Sheet 7

Approximation and Dirichlet forms

Exercise 1 ($Q^{(D)} \neq Q^{(N)}$) 4 points

Let (b,c) be a graph over (X,m) with vanishing killing term $c = 0$ such that $m(X) = 1$ and $\lambda_0 = \inf \sigma(L^{(D)}) > 0$. Show that $Q^{(D)} \neq Q^{(N)}$.

Solution: As $Q^{(D)}$ is a restriction of $Q^{(N)}$ it is sufficient to show that $D(Q^{(D)}) \neq D(Q^{(N)})$. So finding a function f with $f \in D(Q^{(N)})$ and $f \notin D(Q^{(D)})$ finishes the proof.
We claim that the constant one function $f = 1$ on X satisfies the above.
(i) $f \in D(Q^{(N)})$. First, we note that the finite measure of X implies

$$\sum_{x \in X} f(x)^2 \cdot m(x) = \sum_{x \in X} 1 \cdot m(x) = m(X) = 1 < \infty.$$

Thus, $f \in \ell^2(X,m)$ and $\|f\| = 1$. Due to the vanishing killing term, we also find

$$\mathcal{Q}(f) = \frac{1}{2} \cdot \sum_{x,y \in X} b(x,y) \cdot (f(x) - f(y))^2 + \sum_{x \in X} c(x) \cdot f(x)^2 = \sum_{x,y \in X} b(x,y) \cdot (1-1)^2 = 0,$$

from which we conclude $f \in \mathcal{D}$. Hence, $f \in D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$.
(ii) $f \notin D(Q^{(D)})$. First, we assume $f \in D(Q^{(D)})$ and show that this leads to a contradiction to the given assumptions. Theorem 6.3 yields

$$\lambda_0 = \inf_{g \in D(Q^{(D)}),\, \|g\|=1} Q^{(D)}(g). \tag{1}$$

We have shown in *(i)* that $\|f\| = 1$ and $Q^{(D)}(f) = \mathcal{Q}(f) = 0$. Hence, it follows from the assumption $f \in D(Q^{(D)})$ that f is one of the functions over which the infimum is taken. Thus, we conclude

$$\lambda_0 = \inf_{g \in D(Q^{(D)}),\, \|g\|=1} Q^{(D)}(g) \leq Q^{(D)}(f) = \mathcal{Q}(f) = 0$$

Thus, $\lambda_0 \leq 0$, which is a contradiction to the assumption $\lambda_0 > 0$. Therefore, f cannot be an element of $D(Q^{(D)})$.
In conclusion, the constant one function $f = 1$ lies in $D(Q^{(N)}) \setminus D(Q^{(D)})$ and thus $D(Q^{(D)}) \neq D(Q^{(N)})$. From this, $Q^{(D)} \neq Q^{(N)}$ follows.

Exercise 2 ($Q^{(D)} = Q^{(N)}$) 4 points

Let (b,c) be a graph over (X,m) such that the *weighted degree*

$$\mathrm{Deg}_c \colon X \to [0,\infty), \mathrm{Deg}_c(x) := \frac{1}{m(x)} \left(\sum_{y \in X} b(x,y) + c(x) \right)$$

is bounded. Show that $Q^{(D)} = Q^{(N)}$.

Solution: The Neumann and Dirichlet forms are defined by

$$Q^{(N)} \colon D(Q^{(N)}) \times D(Q^{(N)}) \to \mathbb{R}, (f,g) \mapsto Q^{(N)}(f,g) := \mathcal{Q}(f,g) \text{ and}$$
$$Q^{(D)} \colon D(Q^{(D)}) \times D(Q^{(D)}) \to \mathbb{R}, (f,g) \mapsto Q^{(D)}(f,g) := \mathcal{Q}(f,g),$$

where $D(Q^{(N)}) := \ell^2(X,m) \cap \mathcal{D}$ and $D(Q^{(D)}) := \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$, respectively. Since $Q^{(N)}$ and $Q^{(D)}$ are restrictions of $\mathcal{Q}$, we need to show $D(Q^{(N)}) = D(Q^{(D)})$. Let $D \geq 0$ be such that $\mathrm{Deg}_c(x) \leq D$ for any $x \in X$.

