Solutions for Sheet 10 of ISem 26

Team Hagen[1]

Exercise 1 (Superharmonic functions). Let b be a graph over X. Show the following statements:

(a) The pointwise infimum of a set of superharmonic functions is superharmonic whenever the infimum is a finite function.

(b) The sum of two superharmonic functions is superharmonic.

(c) The limit of any monotonically increasing sequence of superharmonic functions is superharmonic whenever the limit is pointwise finite.

(d) The composition $\varphi \circ u$ of a monotonically increasing concave function $\varphi : [0, \infty) \to [0, \infty]$ with a positive superharmonic function u is a superharmonic function which is non-harmonic whenever φ is strictly concave.

Solution. (i) Let $(u_i)_{i\in I}$ be a family of superharmonic functions such that $u(x) := \inf_{i\in I} u_i(x) \in \mathbb{R}$ exists for every $x \in X$. We want to show that the function $u : X \to \mathbb{R}$, $x \mapsto u(x)$ is superharmonic. First, it is an immediate observation that $u \in \mathcal{F}$ (note that according to the local Harnack inequality (Theorem 8.1) each u_i is either non-negative or strictly negative on each connected component).
Next, we have to show that $\mathcal{L}\, u \geq 0$. In the case that $\sum_{y\in X} b(x, y) = 0$ for some $x \in X$ it follows that $b(x, y) = 0$ for every $y \in X$ and therefore $(\mathcal{L}\, u)(x) = 0$. Thus, we only have to consider those $x \in x$, where $\sum_{y\in X} b(x, y) > 0$. Assume that $(\mathcal{L}\, u)(x) < 0$ for some $x \in X$ and put $\varepsilon := -\frac{(\mathcal{L}\, u)(x)m(x)}{2\sum_{y\in X} b(x,y)} > 0$ (note that we suppose that $\sum_{y\in X} b(x, y) > 0$). Choose $i_0 \in I$ such that $u(x) = \inf_{i\in I} u_i(x) > u_{i_0}(x) - \varepsilon$. Then:

$$
\begin{aligned}
(\mathcal{L}\, u)(x) &= \frac{1}{m(x)} \sum_{y\in X} b(x, y)(u(x) - u(y)) \geq \frac{1}{m(x)} \sum_{y\in X} b(x, y)(u(x) - u_{i_0}(y)) \\
&> \frac{1}{m(x)} \sum_{y\in X} b(x, y)(u_{i_0}(x) - u_{i_0}(y) - \varepsilon) = (\mathcal{L}\, u_{i_0})(x) - \frac{\varepsilon}{m(x)} \sum_{y\in X} b(x, y) \\
&= (\mathcal{L}\, u_{i_0})(x) + \frac{(\mathcal{L}\, u)(x)}{2} \geq \frac{(\mathcal{L}\, u)(x)}{2} \quad \Longleftrightarrow \quad \frac{(\mathcal{L}\, u)(x)}{2} \geq 0,
\end{aligned}
$$

i.e., $(\mathcal{L}\, u)(x) \geq 0$, a contradiction. Thus it follows that $(\mathcal{L}\, u)(x) \geq 0$ for every $x \in X$, in other words $u \in \mathcal{F}$ is superharmonic.
(ii) Let $u, v \in \mathcal{F}$ be two superharmonic functions, i.e. $\mathcal{L}\, u, \mathcal{L}\, v \geq 0$. Then, for $x \in X$, we clearly have that

$$\sum_{y\in X} b(x, y)|u(y) + v(y)| \leq \sum_{y\in X} b(x, y)|u(y)| + \sum_{y\in X} b(x, y)|v(y)| < \infty.$$

Thus, $u + v \in \mathcal{F}$. Furthermore, we have

$$\mathcal{L}(u + v) = \mathcal{L}\, u + \mathcal{L}\, v \geq 0,$$

in other words, $u + v$ is superharmonic.
(iii) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of superharmonic, pointwise increasing functions such that $u_n \nearrow u$ pointwise to some function $u : X \to \mathbb{R}$. Without loss of generality, we suppose that $u_n \geq 0$ for every $n \in \mathbb{N}$ (otherwise one can turn to the sequence $w_n := u_n - u_1$). We have to show that u is superharmonic. Let $x \in X$. Since each u_n is superharmonic, we have that $(\mathcal{L}\, u_n)(x) \geq 0$ for every $n \in \mathbb{N}$, equivalently

$$\sum_{y\in X} b(x, y)u_n(y) \leq \sum_{y\in X} b(x, y)u_n(x)$$

for every $n \in \mathbb{N}$ which implies according to Fatou's lemma

$$\sum_{y\in X} b(x, y)u(y) \leq \liminf_{n\to\infty} \sum_{y\in X} b(x, y)u_n(y) \leq \liminf_{n\to\infty} \sum_{y\in X} b(x, y)u_n(x) = u(x)m(x) \operatorname{Deg}(x)$$

