Solutions for Lecture 05 of ISEM 26

Team Dresden

Exercise 1 (**Forms of multiplication operators**). Let (X, μ) be a measure space and $u\colon X \to [0, \infty)$ a measurable function. Let

$$D(Q) := \left\{ f \in L^2(X, \mu) \;\middle|\; \int_X u|f|^2 \,\mathrm{d}\mu < \infty \right\}$$

and

$$Q(f, g) := \int_X u\overline{f} g \,\mathrm{d}\mu$$

for $f, g \in D(Q)$.

(a) Show that Q is a closed form.

(b) Show that Q is bounded if $u \in L^\infty(X, \mu)$.

(c) The associated operator to Q is the operator of multiplication by u denoted by M_u. Moreover,

$$D(Q) = D(M_{\sqrt{u}}).$$

(d) Compute $Q'\colon L^2(X, \mu) \to [0, \infty]$ which is defined as $Q'(f) := Q(f, f)$ for $f \in D(Q)$ and ∞ otherwise.

Solution. We prove all assertions separately.

(a) First of all, we prove that Q is a form which in the first place requires Q to be densely defined. To show this, let f in $L^2(X, \mu)$ be arbitrary. For $n \in \mathbb{N}$, we define $X_n := \{x \in X \mid u(x) \leq n\}$ and consider the characteristic functions $\mathbb{1}_{X_n}$. We then have $\mathbb{1}_{X_n} f \in D(Q)$ and $\mathbb{1}_{X_n} f \to f$ pointwise everywhere. Furthermore, we observe $|\mathbb{1}_{X_n} f| \leq |f|$ for all $n \in \mathbb{N}$. By Lebesgue's dominated convergence, this implies $\mathbb{1}_{X_n} f \to f$ in $L^2(X, \mu)$ and shows $\overline{D(Q)} = L^2(X, \mu)$.

Secondly, we verify that Q is symmetric, linear and positive. Let $\alpha, \beta \in \mathbb{C}$ and $f, g, h \in D(Q)$. Then

$$\begin{aligned} Q(f, \alpha g + \beta h) &= \int_X u\overline{f}(\alpha g + \beta h)\,\mathrm{d}\mu \\ &= \alpha \int_X u\overline{f}g\,\mathrm{d}\mu + \beta \int_X u\overline{f}h\,\mathrm{d}\mu \\ &= \alpha Q(f, g) + \beta Q(f, h), \end{aligned}$$

i.e. Q is linear in the second argument;

$$\overline{Q(g, f)} = \overline{\int_X u\overline{g}f\,\mathrm{d}\mu} = \int_X \overline{u}\,\overline{\overline{g}}\,\overline{f}\,\mathrm{d}\mu = \int_X ug\overline{f}\,\mathrm{d}\mu = Q(f, g),$$

i.e. Q is symmetric;

$$Q(f, f) = \int_X u|f|^2\,\mathrm{d}\mu \geq 0,$$

i.e. Q is positive. Here, we repeatedly used that u maps into the non-negative real numbers.

It remains to show that Q is closed. Let $f \in L^2(X, \mu)$. As

$$\|f\|_Q^2 = Q(f) + \|f\|_{L^2}^2 = \int_X u\,|f|^2\,\mathrm{d}\mu + \int_X |f|^2\,\mathrm{d}\mu = \int_X (1+u)\,|f|^2\,\mathrm{d}\mu,$$

one concludes that $(D(Q), \|\cdot\|_Q)$ is just the Hilbert space $L^2(X, (1+u)\,\mathrm{d}\mu)$.

(b) If $u \in L^\infty(X, \mu)$ we have $\|u\|_{L^\infty} < \infty$. Thus, for arbitrary $f, g \in D(Q)$ we compute

$$\begin{aligned} |Q(f, g)| = \left| \int_X u\overline{f}g\,\mathrm{d}\mu \right| &\leq \int_X |u|\,\left|\overline{f}\right|\,|g|\,\mathrm{d}\mu \\ &\leq \|u\|_{L^\infty} \int_X |f||g|\,\mathrm{d}\mu \leq \|u\|_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}, \end{aligned}$$

where we applied Hölder's inequality in the last step. Consequently, Q is bounded with constant $C := \|u\|_{L^\infty}$.