$\supseteq$: We have $\mathcal{D}_0 \coloneqq \{f \in \mathcal{D} \mid \exists (\varphi_n)_{n\in\mathbb{N}} \subset C_c(X)\colon \varphi_n \to f$ pointwise, $Q(f-\varphi_n) \to 0\} \subseteq \mathcal{D}$ directly from the definition. The inclusion then follows from Thm. 4.1, which asserts $D(Q^{(D)}) = \ell^2(X,m) \cap \mathcal{D}_0$, and by $\ell^2(X,m) \cap \mathcal{D}_0 \subseteq \ell^2(X,m) \cap \mathcal{D} = D(Q^{(N)})$.

$\subseteq$: Let $f \in D(Q^{(N)})$. We show $f \in \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$ by finding a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(X)$ which converges to f w.r.t. $\|\cdot\|_{\mathcal{Q}}$. Set $\varphi_n \coloneqq f|_{K_n}$, where $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of X with $\bigcup_{n\in\mathbb{N}} K_n = X$. Continuity of f implies $\varphi_n \in C_c(X)$ for every $n \in \mathbb{N}$. Furthermore, $\varphi_n \to f$ pointwise. We calculate:

$$\begin{aligned}
\mathcal{Q}(\varphi_n - f) &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi_n(x) - f(x) - \varphi_n(y) + f(y))^2 + \sum_{x\in X} c(x)(\varphi_n(x) - f(x))^2 \\
&= \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi_n(x) - f(x) - (\varphi_n(y) - f(y)))^2 + \sum_{x\in X} c(x)(\varphi_n(x) - f(x))^2 \\
&\le \sum_{x,y\in X} b(x,y)((\varphi_n(x) - f(x))^2 + (\varphi_n(y) - f(y))^2) + \sum_{x\in X} c(x)(\varphi_n(x) - f(x))^2 \\
&= \sum_{x,y\in X} b(x,y)(\varphi_n(x) - f(x))^2 + \sum_{x,y\in X} b(x,y)(\varphi_n(y) - f(y))^2 \\
&\qquad + \sum_{x\in X} c(x)(\varphi_n(x) - f(x))^2 \\
&= 2\sum_{x\in X}\left(\sum_{y\in X} b(x,y)\right)(\varphi_n(x) - f(x))^2 + \sum_{x\in X} c(x)(\varphi_n(x) - f(x))^2 \\
&= 2\sum_{x\in X}\left(\sum_{y\in X} b(x,y) + c(x)\right)(\varphi_n(x) - f(x))^2 \\
&\le \sum_{x\in X} 2Dm(x)(\varphi_n(x) - f(x))^2, \\
\|\varphi_n - f\|_{\mathcal{Q}}^2 &= \|\varphi_n - f\|_2^2 + \mathcal{Q}(\varphi_n - f) \\
&\le \sum_{x\in X} m(x)(\varphi_n(x) - f(x))^2 + \sum_{x\in X} 2Dm(x)(\varphi_n(x) - f(x))^2 \\
&= \sum_{x\in X} (2D+1)m(x)(\varphi_n(x) - f(x))^2 \\
&\xrightarrow{n\to\infty} 0,
\end{aligned}$$

which proves the claim.

Exercise 3 (Regularity and resolvent convergence) 4 points

Let (X,m) be a discrete measure space. Let Q be a Dirichlet form on (X,m) such that $C_c(X) \subseteq D(Q)$ and let L be the self-adjoint operator associated to Q. For an increasing sequence of finite sets $K_n \subseteq X$ such that $X = \bigcup_n K_n$, let L_{K_n} be the operators corresponding to the restriction of Q to $C_c(K_n)$. Assume

$$\lim_{n\to\infty} (L_{K_n} + \alpha)^{-1}\varphi = (L+\alpha)^{-1}\varphi$$

for all $\alpha > 0$ and $\varphi \in C_c(X)$. Show that Q is regular.

Solution: We have to show that (see Lecture 6, p. 74)

(i) $D(Q) \cap C_c(X)$ is dense in $C_c(X)$ w.r.t. $\|\cdot\|_\infty$,

(ii) $D(Q) \cap C_c(X)$ is dense in $D(Q)$ w.r.t. $\|\cdot\|_Q$.

Since $C_c(X) \subseteq D(Q)$, we have $D(Q) \cap C_c(X) = C_c(X)$. Thus, (i) is clearly fulfilled.
Let us turn to (ii). Due to Corollary 3.37, we know that $D(L)$ is dense in $D(Q)$ w.r.t. $\|\cdot\|_Q$. Hence, it is sufficient to prove that $C_c(x)$ is dense in $D(L)$ w.r.t. $\|\cdot\|_Q$.

