ISEM 26, lecture 9 - Exercises

Team Karlsruhe

Exercise 1 (Recovering the combinatorial metric for trees)**.**
Let b be a tree with standard weights over a countable measure space X with the counting measure m. Set

$$\sigma(x,y) = \sup\{|f(x)-f(y)| : f \in A_1(X)\}\,.$$

Then

$$\begin{aligned}
\sigma(x,y) &= 1\,, && \text{if } d_{comb}(x,y) = 1\,,\\
\sigma(x,y) &= d_{comb}(x,y)/\sqrt{2}\,, && \text{if } d_{comb}(x,y) \text{ is even}\,,\\
\sigma(x,y) &= \sqrt{5}\,, && \text{if } d_{comb}(x,y) = 3\,,\\
\sqrt{\frac{d_{comb}(x,y)^2+1}{2}} \le \sigma(x,y) &\le \sqrt{\frac{d_{comb}(x,y)(d_{comb}(x,y)+1)}{2}}\,, && \text{if } d_{comb}(x,y) \ge 5 \text{ is odd}\,.
\end{aligned}$$

Conjecture: *when $d_{comb}(x,y)$ is odd, there holds equality*

$$\sigma(x,y) = \sqrt{\frac{d_{comb}(x,y)^2+1}{2}}\,.$$

Solution. Recall that a tree is a connected graph without any cycles, and standard weights means $b(x,y) \in \{0,1\}$ for all $x,y \in X$. In a tree, from one vertex to another, there exists exactly one path (without repeating vertices). Moreover, since we consider standard weights and the counting measure, we have

$$|\nabla f|^2(x) = \sum_{y\sim x} |f(x)-f(y)|^2\,, \quad x \in X\,.$$

Consider first two neighboring vertices $x,y \in X$, i.e. $d_{\mathrm{comb}}(x,y) = 1$. Define $f \in C(X)$ by

$$f(z) = \begin{cases} 0, & \text{if } z \text{ is closer to } x \text{ (w.r.t. } d_{\mathrm{comb}})\,,\\ 1, & \text{if } z \text{ is closer to } y \text{ (w.r.t. } d_{\mathrm{comb}})\,. \end{cases}$$

Since X is a tree (note that we often identify X and the graph b), f is well-defined and the only edge between the subsets $\{f=1\} \subseteq X$ and $\{f=0\} \subseteq Y$ is (x,y), so that $|\nabla f|^2(x) = |\nabla f|^2(y) = 1$ and $|\nabla f|^2(z) = 0$ for $z \in X \setminus \{x,y\}$. Thus, $f \in A_1(X)$, which gives $\sigma(x,y) \ge |f(x)-f(y)| = 1$. Conversely, $\sigma(x,y) \le 1$ follows immediately from

$$|g(x)-g(y)|^2 \le |\nabla g|^2(x) \le 1$$

for any $g \in A_1(X)$.
Consider $x, y \in X$ non-adjacent, i.e. $d_{\text{comb}}(x,y) = n \geq 2$, with a path $x = x_0 \sim x_1 \sim \ldots \sim x_n = y$ in b. Fix $f \in A_1(X)$. Using the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} |f(y) - f(x)|^2 &= \left(\sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \right)^2 \\ &\leq n \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^2 \leq n \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} |\nabla f|^2 (x_{2i-1}) \leq n \left\lfloor \frac{n+1}{2} \right\rfloor . \end{aligned}$$

Consider the case n even. We have proven $\sigma(x,y) \leq n/\sqrt{2}$ above. We show equality by finding a function $f \in A_1(X)$ which attains this upper bound. For the path $x = x_0 \sim x_1 \sim \ldots \sim x_n = y$, we denote by

$$X_i = \{ z \in X : z \text{ is closer (w.r.t. to } d_{\text{comb}}) \text{ to } x_i \text{ than to any other } x_j \} \subseteq X$$

the connected component of x_i in X without the path edges (recall that X is a tree, so that removing the path edges splits X into $n+1$ non-empty connected components), and define $f \in C(X)$ by

$$f(z) = i/\sqrt{2}\,, \quad z \in X_i\,.$$

Then, for $z \in X$ not on the path, we have $|\nabla f|^2(z) = 0$, and

$$\begin{gathered} |\nabla f|^2 (x) = (f(x) - f(x_1))^2 = 1/2\,, \qquad |\nabla f|^2 (y) = (f(x_{n-1}) - f(y))^2 = 1/2\,, \\ |\nabla f|^2 (x_i) = (f(x_i) - f(x_{i-1}))^2 + (f(x_i) - f(x_{i+1}))^2 = 1\,, \quad i = 1, \ldots, m-1\,, \end{gathered}$$

so that $f \in A_1(X)$ and $f(y) - f(x) = n/\sqrt{2}$. We conclude that, when $d_{\text{comb}}(x,y)$ is even, we have $\sigma(x,y) = d_{\text{comb}}(x,y)/\sqrt{2}$.
Of course, when n is odd, the above function still lies in $A_1(X)$ and gives the lower bound

$$\frac{d_{\text{comb}}(x,y)}{\sqrt{2}} \leq \sigma(x,y) \leq \sqrt{\frac{d_{\text{comb}}(x,y)(d_{\text{comb}}(x,y)+1)}{2}}\,.$$