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which proves that $u \in \mathcal{F}$ is superharmonic as $x \in X$ was arbitrary.
(iv) As $u \in \mathcal{F}$ is superharmonic, we have that $(\mathcal{L}\, u)(x) \geq 0$, equivalently

$$\sum_{y \in X} b(x,y)(u(x) - u(y)) \geq 0 \iff u(x) \geq \sum_{y \in X} \frac{b(x,y)}{\sum_{z \in X} b(x,z)} u(y).$$

As φ is monotonically increasing as well as concave (or more precisely, by Jensen's inequality for concave functions), this implies at first that

$$\varphi(u(x)) \geq \varphi\Big(\sum_{y \in X} \frac{b(x,y)}{\sum_{z \in X} b(x,z)} u(y)\Big) \geq \sum_{y \in X} \frac{b(x,y)}{\sum_{z \in X} b(x,z)} \varphi(u(y)). \tag{1}$$

Hence $\varphi \circ u \in \mathcal{F}$ with

$$\begin{aligned}(\mathcal{L}(\varphi \circ u))(x) &= \frac{1}{m(x)} \sum_{y \in X} b(x,y)(\varphi(u(x)) - \varphi(u(y))) \\ &= \frac{1}{m(x)} \sum_{y \in X} b(x,y)\varphi(u(x)) - \frac{1}{m(x)} \sum_{y \in X} b(x,y)\varphi(u(y)) \\ &\geq \frac{1}{m(x)} \sum_{y \in X} b(x,y)\varphi(u(y)) - \frac{1}{m(x)} \sum_{y \in X} b(x,y)\varphi(u(y)) = 0\end{aligned}$$

for every $x \in X$ and one can observe that $\mathcal{L}(\varphi \circ u) > 0$ if and only if φ is strictly concave (and thus we have a strict inequality in (1)). □

Exercise 2 (Positive eigenfunctions). Let (b,c) be a connected graph over a measure space (X,m). Let L denote the Laplacian associated to (b,c) over (X,m). If there is an eigenvalue λ of L with a positive eigenfunction, then $\lambda = \lambda_0(L)$. Show that this statement is false if the graph is not connected.

Solution. We just make use of the Agmon-Allegretto-Piepenbrink theorem (Theorem 8.10). So let λ be an eigenvalue of L with a nontrivial positive eigenfunction $u \in D(L)$, in particular $\lambda_0(L) \leq \lambda$ due to the variational characterization of the first eigenvalue (see Theorem 6.3). As we have that $D(L) \subset \mathcal{F}$ (see Theorem 4.9) we have that $u \in \mathcal{F}$ with $(\mathcal{L} - \lambda)u = 0$. According to the Agmon-Allegretto-Piepenbrink theorem, it follows that $\lambda \leq \lambda_0(L)$, in particular $\lambda = \lambda_0(L)$.
Next, consider $X = \{x,y,z\}$ with $b(x,y) = 1$ and $b(y,z) = b(x,z) = 0$ as well as $c(x) = c(z) = 0$ as well as $c(y) = \frac{3}{2}$ and $m = \mathbf{1}$ the counting measure. Then (b,c) is a non-connected graph over (X,m) and for $f : X \to \mathbb{R}$ defined by $f(x) := 2$, $f(y) := 1$ and $f(z) = 0$, we have

$$\begin{aligned}(L_{b,c}f)(x) &= f(x) - f(y) + c(x)f(x) = 1 = \frac{1}{2}f(x), \\ (L_{b,c}f)(y) &= f(y) - f(x) + c(y)f(y) = -1 + \frac{3}{2} = \frac{1}{2} = \frac{1}{2}f(y), \\ (L_{b,c}f)(z) &= 0 = \frac{1}{2}f(z).\end{aligned}$$

Thus, $\frac{1}{2}$ is an eigenvalue of $L_{b,c}$ with positive eigenfunction f. However, by considering $g : X \to \mathbb{R}$ given by $g(x) = g(y) = 0$ and $g(z) = 1$, we see that $\lambda_0(L) = 0$ is the minimal eigenvalue of $L_{b,c}$. Thus, the statement does not hold for non-connected graphs. □

Exercise 3 (Finite measure and bounded degree implies recurrent). Let b be a graph over (X,m) such that $m(X) < \infty$ and Deg is bounded. Show that the graph is recurrent.