(c) We have to verify that $D\left(\sqrt{M_u}\right) = D(Q)$ and

$$Q(f, g) = \left\langle \sqrt{M_u}f, \sqrt{M_u}g \right\rangle$$

for $f, g \in D(Q)$. To start with, we note that by definition of the functional calculus and as a consequence of the spectral theorem we have $\sqrt{M_u} = M_{\sqrt{u}}$. Furthermore, we recall that $D\left(M_{\sqrt{u}}\right) = \{f \in L^2(X,\mu) \mid \sqrt{u} f \in L^2(X,\mu)\}$. Hence, for an arbitrary $f \in L^2(X,\mu)$ we obtain

$$\begin{aligned} f \in D\left(M_{\sqrt{u}}\right) &\iff \sqrt{u} f \in L^2(X,\mu) \\ &\iff \int_X \left|\sqrt{u} f\right|^2 \mathrm{d}\mu < \infty \\ &\iff \int_X u|f|^2 \mathrm{d}\mu < \infty \\ &\iff f \in D(Q). \end{aligned}$$

This proves $D\left(\sqrt{M_u}\right) = D(Q)$. For the second claim we compute

$$\begin{aligned} \left\langle \sqrt{M_u} f, \sqrt{M_u} g \right\rangle = \langle M_{\sqrt{u}} f, M_{\sqrt{u}} g\rangle &= \langle f, M_{\sqrt{u}}^2 g\rangle \\ &= \langle f, ug\rangle = \int_X u\overline{f} g \,\mathrm{d}\mu = Q(f,g), \end{aligned}$$

where we used that $M_{\sqrt{u}}$ is self-adjoint.

(d) Finally, we compute $Q'\colon L^2(X,\mu) \to [0,\infty]$ which is defined as $Q'(f) := Q(f,f)$ for $f \in D(Q)$ and ∞ otherwise. We want to apply Proposition 3.12 with $A = M_u$. Then the unitary operator U received from the spectral theorem is the identity operator I on $L^2(X,\mu)$. We apply Proposition 3.12 with $\varphi = \mathbb{1}_{[0,\infty)} \operatorname{id}_{\mathbb{R}}$ to obtain

$$Q'(f) = \int_X u|f|^2 \,\mathrm{d}\mu = \int_X (\varphi \circ u)|f|^2 \,\mathrm{d}\mu = \int_{[0,\infty)} \operatorname{id}_{\mathbb{R}} \mathrm{d}\mu_f$$

for all $f \in L^2(X,\mu)$. This equation also includes the case $Q'(f) = \infty$ as the occurring integrals are finite if and only if $\varphi \in L^1(\mathbb{R}, \mu_f)$. □

Exercise 2 (Closable forms). Let $Q \geq 0$ be a form on a Hilbert space H which allows for a closed extension $Q^{\#}$.

(a) Let $(f_n)_{n\in\mathbb{N}}$ in $D(Q)$ and $f \in H$. Show that the following statements are equivalent:

  (i) $f \in D(Q^{\#})$ and $f_n \to f$ with respect to $\|\cdot\|_{Q^{\#}}$.

  (ii) $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_Q$ and $f_n \to f$ in H.

(b) Show that

$$D(\overline{Q}) := \left\{ f \in D(Q^{\#}) \;\middle|\; \exists (f_n)_{n\in\mathbb{N}} \subseteq D(Q) \colon \; \|f_n - f\|_{Q^{\#}} \to 0 \right\}$$

and

$$\overline{Q}(f,g) := Q^{\#}(f,g)$$

is a closed form.

(c) Show that $\overline{Q}$ is included in every closed extension of Q.

