Solution key for Lecture 8 of ISEM 26

December 14, 2022

Exercise 1

The definition of g_α yields

$$g_\alpha(x,y) = \frac{\langle 1_x, (L+\alpha)^{-1} 1_y \rangle}{m(x)m(y)}.$$

It follows from Theorem 6.1 that $g_\alpha > 0$ for all $x, y \in X$ if and only if (b, c) is connected.

First, suppose $\lambda_0 = 0$. Then let u be the ground state, and so $1_{\lambda_0}(L) = \langle u, \cdot \rangle u$. For μ from the proof of Theorem 6.6, we get

$$\begin{aligned} m(x)m(y)|\alpha g_\alpha(x,y) - u(x)u(y)| &= |\langle 1_x, (\alpha(L+\alpha)^{-1} - 1_{\lambda_0}(L))1_y \rangle| \\ &= \left| \int_0^\infty (1 - 1_0(s)) \, d\mu(s) \right| \to 0 \qquad \text{as } \alpha \to 0, \end{aligned}$$

by Lebesgure's dominated convergence theorem. In the above second equality, we used $\lambda_0 = 0$.

Next, suppose $\lambda_0 > 0$. Then, by taking $u \equiv 0$,

$$m(x)m(y)|\alpha g_\alpha(x,y) - u(x)u(y)| = |\langle 1_x, \alpha(L+\alpha)^{-1} 1_y \rangle| = \left| \int_{\lambda_0}^\infty \frac{\alpha}{\lambda_0 + \alpha} d\mu(s) \right| \to 0$$

as $\alpha \to 0$. Overall, we proved both claims in the exercise.

Exercise 2

Recall that

$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)|f(x) - f(y)|^2 + \sum_{x \in X} c(x)|f(x)|^2,$$

$$L(f)(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).$$

We see that $Q(1) = 0$ is equivalent to $c \equiv 0$, which clearly implies that $f \equiv C$ for some constant C is an eigenfunction of L with eigenvalue 0.

For the other direction, suppose $\lambda_0 = 0$. Then there exists an eigenfunction $f \not\equiv 0$ such that $L(f) \equiv 0$. Thus

$$0 = \langle f, L(f) \rangle = Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)|f(x) - f(y)|^2 + \sum_{x \in X} c(x)|f(x)|^2.$$

This implies that $f \equiv C$ for some $C \neq 0$, and then $c \equiv 0$.

Exercise 3

Since $m(X) = \infty$, by Example 6.5, we have $\lambda_0 > 0$. The conclusion then follows immediately from the theorem of Chavel-Karp, which says

$$\lim_{t \to \infty} \frac{1}{t} \log p_t(x, y) = -\lambda_0 < 0.$$

Exercise 4

Let $(d, 0)$ be the standard weights on $l^2(\mathbb{Z})$ (then $d \in \{0, 1\}$), and Δ be the associated Laplacian. Let us assume that $(d, 0)$ is connected. By Remark 1.22, $e^{-t\Delta}1 = 1$ for all $t > 0$. (Though the remark is stated for finite graphs, the same holds on $l^2(\mathbb{Z})$ by approximation Lemma 5.3.

Note that the approximation argument is wrong; however, one can show the equality given in the hint directly.

Note that it follows from the formula given in Exercise 3 on sheet 5,

$$\Delta^s f = -\frac{s}{|\Gamma(1-s)|} \int_0^\infty (e^{-t\Delta} - I) f \frac{dt}{t^{1+s}}.$$

Thus $e^{-t\Delta}1 = 1$ yields $\Delta^s 1 = 0$ which implies that $c \equiv 0$. Moreover, for any $x \neq y$,

$$\begin{aligned} \Delta^s 1_y(x) &= -\frac{1}{m(x)} b(x, y) = -\frac{s}{|\Gamma(1-s)|} \int_0^\infty (e^{-t\Delta} - I) 1_y(x) \frac{dt}{t^{1+s}} \\ &= -\frac{s}{|\Gamma(1-s)|} \int_0^\infty p_t(x, y) m(y) \frac{dt}{t^{1+s}}. \end{aligned}$$

So to prove $b(x, y) > 0$, it suffices to verify $p_t(x, y) > 0$. Indeed, since $(d, 0)$ is connected, Theorem 6.1 yields that Δ is positivity improving, and therefore

$$p_t(x, y) = e^{-t\Delta} 1_y(x) / m(y) > 0.$$