We want to improve this lower bound. Fix $x, y \in X$ with $n = d_{\text{comb}}(x,y) = 2k+1$ odd and at least 3. From the above reasoning (when n was even), it is clear that we are looking for a function $f \in A_1(X)$ which increases as much as possible *along the path* $P = (x_0, x_1, \ldots, x_n)$ from x to y. Thus, w.l.o.g. we restrict our attention to functions which are constant on each X_i (see the definition above), which we identify with functions in $C(P)$. If we denote by $a_i = f(x_i) - f(x_{i-1})$ the increment of such a function f along an edge of P, we are looking to maximize

$$I(f) = f(y) - f(x) = a_1 + a_2 + \ldots + a_n$$

over the subset

$$V = \{ f \in C(P) : a_i^2 + a_{i+1}^2 \leq 1 \ \text{ for } \ i = 1, \ldots, n-1 \} \subseteq C(P)\,,$$

where the constraints correspond to the condition $|\nabla f|^2 \leq 1$ (this requirement can be dropped at x_0 and x_n if it holds for x_1 resp. x_{n-1}). Forgetting for a moment the underlying function f (i.e.

identifying $C(P)$ modulo the constant functions with $\mathbb{R}^n$ via $f \mapsto (a_i)_i$), this is a purely analytical maximization problem with constraints

$$\text{maximize} \;\; a_1 + a_2 + \ldots + a_n$$
$$\text{under the } n-1 \text{ constraints} \;\; a_i^2 + a_{i+1}^2 \le 1\,, \quad i = 1, \ldots, n-1\,.$$

Clearly, any maximizer will satisfy $a_i \ge 0$ for all $i = 1, \ldots, n$. By compactness of the subset of $\mathbb{R}^n$ given by these constraints and continuity, there exists a global maximum, which by the above reasoning will correspond to $\sigma(x, y)$.
We don't know how to solve this problem exactly, but one approach is to replace the inequality constraints by the equality constraints

$$a_i^2 + a_{i+1}^2 = 1\,, \quad i = 1, \ldots, n-1\,.$$

This condition seems plausible, since it is easily seen that, if $a \in \mathbb{R}^n$ is a solution of the optimization problem, then $a_1^2 + a_2^2 = 1$ and $a_{n-1}^2 + a_n^2 = 1$ must hold, and it cannot occur that two consecutive conditions are not binding in the sense that $a_i^2 + a_{i+1}^2 < 1$ and $a_{i+1}^2 + a_{i+2}^2 < 1$, otherwise one could increase a_{i+1} to get a bigger value. However we offer no proof that this simplification really leads to a maximum (except when $k = 1$, i.e. $n = 3$), so we get only a lower bound for $\sigma(x, y)$.
The equality constraints $a_i^2 + a_{i+1}^2 = 1$ (and the condition $a_i \ge 0$) give a direct relation between the a_i's, from which we can read off

$$a_0 = a_2 = \ldots = a_{2k} =: t \in [0, 1]\,, \quad a_1 = a_3 = \ldots = a_n = \sqrt{1 - t^2}\,.$$

Thus, maximizing $a_1 + \ldots + a_n$ under the equality constraints boils down to maximizing the scalar function $\theta(t) = (k+1)t + k\sqrt{1-t^2}$ over the interval $[0, 1]$. The maximum is attained at $t^* = (k+1)/\sqrt{k^2 + (k+1)^2}$, and has the value $\theta(t^*) = \ldots = \sqrt{2k^2 + 2k + 1}$. This leads to the improved lower bound (write d short for $d_{\text{comb}}(x, y)$ and recall $d = 2k + 1$)

$$\frac{d}{\sqrt{2}} = \sqrt{2k^2 + 2k + \frac{1}{2}} < \sqrt{2k^2 + 2k + 1} \le \sigma(x, y) \le \sqrt{2k^2 + 3k + 1} = \sqrt{\frac{d(d+1)}{2}}\,.$$