Solution. (a) $(i) \Rightarrow (ii)$: Let $(f_n)_{n\in\mathbb{N}} \subseteq D(Q)$ and $f \in D(Q^{\#})$ such that $f_n \to f$ with respect to $\|\cdot\|_{Q^{\#}}$. By definition of $\|\cdot\|_{Q^{\#}}$ it directly follows

$$\|f_n - f\|_H \le \|f_n - f\|_{Q^{\#}} \longrightarrow 0$$

which means $f_n \to f$ in H. Since $Q^{\#}|_{D(Q)} = Q$ the norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q^{\#}}$ are equal on $D(Q)$. Since $(f_n)_{n\in\mathbb{N}}$ is convergent in $(D(Q^{\#}), \|\cdot\|_{Q^{\#}})$ and since $f_n \in D(Q)$, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(D(Q), \|\cdot\|_Q)$.
$(ii) \Rightarrow (i)$: Let $(f_n)_{n\in\mathbb{N}} \subseteq D(Q)$ be a Cauchy sequence with respect to $\|\cdot\|_Q$ and let $f \in H$ such that $f_n \to f$ in H. Since $Q^{\#}$ is lower semi-continuous and since $(f_n)_{n\in\mathbb{N}} \subseteq D(Q)$ is a Cauchy sequence, we compute

$$\begin{aligned}
\|f_n - f\|_{Q^{\#}}^2 &= Q^{\#}(f_n - f) + \|f_n - f\|_H^2 \\
&\le \liminf_{m\to\infty} \Big(Q^{\#}(f_n - f_m) + \|f_n - f_m\|_H^2 \Big) \\
&= \liminf_{m\to\infty} \|f_n - f_m\|_{Q^{\#}}^2 \\
&= \liminf_{m\to\infty} \|f_n - f_m\|_Q^2 \,.
\end{aligned}$$

The right hand side converges to 0 as $n \to \infty$, and we have shown $f_n \to f$ with respect to $\|\cdot\|_{Q^{\#}}$.

(b) By definition we have $D(\overline{Q}) \subseteq D(Q^{\#})$ and $\overline{Q} = Q^{\#}$ on $D(\overline{Q})$. Hence, $\overline{Q}$ inherits the linearity, symmetry and positivity from $Q^{\#}$. To prove that $\overline{Q}$ is closed, we show that $\big(D(\overline{Q}), \langle\cdot,\cdot\rangle_{\overline{Q}}\big)$ is a Hilbert space. Here, we only verify that $D(\overline{Q})$ is complete with respect to $\|\cdot\|_{\overline{Q}}$. So, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $D(\overline{Q})$. By definition of $D(\overline{Q})$, for every $n \in \mathbb{N}$ there exists a $g_n \in D(Q)$ such that

$$\|f_n - g_n\|_{Q^{\#}} < 2^{-n}.$$

By this estimate, we observe that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $D(Q)$. Furthermore, as H is complete, there exists a $g \in H$ such that $g_n \to g$ in H. From part (a) of this exercise it now follows $g_n \to g$ with respect to $\|\cdot\|_{Q^\#}$. Hence, $g \in D(\overline{Q})$. Finally, we estimate

$$\begin{aligned}\|f_n - g\|_{\overline{Q}} &\le \|f_n - g_n\|_{\overline{Q}} + \|g_n - g\|_{\overline{Q}} \\ &= \underbrace{\|f_n - g_n\|_{Q^\#}}_{<\,2^{-n}} + \underbrace{\|g_n - g\|_{Q^\#}}_{\longrightarrow\, 0} \longrightarrow 0.\end{aligned}$$

This means $f_n \to g$ in $D(\overline{Q})$ and proves $\big(D(\overline{Q}), \langle\cdot,\cdot\rangle_{\overline{Q}}\big)$ to be complete.

(c) Let Q^* be an arbitrary closed extension of Q, i.e. $D(Q) \subseteq D(Q^*)$, $Q^*|_{D(Q)} = Q$ and $(D(Q^*), \|\cdot\|_{Q^*})$ is complete. We have to show that $D(\overline{Q}) \subseteq D(Q^*)$ as well as $Q^*|_{D(\overline{Q})} = \overline{Q}$. As in part (b) of this exercise $Q^\#$ was an arbitrary closed extension of Q, we apply the definition for $Q^\# = Q^*$ and obtain $D(\overline{Q}) \subseteq D(Q^*)$ as well as $Q^*(f,g) = \overline{Q}(f,g)$ for all $f, g \in D(\overline{Q})$. □

