(x)}{\sum_{y\in X} b(x,y)})f(x) \leq M$$

holds. We set $c(x) = 2^x \sum_{y\in X} b(x,y)$ and conclude

$$0 \leq (1 + 2^x)f(x) \leq M$$

for all $x \in X$. Therefore f has to vanish at infinity. Since X is countable there exist a finite set $K \subset X$ where f takes its supremum. So we find $x_0 \in K$ such that $f(x_0) = M$. Putting x_0 in the inequality above yields

$$0 \leq (1 + 2^{x_0})M \leq M$$

and $f = 0$ folows.

Bonus Exercise 2 (**Generalized conservation property**). Let (b, c) be a graph over (X, m) with Laplacian $L = L_{b,c,m}$. Assume that b (without c) is stochastically complete and that c/m is bounded. Prove the formulae

$$\mathbb{1} = \alpha(L+\alpha)^{-1}\mathbb{1} + (L+\alpha)^{-1}\frac{c}{m}$$

and

$$\mathbb{1} = e^{-tL}\mathbb{1} + \int_0^t e^{-sL}\frac{c}{m}ds,$$

where the integral is understood pointwise.

Solution. Suppose there exists $u, f \in \ell^\infty(X)$ such that

$$(L_{b,c} + \alpha)u = f$$

holds for any $\alpha > 0$. Then u is uniquely determined by f. This follows directly by assuming the existence of two different solutions u_1, u_2 and setting $u = u_1 - u_2$. Since $(b, 0)$ is stochastically complete we conclude with Exercise 4 that

$$(L_{b,c} + \alpha)u = 0$$

only has the trivial solution $u = 0$ and $u_1 = u_2$ holds. We set $f = \alpha\mathbb{1} + \frac{c}{m}$ and get $f \in \ell^\infty(X)$ since c/m is bounded. Moreover

$$(L_{b,c} + \alpha)\mathbb{1} = f$$

holds. Since $u = \mathbb{1}$ is an uniquely solution we infer

$$\mathbb{1} = \alpha(L_{b,c} + \alpha)^{-1}\mathbb{1} + (L_{b,c} + \alpha)^{-1}\frac{c}{m}.$$

Using Laplace transformation gives

$$\mathbb{1} = \int_0^\infty \alpha e^{-t\alpha}(e^{-tL}\mathbb{1})dt + \int_0^\infty e^{-t\alpha}(e^{-tL}\frac{c}{m})dt$$

and with integration by parts

$$\mathbb{1} = \left(e^{-t\alpha} \int_0^t (e^{-sL} \frac{c}{m}) ds \right) \Big|_0^\infty + \int_0^\infty \alpha e^{-t\alpha} \left(\int_0^t e^{-sL} \frac{c}{m} ds \right) dt + \int_0^\infty \alpha e^{-t\alpha} (e^{-tL} \mathbb{1}) dt$$

follows. Obviously the first term is vanishing. Setting

$$M_t(x) := \left(e^{-tL} \mathbb{1} \right)(x) + \int_0^t e^{-sL} \frac{c}{m}(x) ds$$

with $t \geq 0$, yields

$$1 = \int_0^\infty \alpha e^{-t\alpha} M_t(x) dt.$$

for any $x \in X$. To show $M_t = 1$ for all $t \geq 0$ we assume there exists $t_0 > 0$ such that $M_{t_0} < 1$. Since the map $t \mapsto M_t$ is continuous and

$$0 \leq M_s \leq M_t \leq 1$$

holds for all $s \geq t \geq 0$ (see [KLW21], Chapter 7, Theorem 7.14) we infer $M_t < 1$ for all $t > 0$. Therefore

$$\int_0^\infty \alpha e^{-t\alpha} (1 - M_t) dt > 0$$

and

$$1 = \int_0^\infty \alpha e^{-t\alpha} dt > \int_0^\infty \alpha e^{-t\alpha} M_t dt$$

follows. This contradicts $\int_0^\infty \alpha e^{-t\alpha} M_t dt = 1$ and the claim $M_t = 1$ follows.

Literatur

[KLW21] M. Keller, D. Lenz, and R. Wojciechowski. *Graphs and Discrete Dirichlet Spaces.* Springer, 2021.