Solution. Let $(K_n)_{n \in \mathbb{N}}$ an increasing sequence of finite subsets of X such that $X = \bigcup_{n \in \mathbb{N}} K_n$. We consider the sequence $(e_n)_{n \in \mathbb{N}} \subseteq C_c(X)$ given by

$$e_n := \mathbf{1}_{K_n} \quad \text{for } n \in \mathbb{N}.$$

Then, on the one hand, $0 \leq e_n \leq 1$ such that $e_n \to \mathbf{1}$ pointwise. On the other hand, according to the boundedness of the degree Deg, we find a constant $M \geq 0$ such that $\mathrm{Deg}(x) \leq M$ for every $x \in X$ which is equivalent to the fact that $\sum_{y\in X} b(x,y) \le Mm(x)$ for every $x \in X$. This implies

$$0 \le Q(e_n) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(e_n(x)-e_n(y))^2 = \sum_{x\in X\setminus K_n, y\in K_n} b(x,y)$$
$$= \sum_{x\in X\setminus K_n}\sum_{y\in K_n} b(x,y) \le M \sum_{x\in X\setminus K_n} m(x) \to 0 \quad \text{as } n\to\infty,$$

since $m(X) = \sum_{x\in X} m(x) < \infty$. Hence $Q(e_n) \to 0$ also holds. □

Exercise 4 (Khasminskii criterion). Let b be a connected graph over X. Show that b is recurrent if and only if there exists a function $f \in \mathcal{D}$ such that $f(x) \to \infty$ as $x \to \infty$ where $X \cup \{\infty\}$ is the one point compactification of X.

Solution. „$\Rightarrow$“: Suppose that b is recurrent, i.e., there exists a sequence $(e_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ such that $e_n \to \mathbf{1}$ pointwise and $Q(e_n) \to 0$. We claim that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $Q(e_{n_k})^{\frac{1}{2}} < \frac{1}{k^2}$ and for every $x \in X$

$$1 - e_{n_k}(x) < \frac{1}{k^2} \quad \text{for almost every } k \in \mathbb{N}.$$

First, as $Q(e_n) \to 0$, we can choose a subsequence $(n_k^{(0)})_{k\in\mathbb{N}}$ such that $Q(e_{e_{n_k}^{(0)}})^{\frac{1}{2}} < \frac{1}{k^2}$ for every $k \in \mathbb{N}$. As X is countable, we can write $X = \{x_n \mid n \in \mathbb{N}\}$. First, let $n = 1$. As $e_n(x_1) \to 1$, there exists a subsequence $(n_k^{(1)})_{k\in\mathbb{N}}$ of $(n_k^{(0)})_{k\in\mathbb{N}}$ such that

$$1 - e_{n_k^{(1)}}(x_1) < \frac{1}{k^2} \quad \text{for every } k \in \mathbb{N}.$$

Suppose next that for $x_1,\ldots,x_{l-1} \in X$, $l \ge 2$ there exist subsequences $(n_k^{(i)})_{k\in\mathbb{N}}$, $i = 1,\ldots,l-1$ such that $(n_k^{(i+1)})_{k\in\mathbb{N}}$ is a subsequence of $(n_k^{(i)})_{k\in\mathbb{N}}$ for every $k = 1,\ldots,l-2$ as well as

$$1 - e_{n_k^{(i)}}(x_i) < \frac{1}{k^2} \quad \text{for every } k \in N$$

and $i = 1,\ldots,l-1$. Then, as also $e_{n_k^{(l-1)}}(x_l) \to 1$, we can choose a subsequence $(n_k^{(l)})_{k\in\mathbb{N}}$ of $(n_k^{(l-1)})_{k\in\mathbb{N}}$ such that

$$1 - e_{n_k^{(l)}}(x_l) < \frac{1}{k^2} \quad \text{for every } k \in \mathbb{N}.$$