Observe also $\sqrt{2k^2 + 2k + 1} = \sqrt{(d^2 + 1)/2}$.
From the solution of the optimization problem (with equality constraints), we easily reconstruct the corresponding function $f \in A_1(X)$, which is given by (up to an additive constant)

$$f(x) = 0\,, \quad f(x_1) = t^*\,, \quad f(x_2) = t^* + \sqrt{1 - t^{*2}} \quad f(x_3) = 2t^* + \sqrt{1 - t^{*2}}\,,$$
$$\ldots\,, \quad f(y) = (k+1)t^* + k\sqrt{1 - t^{*2}} = \sqrt{2k^2 + 2k + 1}\,,$$

and $f(z) = f(x_i)$ for $z \in X_i$.
In particular, for $n = 3$, the function (which in this case is a true maximizer) is

$$f(x) = 0\,, \quad f(x_1) = 2/\sqrt{5}\,, \quad f(x_2) = 3/\sqrt{5}\,, \quad f(y) = 5/\sqrt{5} = \sqrt{5} > 3/\sqrt{2}\,.$$

Moreover, note that the approach with the optimization problem also works for n even, and in that case, replacing the inequality constraints with equalities yields the maximizing function we found before, which is indeed a true maximizer. This comforts the idea that the simplification is sensible. □

Exercise 2 (Finite sets have finite boundary area)**.**
Let b be a graph over (X,m) with an intrinsic metric ϱ. Then, any finite set $W \subseteq X$ has finite boundary area, and, more precisely,

$$A_{b\varrho}(\partial W) \le (mn)^{1/2}(W)\,,$$

where n denotes the normalizing measure $n(x) = \sum_{y\in X} b(x,y)$.

Remark. This is the statement from the book *Graphs and Dirichlet spaces*, Exercise 13.1, which is weaker than the one suggested in the ISEM lecture, since one has $(mn)(W)^{1/2} \le (mn)^{1/2}(W)$, and the inequality is sometimes strict.

Solution. We compute

$$\begin{aligned}
A_{b\varrho}(\partial W) &= \frac{1}{2}\sum_{(x,y)\in\partial W} b(x,y)\varrho(x,y) \\
&= \sum_{x\in W}\sum_{y\notin W} b(x,y)\varrho(x,y) && \text{(by symmetry of } b \text{ and } \varrho) \\
&\le \sum_{x\in W}\sum_{y\in X} b(x,y)\varrho(x,y) && \text{(by non-negativity)} \\
&\le \sum_{x\in W}\Big(\sum_{y\in X} b(x,y)\varrho(x,y)^2\Big)^{1/2}\Big(\sum_{y\in X} b(x,y)\Big)^{1/2} && \text{(by Cauchy-Schwarz)} \\
&\le \sum_{x\in W} m(x)^{1/2}\; n(x)^{1/2} && \text{(by } \rho \text{ being intrinsic)} \\
&= (mn)^{1/2}(W)\,.
\end{aligned}$$

□

Exercise 3 (Upper bound via the Cheeger constant h)**.**
Let b be a graph over (X,m) with an intrinsic metric ϱ satisfying $\varrho(x,y) \ge C > 0$ for all neighbouring $x \sim y$. Then, there holds the upper bound for the bottom of the spectrum of L

$$\lambda_0(L) \le \frac{h}{C}\,.$$

Solution. Recall the definition of the Cheeger constant $h = \inf_{W\subseteq X \text{ finite}} A_{b\varrho}(\partial W)/m(W)$, and the characterization of the smallest spectral value from Theorem 6.3,

$$\lambda_0(L) = \inf_{f\in D(Q)} \frac{Q(f)}{\|f\|^2}\;.$$

For $W \subseteq X$ finite, we consider its indicator function $f = \mathbf{1}_W \in C_c(X) \subseteq D(Q)$ and estimate

$$\begin{aligned}
\lambda_0(L) &\leq \frac{Q(\mathbf{1}_W)}{\|\mathbf{1}_W\|^2} \\
&\leq \frac{1}{2} \sum_{x,y\in X} b(x,y) \frac{\varrho(x,y)}{C} (\mathbf{1}_W(x) - \mathbf{1}_W(y))^2 \Big/ \Big(\sum_{x\in X} \mathbf{1}_W(x)^2 m(x) \Big) \\
&= \frac{1}{C} \sum_{(x,y)\in\partial W} b(x,y)\varrho(x,y) \Big/ \Big(\sum_{x\in W} m(x) \Big) \qquad \text{(since } (\mathbf{1}_W(x) - \mathbf{1}_W(y))^2 = \mathbf{1}_{\partial W}(x,y)) \\
&= \frac{A_{b\varrho}(W)}{C\, m(W)}\,.
\end{aligned}$$

The assertion now follows by taking the infimum over all finite subsets $W \subseteq X$. □

Exercise 4.
a) There is a graph b over (X, m) with bounded Degree and positive Cheeger constant h.
b) If b is a graph over (X, m') with finite measure $m'(X) < \infty$ and positive Cheeger constant $h > 0$, then $Q^{(D)}_{b,m'} \neq Q^{(N)}_{b,m'}$.