Let $f \in D(L)$ and $\varepsilon > 0$. First, since $f \in D(L)$, there is $\psi \in \ell^2(X,m)$ such that $f = (L+1)^{-1}\psi$. From the continuity of $(L+1)^{-1}$ and the density of $C_c(X)$ in $\ell^2(X,m)$ w.r.t. $\|\cdot\|_{\ell^2(X,m)}$, we deduce that there is $\varphi \in C_c(X)$ such that

$$\|\varphi - \psi\|_{\ell^2(X,m)} \leq \frac{\varepsilon}{\|(L+1)^{-1}\|_{B(\ell^2(X,m))}} \tag{2}$$

and

$$\begin{aligned}\left\|(L+1)^{-1}\varphi - f\right\|_{\ell^2(X,m)} &= \left\|(L+1)^{-1}(\varphi - \psi)\right\|_{\ell^2(X,m)} \leq \left\|(L+1)^{-1}\right\|_{B(\ell^2(X,m))} \|\varphi - \psi\|_{\ell^2(X,m)} \\ &\underset{(2)}{\leq} \varepsilon. \end{aligned} \tag{3}$$

Due to $\lim_{n\to\infty}(L_{K_n} + \alpha)^{-1}\varphi = (L+\alpha)^{-1}\varphi$ for all $\alpha > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left\|(L_{K_n} + 1)^{-1}\varphi - (L+1)^{-1}\varphi\right\|_{\ell^2(X,m)} \leq \varepsilon \tag{4}$$

and

$$\begin{aligned}\left\|(L_{K_n} + 1)^{-1}\varphi\right\|_{\ell^2(X,m)} &\leq \left\|(L_{K_n} + 1)^{-1}\varphi - (L+1)^{-1}\varphi\right\|_{\ell^2(X,m)} + \left\|(L+1)^{-1}\varphi\right\|_{\ell^2(X,m)} \\ &\underset{(4)}{\leq} \varepsilon + \left\|(L+1)^{-1}\varphi\right\|_{\ell^2(X,m)} \leq \varepsilon + \left\|(L+1)^{-1}\right\|_{B(\ell^2(X,m))} \|\varphi\|_{\ell^2(X,m)}. \end{aligned} \tag{5}$$

Second, we set $(Q+1)(g,h) := Q(g,h) + \langle g, h\rangle$ for $g,h \in D(Q)$. We note that

$$(Q+1)(g) = Q(g) + \langle g, g\rangle = \|g\|_Q^2 \tag{6}$$

for all $g \in D(Q)$, and

$$\begin{aligned}(Q+1)((L+1)^{-1}g,h) &= (Q+1)(h,(L+1)^{-1}g) = Q(h,(L+1)^{-1}g) + \left\langle h, (L+1)^{-1}g\right\rangle \\ &= \left\langle h, L(L+1)^{-1}g\right\rangle + \left\langle h, (L+1)^{-1}g\right\rangle = \left\langle h, (L+1)(L+1)^{-1}g\right\rangle \\ &= \langle h, g\rangle = \langle g, h\rangle \end{aligned} \tag{7}$$

for all $g \in \ell^2(X,m)$ and $h \in D(Q)$ by Corollary 3.37. Further, for $n \in \mathbb{N}$, we remark that

$$(L_{K_n} + 1)^{-1}\phi := i_{K_n}((L_{K_n} + 1)^{-1}(\phi_{|K_n}))$$

for all $\phi \in C_c(X)$ (analogously to Notation, Lecture 7, p. 83). Thus, $(L_{K_n} + 1)^{-1}\varphi \in C_c(X)$ and

$$\begin{aligned}&(Q+1)(i_{K_n}(h),(L_{K_n} + 1)^{-1}\varphi) \\ &= Q(i_{K_n}(h),(L_{K_n} + 1)^{-1}\varphi) + \left\langle i_{K_n}(h), (L_{K_n} + 1)^{-1}\varphi\right\rangle \\ &= Q_{K_n}(h,(L_{K_n} + 1)^{-1}(\varphi_{|K_n})) + \left\langle i_{K_n}(h), (L_{K_n} + 1)^{-1}\varphi\right\rangle \\ &= \left\langle h, L_{K_n}(L_{K_n} + 1)^{-1}(\varphi_{|K_n})\right\rangle_{\ell^2(K_n,m_{K_n})} + \left\langle i_{K_n}(h), (L_{K_n} + 1)^{-1}\varphi\right\rangle \\ &= \langle i_{K_n}(h), \varphi\rangle \end{aligned} \tag{8}$$