Thus, we have shown inductively that there exist subsequences $(n_k^{(l)})_{k\in\mathbb{N}}$ for $l \in \mathbb{N}$ such that $(n_k^{(i+1)})_{k\in\mathbb{N}}$ is a subsequence $(n_k^{(i)})_{k\in\mathbb{N}}$ and $1 - e_{n_k^{(l)}}(x_l) < \frac{1}{k^2}$ for all $k, l \in \mathbb{N}$. Now put $n_k := n_k^{(k)}$. Then, for $x = x_l \in X$ and $k > l$ it follows by construction that n_k is a subsequence of $n_k^{(l)}$ and therefore $1 - e_{n_k}(x) < \frac{1}{k^2}$ for every $k > l$. Also note that $Q(e_{n_k})^{\frac{1}{2}} < \frac{1}{k^2}$ for every $k \in \mathbb{N}$ as $(e_{n_k})_{k\in\mathbb{N}}$ is clearly a subsequence of $(e_{n_k^{(0)}})_{k\in\mathbb{N}}$. This shows the first claim.
Next, define $f := \sum_{k=1}^{\infty}(\mathbf{1} - e_{n_k})$. We show that $f \in \mathcal{D}$ with $f(x) \to \infty$ as $x \to \infty$. To this end, let $x \in X$ and choose $k_0 \in \mathbb{N}$ such that $1 - e_{n_k}(x) < \frac{1}{k^2}$ for every $k \ge k_0$. Then:

$$f(x) = \sum_{k=1}^{\infty}(1 - e_{n_k}(x)) = \sum_{k=1}^{k_0-1}(1 - e_{n_k}(x)) + \sum_{k=k_0}^{\infty}(1 - e_{n_k}(x))$$
$$\le \sum_{k=1}^{k_0-1}(1 - e_{n_k}(x)) + \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

Hence, f is well-defined. Put $f_N := \sum_{k=1}^{N}(\mathbf{1} - e_{n_k})$. Since $f_N \to f$ pointwise as $N \to \infty$, we obtain by Proposition 2.4 (or in other words, the lower semi-continuity of Q in $C(X)$) that

$$Q(f) \le \liminf_{N\to\infty} Q(f_N). \tag{2}$$

Moreover, using Cauchy-Schwarz inequality, we observe that

$$\begin{aligned} Q(f_N) &= Q\Big(\sum_{k=1}^{N}(\mathbf{1}-e_{n_k})\Big) = \sum_{k=1}^{N}\sum_{l=1}^{N} Q(\mathbf{1}-e_{n_k},\mathbf{1}-e_{n_l}) \\ &\le \sum_{k=1}^{N}\sum_{l=1}^{N} Q(\mathbf{1}-e_{n_k})^{\frac12} Q(\mathbf{1}-e_{n_l})^{\frac12} = \Big(\sum_{k=1}^{N} \underbrace{Q(\mathbf{1}-e_{n_k})^{\frac12}}_{=Q(e_{n_k})^{\frac12}}\Big)^2 \\ &= \Big(\sum_{k=1}^{N} Q(e_{n_k})^{\frac12}\Big)^2 \le \Big(\sum_{k=1}^{N}\frac{1}{k^2}\Big)^2 \end{aligned}$$

for every $N \in \mathbb{N}$. Together with (2), this implies that $Q(f) \le \liminf_{N\to\infty} Q(f_N) < \infty$, i.e. $f \in \mathcal{D}$. Lastly, we show that $f(x)$ tends to ∞ as $x \to \infty$. To do so, let $C > 0$ and choose $k_0 \in \mathbb{N}$ with $k_0 > C$. Putting $K := \bigcup_{k=1}^{k_0} \operatorname{supp}(e_{n_k})$ which is finite by the fact that each e_{n_k} belongs to $C_c(X)$, we obtain for $x \in X \setminus K$:

$$f(x) = \sum_{k=1}^{k_0}(1-\underbrace{e_{n_k}(x)}_{=0}) + \sum_{k=k_0+1}^{\infty}(1-e_{n_k}(x)) = k_0 + \sum_{k=k_0+1}^{\infty}(1-e_{n_k}(x)) > C.$$

Hence, $f(x) \to \infty$ as $x \to \infty$.
„$\Leftarrow$“: Let conversely $f \in \mathcal{D}$ such that $f(x) \to \infty$ as $x \to \infty$. For $n \in \mathbb{N}$ we define

$$e_n := \Big(\Big(\mathbf{1} - \frac{1}{n}f\Big) \vee 0\Big) \wedge \mathbf{1}.$$