Solution. a) Consider the standard 3-regular tree with the counting measure, which is the tree such that each vertex has degree 3. Endow it with the (scaled down) combinatorial distance $\varrho = d_{\mathrm{comb}}/3$. Due to standard weights and $m \equiv 1$, each node has Deg 3, so that ϱ is intrinsic by Example 7.5. We observe that

$$m(W) = |W| \quad \text{ and } \quad A(W) = A_{b\varrho}(W) = \frac{1}{3}\,|\{\text{edges from } W \text{ to } X \setminus W\ \}|$$

for all finite subsets $W \subseteq X$. We identify a subset W with the subgraph (which is a forest, i.e. a disjoint union of trees) of b generated by its nodes.

We prove by induction that any finite and connected subtree $W \subseteq X$ satisfies $A(W) \geq m(W)/3$, which yields the assertion. For a general finite W generating a subforest F, the claim follows via $m(T_1 \cup T_2) = m(T_1) + m(T_2)$ and

$$A(T_1 \,\dot\cup\, T_2) = |\partial T_1 \,\dot\cup\, \partial T_2|\,/3 = A(T_1) + A(T_2)$$

for two disjoint trees $T_1, T_2 \subseteq F$ due to $d_{\mathrm{comb}}(T_1, T_2) \geq 2$ (else T_1 would be connected to T_2).

Let $W_0 = \{o\} \subseteq X$. Then $m(W_0) = 1$ and $A(W_0) = 3/3 = 1$. Consider a subtree $W_1 \subseteq X$ with $o \in W_1$ and such that nodes in W_1 have at most combinatorial distance 1 from o, so that we can write $W_1 = W_0 \cup L_1$ and each $x \in L_1$ is a leaf of W_1 (i.e. a vertex of degree 1 in W_1) which is added to W_0. Adding a leaf x to W_0 removes the edge (o, x) from the boundary, but creates two new edges from x to outside the tree (draw a picture!). Thus, we have $m(W_1) = m(W_0) + m(L_1)$, and $A(W_1) = A(W_0) - m(L_1)/3 + 2m(L_1)/3 = A(W_0) + m(L_1)/3$, from which we get

$$A(W_1) = A(W_0) + m(L_1)/3 \geq m(W_0) + m(L_1)/3 \geq m(W_0)/3 + m(L_1)/3 = m(W_1)/3$$

By iteratively adding layers, the same reasoning gives

$$A(W_k) = A(W_{k-1}) + m(L_k)/3 \geq m(W_{k-1})/3 + m(L_k)/3 = m(W_k)/3\,,$$

for connected subtrees W_k with $o \in W_k$ with nodes at combinatorial distance at most k from o. Therefore, we arrive at the assertion $h \geq 1/3$.
By considering subtrees $T_n = B_n^{d_{\text{comb}}}(o) = \{z \in X : d_{\text{comb}}(z, o) \leq n\} \subseteq X$, a direct computation gives

$$\frac{A(T_n)}{m(T_n)} = \frac{1}{3}\frac{3 \cdot 2^n}{3 \cdot 2^n - 2} \to \frac{1}{3}, \quad n \to \infty,$$

which gives $h \leq 1/3$, and thus $h = 1/3$.
Note: the same reasoning works for the k-regular tree and gives $h_k = 1/k$, $k \geq 3$ (but for $k = 2$, there holds $h_2 = 0$).

b) Since $m'(X) < \infty$, the constant functions lie in $\ell^2(X)$ and thus in $D(Q^{(N)}_{b,m'})$ (recall the definition from Chapter 4.1), and any constant $f \in C(X)$ satisfies $Q^{(N)}_{b,m'}(f) = 0$. But no constant function lies in $D(Q^{(D)}_{b,m'})$, because for any nonzero function $\varphi \in D(Q^{(D)}_{b,m'})$ we have

$$Q^{(D)}_{b,m'}(\varphi) \geq \|\varphi\|^2 \lambda_0(L^{(D)}) \geq \|\varphi\|^2 h^2/2 > 0,$$

by the variational characterization Theorem 6.3, the Cheeger inequality Theorem 7.8 and the assumption $h > 0$. This proves $D(Q^{(D)}_{b,m'}) \neq D(Q^{(N)}_{b,m'})$. □