Solutions to the Exercises of Lecture 11

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Exercise 1 (Semigroups with bounded generators on $\ell^p(X,m)$)**.** Let $p \in [1,\infty]$ and $A \in B(\ell^p(X,m))$. For $t \in \mathbb{R}$ define

$$e^{-tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!}(-A)^n.$$

(a) Show that $(e^{-tA})_{t\geq 0}$ is a strongly continuous semigroup.

(b) Show that $(e^{-tA})_{t\geq 0}$ is even norm continuous, i.e. $t \mapsto e^{-tA} \in B(\ell^p(X,m))$ is continuous.

(c) Show that $t \mapsto e^{-tA} \in B(\ell^p(X,m))$ is continuously differentiable with derivative $t \mapsto -Ae^{-tA}$.

(d) Show that for $t \in \mathbb{R}$ we have that e^{-tA} is invertible with inverse e^{tA}.

Solution. First note that we fixed the definition of $\exp(-tA)$ compared to the lecture with $-A$ instead of A on the right-hand side.

(a) We first show the semigroup property. Let $t, s \in \mathbb{R}$. Using the binomial theorem and the Cauchy product formula, we obtain

$$\begin{aligned} e^{-(t+s)A} &= \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!}(-A)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k s^{n-k} \frac{(-A)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-A)^k}{k!} \frac{(-A)^{n-k}}{(n-k)!} \\ &= \left(\sum_{n=0}^{\infty} \frac{-(tA)^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{-(sA)^n}{n!}\right) \\ &= e^{-tA}e^{-sA}. \end{aligned}$$

Using the convention that A^0 is the identity matrix, $e^{0\cdot A} = 1$ is obvious. The strong continuity is an implication of (b).

(b) For all $x \in \ell^p(X,m)$ we find

$$\left\|\sum_{n=0}^{\infty}\frac{(t)^n}{n!}(-A)^n x - x\right\| \le \sum_{n=1}^{\infty}\frac{t^n}{n!}\|A\|^n\|x\| = \|x\|\left(e^{-t\|A\|}-1\right) \to 0 \ (t\to 0).$$

(c) We have

$$\begin{aligned}\left\|\frac{1}{h}\left(e^{-(t+h)A}-e^{tA}\right)+Ae^{-tA}\right\| &= \|e^{-tA}\|\left\|\frac{1}{h}\left(e^{-hA}-1\right)+A\right\| \\ &= \|e^{-tA}\|\left\|\frac{1}{h}\sum_{n=1}^{\infty}\frac{(-Ah)^n}{n!}+A\right\| \\ &= \|e^{-tA}\|\left\|\frac{1}{h}\sum_{n=2}^{\infty}\frac{(-Ah)^n}{n!}\right\| \\ &\le \|e^{-tA}\|\|A\|\left\|\sum_{n=1}^{\infty}\frac{(-Ah)^n}{(n+1)!}\right\| \\ &\le \|e^{-tA}\|\|A\|(e^{-h\|A\|}-1) \to 0 \ (h\to 0).\end{aligned}$$

(d) Noticing that we did not require s and t to be positive in (a), we use (a) to obtain

$$e^{tA}e^{-tA} = e^{0\cdot A} = I. \qquad \square$$

Exercise 2 (Norm continuous semigroups on $\ell^p(X,m)$)**.** Let $p\in[1,\infty]$ and $(S(t))_{t\ge 0}$ a norm continuous semigroup on $\ell^p(X,m)$, i.e. $t\mapsto e^{-tA}\in B(\ell^p(X,m))$ is continuous. Let A be the generator of $(S(t))_{t\ge0}$. Show that $A\in B(\ell^p(X,m))$.

Hint: You can use the following fact without proof: For $f\in\ell^p(X,m)$ and $t>0$ we have $\int_0^t S(s)f ds\in D(A)$ and

$$A\int_0^t S(s)f ds = S(t)f - f.$$

Solution. Let $f\in\ell^p(X,m)$. We claim that there exists $t>0$ such that $\int_0^t S(s)ds$ is invertible (see below). Set $B := \left(\int_0^t S(s)ds\right)^{-1}\in B(\ell^p(X,m))$. It follows that

$$Af = A\int_0^t S(s)dsBf = A\int_0^t S(s)Bf ds = (S(t)-I)Bf,$$

where we have used the hint for the last equality. This already yields $A\in B(\ell^p(X,m))$.