Then clearly $0 \le e_n \le 1$ and each e_n belongs to $C_c(X)$ as for every $n \in \mathbb{N}$ there exists a finite subset K of X such that $f|_{X\setminus K} > n$, equivalently, $\mathbf{1} - \frac{1}{n}f > 0$ can only occur finitely often. Furthermore, it clearly follows that $e_n \to \mathbf{1}$ pointwise. Thus, it is only left to show that $Q(e_n) \to 0$. Indeed, by Proposition 2.3, Q is compatible with normal contractions, which yields (as $f \in \mathcal{D}$)

$$Q(e_n) = Q\Big(\Big(\Big(\mathbf{1} - \frac{1}{n}f\Big) \vee 0\Big) \wedge \mathbf{1}\Big) \le Q\Big(\mathbf{1} - \frac{1}{n}f\Big) = \frac{1}{n^2}Q(f) \to 0$$

as $n \to \infty$ and the proof is complete. □

Bonus Exercise 1 ($\operatorname{Deg} \ge w$**).** Let (b,c) be a connected graph over (X,m) and $w : X \longrightarrow \mathbb{R}$ such that there is non-trivial $u \in \mathcal{F}$, $u \ge 0$ such that $(\mathcal{L} - w)u \ge 0$. Show that

$$\operatorname{Deg} \ge w.$$

Solution. Suppose that there exists a point $x \in X$ such that $\operatorname{Deg}(x) < w(x)$. If $u(x) \neq 0$ it follows that $\operatorname{Deg}(x)u(x) < w(x)u(x) \le (\mathcal{L}u)(x)$, in particular

$$\frac{1}{m(x)}\Big(\sum_{y\in X} b(x,y)u(x) + c(x)u(x)\Big) < \frac{1}{m(x)}\Big(\sum_{y\in X} b(x,y)(u(x)-u(y)) + c(x)u(y)\Big),$$

which is equivalent to the fact that

$$\sum_{y\in X} b(x,y)u(y) < 0,$$

a contradiction. Thus, it follows that $u(x) = 0$. Therefore, as $(\mathcal{L} - w)u \ge 0$, it follows that $(\mathcal{L}u)(x) \ge 0$, i.e.

$$\frac{1}{m(x)}\Big(\sum_{y\in X} b(x,y)(u(x)-u(y)) + c(x)u(x)\Big) = -\frac{1}{m(x)}\sum_{y\in X} b(x,y)u(y) \ge 0.$$

This implies that $u(y) = 0$ for all $y \sim x$. Lastly, let $z \in X$ be arbitrary. By connectedness of the graph, we find a path $z = x_0, \ldots, x_n$ where x_n is adjacent to some $y \sim x$. As $u(y) = 0$, we can do the same argumentation as above by using $u(y) = 0$ and $(\mathcal{L}u)(y) \ge 0$ respectively and obtain that also $u(x_n) = 0$. Inductively, we reach at $u(z) = 0$. Thus, $u = 0$ which is a contradiction to the assumption that u is non-trivial, i.e. $\operatorname{Deg} \ge w$. □

An additional note: The assumptions that the graph is connected and that $u \in \mathcal{F}$ is non-trivial are indeed necessary. On the one hand, one can consider $X = \{x, y, z\}$ with $m = \mathbf{1}$, $b(x, y) = b(y, x) = 1$, $b(x, z) = b(y, z) = 0$ as well as $c \equiv 0$. Furthermore, one can choose $u > 0$ with $u(x) = u(y) = 1$ and $u(z) = 0$. Then for $w : X \to \mathbb{R}$ defined by $w(x) = w(y) = 0$ and $w(z) = 1$, we observe that

$$((\mathcal{L} - w)u)(x) = -w(x)u(x) = 0, \quad ((\mathcal{L} - w)u)(y) = -w(y)u(y) = 0,$$

as well as $(\mathcal{L} - w)u)(z) = -w(z)u(z) = 0$. But we also observe that $\mathrm{Deg}(z) = 0 < w(z)$.
On the other hand one can consider the graph b which is just given by one edge with two vertices and standard weights, i.e. $X = \{x, y\}$, $b(x, y) = b(y, x) = 1$ as well as $c \equiv 0$ and $m = \mathbf{1}$, in particular $\mathrm{Deg}(x) = \mathrm{Deg}(y) = 1$. Then, $u \equiv 0$ trivially solves $(\mathcal{L} - w)u \geq 0$ for every $w : X \to \mathbb{R}$ but we do not have $\mathrm{Deg} \geq w$ whenever w is chosen larger than 1 in some vertex $x, y \in X$.
Overall, the claim does not hold if the graph is either not connected or either $u \geq 0$ is trivial.