for all $h \in C(K_n)$. Hence, $(L_{K_n}+1)^{-1}\varphi - f \in D(Q)$ and we have for all $n \geq N$

$$
\begin{aligned}
&\left\|(L_{K_n}+1)^{-1}\varphi - f\right\|_Q^2 \\
\underset{(6)}{=}\ & (Q+1)((L_{K_n}+1)^{-1}\varphi - f) \\
=\ & (Q+1)((L_{K_n}+1)^{-1}\varphi - (L+1)^{-1}\varphi + (L+1)^{-1}\varphi - f) \\
=\ & (Q+1)((L_{K_n}+1)^{-1}\varphi - (L+1)^{-1}\varphi) \\
&+ 2(Q+1)((L_{K_n}+1)^{-1}\varphi - (L+1)^{-1}\varphi, (L+1)^{-1}\varphi - f) + (Q+1)((L+1)^{-1}\varphi - f) \\
=\ & (Q+1)((L_{K_n}+1)^{-1}\varphi) - 2(Q+1)((L_{K_n}+1)^{-1}\varphi, (L+1)^{-1}\varphi) + Q((L+1)^{-1}\varphi) \\
&+ 2((Q+1)((L_{K_n}+1)^{-1}\varphi, (L+1)^{-1}\varphi) - (Q+1)((L_{K_n}+1)^{-1}\varphi, f) \\
&- (Q+1)((L+1)^{-1}\varphi) + (Q+1)((L+1)^{-1}\varphi, f)) + (Q+1)((L+1)^{-1}(\varphi - \psi)) \\
\underset{(7),(8)}{=} & \left\langle (L_{K_n}+1)^{-1}\varphi, \varphi \right\rangle - 2\left\langle (L_{K_n}+1)^{-1}\varphi, \varphi \right\rangle + \left\langle \varphi, (L+1)^{-1}\varphi \right\rangle \\
&+ 2\left(\left\langle (L_{K_n}+1)^{-1}\varphi, \varphi \right\rangle - \left\langle (L_{K_n}+1)^{-1}\varphi, \psi \right\rangle - \left\langle \varphi, (L+1)^{-1}\varphi \right\rangle + \langle \varphi, f \rangle \right) \\
&+ \left\langle \varphi - \psi, (L+1)^{-1}(\varphi - \psi) \right\rangle \\
=\ & \left\langle (L_{K_n}+1)^{-1}\varphi, \varphi \right\rangle - \left\langle (L_{K_n}+1)^{-1}\varphi, \varphi \right\rangle - \left\langle \varphi, (L_{K_n}+1)^{-1}\varphi \right\rangle + \left\langle \varphi, (L+1)^{-1}\varphi \right\rangle \\
&+ 2\left(\left\langle (L_{K_n}+1)^{-1}\varphi, \varphi \right\rangle - \left\langle (L_{K_n}+1)^{-1}\varphi, \psi \right\rangle - \left\langle \varphi, (L+1)^{-1}\varphi \right\rangle + \langle \varphi, f \rangle \right) \\
&+ \left\langle \varphi - \psi, (L+1)^{-1}(\varphi - \psi) \right\rangle \\
=\ & \left\langle \varphi, (L+1)^{-1}\varphi - (L_{K_n}+1)^{-1}\varphi \right\rangle + 2\left(\left\langle (L_{K_n}+1)^{-1}\varphi, \varphi - \psi \right\rangle + \left\langle \varphi, f - (L+1)^{-1}\varphi \right\rangle \right) \\
&+ \left\langle \varphi - \psi, (L+1)^{-1}(\varphi - \psi) \right\rangle \\
\leq\ & \|\varphi\|_{\ell^2(X,m)} \underbrace{\left\|(L+1)^{-1}\varphi - (L_{K_n}+1)^{-1}\varphi\right\|_{\ell^2(X,m)}}_{\underset{(4)}{\leq} \varepsilon} \\
&+ 2\Big(\underbrace{\left\|(L_{K_n}+1)^{-1}\varphi\right\|_{\ell^2(X,m)}}_{\underset{(5)}{\leq} \varepsilon + \|(L+1)^{-1}\|_{B(\ell^2(X,m))}\|\varphi\|_{\ell^2(X,m)}} \underbrace{\|\varphi - \psi\|_{\ell^2(X,m)}}_{\underset{(2)}{\leq} \frac{\varepsilon}{\|(L+1)^{-1}\|_{B(\ell^2(X,m))}}} + \|\varphi\|_{\ell^2(X,m)} \underbrace{\left\|f - (L+1)^{-1}\varphi\right\|_{\ell^2(X,m)}}_{\underset{(3)}{\leq} \varepsilon} \Big) \\
&+ \underbrace{\|\varphi - \psi\|_{\ell^2(X,m)}}_{\underset{(2)}{\leq} \frac{\varepsilon}{\|(L+1)^{-1}\|_{B(\ell^2(X,m))}}} \underbrace{\left\|(L+1)^{-1}(\varphi - \psi)\right\|_{\ell^2(X,m)}}_{\underset{(3)}{\leq} \varepsilon} \\
\leq\ & \Big(5 \underbrace{\|\varphi\|_{\ell^2(X,m)}}_{\underset{(2)}{\leq} \frac{\varepsilon}{\|(L+1)^{-1}\|_{B(\ell^2(X,m))}} + \|\psi\|_{\ell^2(X,m)}} + \frac{3\varepsilon}{\|(L+1)^{-1}\|_{B(\ell^2(X,m))}} \Big) \varepsilon \\
\leq\ & \left(5\|\psi\|_{\ell^2(X,m)} + \frac{8\varepsilon}{\|(L+1)^{-1}\|_{B(\ell^2(X,m))}} \right) \varepsilon.
\end{aligned}
$$

