كليـــــــة العلوم
السملالية - مراكش
FACULTÉ DES SCIENCES
SEMLALIA - MARRAKECH
MARRAKECH
جامعة القاضي عياض
UNIVERSITÉ CADI AYYAD

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Exercises solutions for Lecture 06

TEAM OF MARRAKESH

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Exercise 1. Let (X, m) be an infinite discrete measure space and $p \in [1, \infty]$. Show that $C_c(X)$ is dense in $\ell^p(X, m)$ if and only if $p \in [1, \infty)$.

Solution. Consider $1 \leq p < \infty$. Since X is countable, we may write $X = \{x_0, x_1, \cdots\}$. Let $f \in \ell^p(X, m)$, then $\sum_{n\geq 0}|f(x_n)|^p m(x_n) < \infty$. Therefore, for $n \in \mathbb{N}$, we set

$$f_n = \sum_{k=0}^{n} f(x_k)\ \mathbb{1}_{x_k}.$$

It is clear that $f_n \in C_c(X)$, and

$$\|f_n - f\|_p = \|R_n\|_p = \sum_{k\geq n+1} |f(x_k)|^p m(x_k) \to 0 \text{ as } n \to \infty.$$

Now, we assume that $p = \infty$. We shall prove that the closure of $C_c(X)$ in $\ell^\infty(X)$ is the space $C_0(X)$ of continuous functions vanishing at infinity. Let $f \in C_0(X)$ and $\varepsilon > 0$, there exists a finite set $A \subset X$ such that

$$|f(x)| < \varepsilon, \text{ for all } x \notin A.$$

Taking $g = \mathbb{1}_A f \in C_c(X)$, we have

$$\|f - g\|_\infty < \varepsilon.$$

Hence, $\overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(X)$. However, the function $f = 1$ is an element of $\ell^\infty(X)$ and not of $C_0(X)$. Thus, $\overline{C_c(X)}^{\|\cdot\|_\infty} \subsetneq \ell^\infty(X)$.

NB:

$$C_0(X) = \{f \in C(X), \forall\varepsilon > 0, \text{ there exists a finite set } A \subset X \text{ with } |f(x)| < \varepsilon, \quad \forall x \notin A\}.$$

Exercise 2. Let (X, m) be a discrete measure space and $p \in [1, \infty)$.

a) Show the equivalence of the following statements:

(i) $\ell^p(X,m) \subseteq \ell^\infty(X)$,

(ii) $\ell^p(X,m) \subseteq C_0(X) := \overline{C_c(X)}^{\|\cdot\|_{\ell^\infty}}$,

(iii) There exists $\alpha > 0$ such that $m \geq \alpha$.

b) Show the equivalence of the following statements:

(i) $\ell^p(X,m) \supseteq \ell^\infty(X,m)$,

(ii) $m(X) < \infty$.

Solution. a) We will prove that (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i), and (i) $\Rightarrow$ (iii).

iii) $\Rightarrow$ ii) Let $f \in \ell^p(X,m)$ then $\sum_{x\in X}|f(x)|^p m(x) < +\infty$. Using (iii) there exists $\alpha > 0$ such that $m(x) \geq \alpha$ for all $x \in X$. Therefore we have

$$\alpha \sum_{x\in X}|f(x)|^p \leq \sum_{x\in X}|f(x)|^p m(x) < +\infty.$$

Since X is countable we let $X = \{x_1, x_2, \ldots\}$. Then the above statement ensures that $\sum_{n=1}^\infty |f(x_n)|^p < \infty$ and hence

$$\lim_{n\to 0} |f(x_n)| = 0. \tag{1}$$

Now let $(f_n)_n$ to be the sequence of $C_c(X)$ defined as follow

$$f_n(x_k) = \begin{cases} f(x_k) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Then for all $n \geq 1$ we have $\|f - f_n\|_{\ell^\infty} = \sup\limits_{k\geq n+1} |f(x_k)|$. Hence (1) implies that

$$\lim_{n\to 0}\|f - f_n\|_{\ell^\infty} = 0.$$

Since $f_n \in C_c(X)$ then (ii) is proved.

ii) $\Rightarrow$ i) It is trivial, since $C_c(X) \subset \ell^\infty(X)$.

i) $\Rightarrow$ iii) By contradiction assume that for all $\alpha > 0$, $\exists\, x \in X$ such that $m(x) < \alpha$. Then let $(x_{n_k})_{k\geq 1}$ be the sequence defined as follow: For $k = 1$ let $n_1 \geq 1$ and $x_{n_1} \in X$ such that $m(x_{n_1}) < 1$, and for $k > 1$ we choose $n_k \geq n_{k-1} + 1$ and $x_{n_k} \in X$ such that

$$m(x_{n_k}) < m(x_{n_{k-1}}) \wedge 1/k^{2+p}.$$

Now let f be the function defined as

$$f(x) = \begin{cases} k & \text{for } x = x_{n_k} \\ 0 & \text{for } x \neq x_{n_k} \end{cases}. \tag{2}$$

Then $f \in \ell^p(X,m)$ but $\|f\|_{\ell^\infty} = +\infty$, which contradict (i).