For the invertibility, we see that

$$\lim_{t\to0}\left\|\frac{1}{t}\int_0^t S(s)ds - I\right\| = \lim_{t\to0}\left\|\frac{1}{t}\int_0^t S(s)-Ids\right\| \le \lim_{t\to0}\sup_{s\in[0,t]}\|S(s)-I\| = 0$$

since $s\mapsto S(s)$ is continuous. Thus, for $t>0$ small enough such that $\left\|\frac{1}{t}\int_0^t S(s)ds - I\right\|<1$, we can use the Neumann series to invert $\frac{1}{t}\int_0^t S(s)ds$ and therefore $\int_0^t S(s)ds$. $\square$

Exercise 3. Let (X,m) be a discrete measure space. Let $(S(t))_{t\ge0}$ be a positivity preserving semigroup on $\ell^\infty(X,m)$, i.e., for all $t\ge0$ we have that $f\ge0$ implies $S(t)f\ge0$. Show that the following assertions are equivalent:

(i) $(S(t))_{t\geq 0}$ is weak*-continuous, i.e., for all $g \in \ell^1(X,m)$ and all $f \in \ell^\infty(X)$, the map

$$[0,\infty) \to \mathbb{R}, \quad t \mapsto \sum_{x\in X} g(x)S(t)f(x)m(x)$$

is continuous.

(ii) For all $f \in \ell^\infty(X)$ and all $x \in X$, the map

$$[0,\infty) \to \mathbb{R}, \quad t \mapsto S(t)f(x)$$

is continuous.

Solution. "(i) $\implies$ (ii)": Given $x \in X$, apply (i) with $g := \frac{1}{m(x)}\mathbf{1}_x$.

"(ii) $\implies$ (i)": Here, we need to make the additional assumption that the semigroup is sub-Markovian, i.e. that for all $t \geq 0$ we have that $0 \leq f \leq 1$ implies $0 \leq S(t)f \leq 1$.

Let $f \in \ell^\infty(X)$ and $g \in \ell^1(X,m)$ and assume w.l.o.g.[1] that $f,g \geq 0$. Since X is discrete, we can write $X = \{x_1, x_2, \dots\}$ and obtain

$$\sum_{i=n}^{\infty} g(x)S(t)f(x)m(x) \leq \|f\|_{\ell^\infty(X)} \sum_{i=n}^{\infty} g(x)m(x) \to 0 \quad (n\to\infty)$$

where we used the sub-Markov property of $(S(t))_{t\geq 0}$ for the inequality. Hence, the series in (i) converges uniformly in $t \geq 0$. We conclude by using the well-known fact that uniform limits of continuous functions are continuous. □

Exercise 4 (Graphs of finite measure)**.** Let b be a connected graph over (X,m). Suppose that m satisfies $m(X) < \infty$. Show that the following statements are equivalent:

(i) b is recurrent.

(ii) $e^{-tL^{(D)}}1 = 1$.

(iii) $Q^{(D)} = Q^{(N)}$.

Solution. "(i) $\implies$ (iii)": Let b be recurrent. By definition this means that $\mathcal{D} = \mathcal{D}_0$. Note that by Theorem 4.1, we have $D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X,m)$. We also know that $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$ simply from the definition of $Q^{(N)}$. Since $Q^{(D)}$ and $Q^{(N)}$ only differ by their domain, the preceding already implies that $Q^{(D)} = Q^{(N)}$.

"(iii) $\implies$ (ii)": Note that by assumption

$$D(Q^{(D)}) = D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$$

and since $m(X) < \infty$, we have $1 \in \ell^2(X,m)$. On the other hand, we also have $1 \in \mathcal{D}$ (by definition of $\mathcal{D}$) and thus also $1 \in D(Q^{(D)})$. Further, remember that

$$Q^{(D)}(f) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))^2$$

[1] To obtain the general case, split f and g into positive and negative part and split the series into four.

which implies $Q^{(D)}(1) = 0$. Using Exercise 8.2, we see that the Dirichlet Laplacian $L^{(D)}$ has eigenvalue 0. The normalized eigenfunction is given by $f \equiv 1$, since

$$0 \overset{(!)}{=} L^{(D)}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y)) \quad \text{for all } x \in X$$

implies f to be constant since b is a connected graph.

Now, by the spectral mapping theorem, we obtain that $e^{-t0} = 1$ is an eigenvalue of the operator $e^{-tL^{(D)}}$ with eigenfunction $f \equiv 1$ which shows (ii).

"(ii) $\implies$ (i)": A quick look at (ii) shows that 1 is an eigenvalue of the operator $e^{-tL^{(D)}}$ with eigenfunction $f \equiv 1$. With the help of spectral calculus, we recover that

$$L^{(D)}1 = \lim_{t \to 0} \frac{1}{t}(I - e^{-tL})1 = 0$$

so that $L^{(D)}$ has eigenvalue 0 with eigenfunction $f \equiv 1$. By Exercise 8.2, this implies that $1 \in D(Q^{(D)})$ and with Theorem 4.1 we immediately see that $1 \in \mathcal{D}_0$. Then b is recurrent by Theorem 9.6. □