We conclude that $C_c(X)$ is dense in $D(L)$ and thus in $D(Q)$ w.r.t. $\|\cdot\|_Q$, which proves part (ii). Therefore, Q is regular.

Exercise 4 (Bounded functions in domain are an algebra) 4 points

Let (X,μ) be a σ-finite measure space and Q be a Dirichlet form on $L^2(X,\mu)$ with domain of Q. Show that $D(Q) \cap L^\infty(X,\mu)$ is an algebra.

Solution: Clearly, $D(Q) \cap L^\infty(X,\mu)$ is a vector space. For it to be an algebra we need to show that for any $f,g \in D(Q) \cap L^\infty(X,\mu)$ we also have $f \cdot g \in D(Q) \cap L^\infty(X,\mu)$.
So let $f,g \in D(Q) \cap L^\infty(X,\mu)$. The fact that $f \cdot g \in L^\infty(X,\mu)$ is straightforward to see, so it remains to show that $f \cdot g \in D(Q)$ as well.
In the following proof one can use either quadratic form Q^α or Q_t (see p.91). We are going to use

$$Q^\alpha(h) = \alpha\langle(I - \alpha(L+\alpha)^{-1})h, h\rangle \qquad \text{for } h \in L^2(X,\mu).$$

By the Second Beurling-Deny criterion (Theorem 1.19) we get that $\alpha(L+\alpha)^{-1}$ is Markov for every $\alpha > 0$. Therefore, Lemma 5.11 is applicable to Q^α which yields the following inequality

$$Q^\alpha(fg) \leq 2\,\|f\|_\infty^2\, Q^\alpha(f) + 2\,\|g\|_\infty^2\, Q^\alpha(g).$$

Using Lemma 5.12 and the above inequality we get

$$\begin{aligned} Q'(fg) &= \lim_{\alpha\to\infty} Q^\alpha(fg) \\ &\leq \lim_{\alpha\to\infty} 2\,\|f\|_\infty^2\, Q^\alpha(f) + 2\,\|g\|_\infty^2\, Q^\alpha(g) \\ &\leq 2\,\|f\|_\infty^2 \lim_{\alpha\to\infty} Q^\alpha(f) + 2\,\|g\|_\infty^2 \lim_{\alpha\to\infty} Q^\alpha(g) \\ &= 2\,\|f\|_\infty^2\, Q'(f) + 2\,\|g\|_\infty^2\, Q'(g) < \infty. \end{aligned}$$

Since, again by Lemma 5.12, $\lim_{\alpha\to\infty} Q^\alpha(fg)$ is finite if and only if $fg \in D(Q)$ we obtain the desired result.