(b) $(i) \implies (ii)$. Assume $\ell^p(X,m) \supseteq \ell^\infty(X,m)$. The function $f := \big(1, 1, \cdots\big)$ is an element

of $\ell^\infty(X,m)$, then $\|f\|_p = \sum_{x\in X} m(x) < \infty$. Hence, $m(X) < \infty$.
$(ii) \implies (i)$. Assume that $m(X) < \infty$. Let $f \in \ell^\infty(X,m)$, one has

$$\begin{aligned}\sum_{x\in X} |f(x)|^p m(x) &\leq \|f\|_\infty^p m(X) \\ &< \infty.\end{aligned}$$

Hence, $f \in \ell^p(X,m)$.

Exercise 3. Let (b,c) be a graph over (X,m). Show that $\mathcal{L}$ is bounded on $\ell^2(X,m)$ if and only if $\mathcal{L}$ is bounded on $\ell^p(X,m)$ for some $p \in [1,\infty]$.

Solution. Assume that $\mathcal{L}$ is bounded on $\ell^p(X,m)$ for some $1 \leq p < \infty$. Let q be the conjugate index of p, we shall prove that $\mathcal{L}$ is a bounded operator on $\ell^q(X,m)$.
$i)$ If $p = 1$ and $q = \infty$; Let $f \in \ell^\infty(X)$ and $x \in X$ be fixed. Since $\ell^\infty(X) \subset \mathcal{F}$, we apply Green's formula for f and $\varphi = 1\!\!1_x/m(x)$ to obtain

$$\begin{aligned}|\mathcal{L}f(x)| &= \left|\sum_{y\in X} (\mathcal{L}f)(y)\varphi(y)m(y)\right| \\ &= \left|\sum_{y\in X} (\mathcal{L}\varphi)(y)f(y)m(y)\right| \\ &\leq \|f\|_\infty \|\mathcal{L}\varphi\|_1 \\ &\leq \|\mathcal{L}\|_1 \|f\|_\infty.\end{aligned}$$

Hence, $\mathcal{L}$ is a bounded operator on $\ell^\infty(X)$.
$ii)$ If $p > 1$, we use the same above idea with Holder's inequality. Let $f \in \ell^p(X,m)$ and $\varphi \in C_c(X) \subset \ell^q(X,m)$, we estimate

$$\begin{aligned}|\langle \mathcal{L}\varphi, f\rangle_{p,q}| &= \left|\sum_{y\in X} (\mathcal{L}\varphi)(y)f(y)m(y)\right| \\ &= \left|\sum_{y\in X} (\mathcal{L}f)(y)\varphi(y)m(y)\right| \\ &\leq \|\mathcal{L}f\|_p \|\varphi\|_q \\ &\leq \|\mathcal{L}\|_p \|f\|_p \|\varphi\|_q.\end{aligned}$$

Therefore, $\mathcal{L}$ is a bounded operator on a dense subspace of $\ell^q(X,m)$. Thus, $\mathcal{L}$ is a bounded operator on $\ell^q(X,m)$. Hence, using Riesz–Thorin interpolation theorem, we conclude that $\mathcal{L}$ is a bounded operator on $\ell^2(X,m)$.

N.B: We can prove that the main result holds also for all $p \in [1,\infty]$. Indeed, it is proved in Theorem 2.18 of Lecture 3 that $\mathcal{L}$ is bounded on $\ell^2(X,m)$ if and only if the weighted degree Deg is a bounded function on X. Assume that the weighted degree is a bounded function.

For $f \in \ell^\infty(X)$ and for a fixed $x \in X$, we estimate

$$\begin{aligned} |\mathcal{L}f(x)| &\leq \frac{1}{m(x)} \sum_{y \in X} b(x,y) \left(|f(x)| + |f(y)|\right) + \frac{c(x)}{m(x)} |f(x)| \\ &\leq 2\|f\|_\infty \frac{1}{m(x)} \sum_{y \in X} b(x,y) + \frac{c(x)}{m(x)} \|f\|_\infty \\ &\leq 2\|\mathrm{Deg}\|_\infty \|f\|_\infty. \end{aligned}$$

Hence, $\mathcal{L}$ is bounded on $\ell^\infty(X)$ and by i), we can prove that $\mathcal{L}$ is bounded on $\ell^1(X,m)$. Again, by Riesz–Thorin interpolation theorem, we deduce that $\mathcal{L}$ is bounded on $\ell^p(X,m)$ for all $p \in [1,\infty]$.

Exercise 4. (Forms between $Q^{(D)}$ and $Q^{(N)}$)
Let (b,c) be a graph over (X,m) and $U \subseteq X$ and

$$\begin{aligned} D\left(Q^{(U)}\right) &= \overline{\{u \in D\left(Q^N\right) \mid U \cap \operatorname{supp} u \text{ is finite }\}}^{\|\cdot\|_{Q^{(N)}}} \\ Q^{(U)}(f,g) &= Q^{(N)}(f,g) \end{aligned}$$

(a) $Q^{(U)}$ is a Dirichlet form.
(b) $Q^{(D)} \subseteq Q^{(U)} \subseteq Q^{(N)}$, where $Q_1 \subseteq Q_2$ if $D\left(Q_1\right) \subseteq D\left(Q_2\right)$ and $Q_1 = Q_2$ on $D\left(Q_1\right)$. Furthermore, $Q^{(X \setminus F)} = Q^{(D)}$ and $Q^{(F)} = Q^{(N)}$ for finite $F \subseteq X$.

Solution.

(a) By Definition 4.3, in order to show that $Q^{(U)}$ is a Dirichlet form, we shall prove the following:

(i) $Q^{(U)}$ is a closed form, and

(ii) $Q^{(U)}$ is compatible with all normal contraction.

For (i), we know that Q^N is a closed form, i.e. $D(Q^N)$ is complete with respect to the norm $\|\cdot\|_{\mathcal{Q}}$. Then $D(Q^U)$ is complete with respect to this norm, as the closed subset of $D(Q^N)$. Hence, the Q^U is closed.
Now, let us prove (ii). First, let $C : \mathbb{R} \to \mathbb{R}$ be a normal contraction and $f \in D(Q^U)$, we need to show that $C \circ f \in D(Q^U)$ with

$$Q\left(C \circ f\right) \leq Q(f). \tag{3}$$

By definition of $D(Q^U)$ there exists a sequence (u_n) in $D(Q^N)$ such that for all $n \in \mathbb{N}$ $\operatorname{supp}(u_n) \cap U$ is finite and $\|u_n - f\|_{\mathcal{Q}} \to 0$. Since $C(0) = 0$, then for all $n \in \mathbb{N}$ $\operatorname{supp}\left(C \circ u_n\right) \cap U$ is finite. Moreover the Lipschitz property of C ensures that $\|C \circ u_n - C \circ f\|_{\mathcal{Q}} \to 0$. Hence $C \circ f$ belongs to $D(Q^U)$.
To show (3), using the compatibility of $Q^{(N)}$ with normal contractions we get

$$Q^{(U)}(C \circ u_n) = Q^{(N)}(C \circ u_n) \leq Q^{(N)}(u_n) = Q^{(U)}(u_n) \to Q^{(U)}(f),$$

as $n \to 0$, where this convergence is assured by the fact that $Q^{(N)}(u_n - f) \to 0$. Then $\left(Q^{(U)}(C \circ u_n)\right)$ is bounded. From Proposition 3.35, we obtain that

$$Q^{(U)}\left(C \circ f\right) \leq \liminf_n Q^{(U)}\left(C \circ u_n\right) \leq Q^{(U)}(f).$$

Hence, $Q^{(U)}$ is compatible with all normal contractions. The proof is then complete.

(b) In order to prove that $Q^{(D)} \subset Q^{(U)}$, we show that $D\left(Q^{(D)}\right) \subset D\left(Q^{(U)}\right)$. For that purpose, we consider $f \in D\left(Q^{(D)}\right)$, which means that there exists a sequence $(\varphi_n)_{n\geq 0}$ in $\mathcal{C}_c(X)$ such that

$$\varphi_n \longrightarrow f \textit{ pointwise and } Q(f - \varphi_n) \longrightarrow 0 \textit{ as } n \longrightarrow \infty.$$

Since we have $\varphi_n \in \mathcal{C}_c(X)$, then $\mathrm{supp}(\varphi_n)$ is finite. It follows that

$$\varphi_n \in \left\{u \in D\left(Q^{(N)}\right) : U \cap \mathrm{supp}(\varphi_n)\right\}$$

Thus, we find that

$$f \in \overline{\{u \in D\,(Q^{(N)}) \mid U \cap \mathrm{supp}\, u \text{ is finite }\}},$$

which means that $f \in D\left(Q^{(U)}\right)$.
Let us consider a finite subset $F \subset X$, then the domain is $D\left(Q^{(U)}\right)$ is exactly $D\left(Q^{N}\right)$ ($Q^{(N)}$ is a closed form). Therefore, $Q^{(F)} = Q^{(N)}$. On the other hand, one has

$$\begin{aligned} D\left(Q^{(X\setminus F)}\right) &= \overline{\left\{u \in D\,(Q^{(N)}) \mid f_{|(X\setminus F)} \in \mathcal{C}_c(X \setminus F)\right\}}^{\|\cdot\|_{Q^{(N)}}} \\ &= \overline{\{u \in D\,(Q^{(N)}) \mid f \in \mathcal{C}_c(X)\}}^{\|\cdot\|_{Q^{(N)}}} \qquad \text{(because } F \text{ is finite)} \\ &= \overline{\mathcal{C}_c(X)}^{\|\cdot\|_{Q^{(N)}}} \\ &= D\left(Q^{(D)}\right). \end{aligned}$$

Hence, $Q^{(X\setminus F)} = Q^{(D)}$ and the proof is completed.

Bonus Exercise 1 (Inclusion of ℓ^p spaces)
Show that the equivalences of Exercise 2 are still true if $\ell^1(X, m)$ is replaced by $\ell^p(X, m)$ with $p \in (1, \infty)$

Solution.

The solution to this exercise is included in Exercise 2.