Exercise 3 (**Fractional powers of operators**)**.** Assume that T is a positive self-adjoint operator and $\varphi\colon [0,\infty) \to [0,\infty)$, $t \mapsto t^s$ for $0 < s < 1$. Let $T^s = \varphi(T)$. Show that $f \in D(T^s)$ for $f \in D(T)$ and

$$\|T^s f\| \le \|Tf\|^s \|f\|^{1-s}.$$

Show furthermore that

$$T^s f = -\frac{s}{|\Gamma(1-s)|} \int_0^\infty (\mathrm{e}^{-tT} - I) f \frac{\mathrm{d}t}{t^{1+s}}$$

for $f \in D(T^s)$ where Γ is Euler's Gamma Function.

Solution. Since T is a self-adjoint and positive operator, there exists, according to the spectral theorem, Theorem 3.6, a σ-finite measure space (X,μ), a measurable function $u\colon X \to [0,\infty)$ and a unitary map $U\colon L^2(X,\mu) \to H$ such that

$$T = U M_u U^{-1}$$

where M_u is the multiplication operator $M_u\colon D(M_u) \to L^2(X,\mu)$ which acts on a function $f \in L^2(X,\mu)$ by multiplication with u, where

$$D(M_u) := \left\{ f \in L^2(X,\mu) \mid uf \in L^2(X,\mu) \right\}.$$

Therefore, applying the functional calculus we obtain

$$\varphi(T) = UM_{\varphi(u)}U^{-1} = UM_{u^s}U^{-1}.$$

Due to the fact that

$$D(M_{u^s}) = \{f \in L^2(X,\mu) \mid u^s f \in L^2(X,\mu)\}$$

and $uf \in L^2(X,\mu)$, we conclude $f \in D(M_{u^s})$ for any $f \in D(M_u)$, since $u^s \leq 1 \vee u$. Finally, $f \in D(T^s)$ for every $f \in D(T)$.

Then, applying Hölder's inequality, we estimate

$$\begin{aligned}\|T^s f\|_{L^2} &\leq \left(\int_0^\infty u^{2s} |f|^2 \,\mathrm{d}\mu\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^\infty u^2 |f|^2 \,\mathrm{d}\mu\right)^{\frac{s}{2}} \left(\int_0^\infty |f|^2 \,\mathrm{d}\mu\right)^{\frac{1-s}{2}} = \|Tf\|_{L^2}^s \|f\|_{L^2}^{1-s}.\end{aligned}$$

Furthermore, we define

$$I := \int_0^\infty (\mathrm{e}^{-t\lambda} - 1) \frac{\mathrm{d}t}{t^{1+s}}.$$

Integrating by parts, we get

$$\begin{aligned} I &= \underbrace{\left[(\mathrm{e}^{-t\lambda} - 1)\frac{t^{-s}}{s}\right]_0^\infty}_{=0} - \int_0^\infty \mathrm{e}^{-t\lambda} \frac{t^{-s}}{s} \,\mathrm{d}(\lambda t) \\ &\overset{r:=\lambda t}{=} -\frac{\lambda^s}{s} \int_0^\infty t^{1-s-1} \mathrm{e}^{-r} \,\mathrm{d}r = -\frac{\Gamma(1-s)}{s}\lambda^s.\end{aligned}$$

Therefore, $\lambda^s = -\frac{s}{\Gamma(1-s)} I$. Note, that $\Gamma(1-s) = |\Gamma(1-s)|$ as $s \in (0,1)$. The conclusion follows by applying the functional calculus. □

Exercise 4 (**Cosine formula for the resolvent**). Let $T \geq 0$ be a positive self-adjoint operator on a Hilbert space H, and let $\lambda > 0$. Then for all $f \in H$,

$$\lambda(\lambda + T)^{-1} f = \int_0^\infty \mathrm{e}^{-s} \cos\left(\lambda^{-1/2} s T^{1/2}\right) f \,\mathrm{d}s.$$

Solution. Let

$$I(x) := \int_0^\infty \mathrm{e}^{-s} \cos\left(\lambda^{-1/2} s x^{1/2}\right) \mathrm{d}s$$

for $x \geq 0$. Integrating by parts, we obtain

$$\begin{aligned} I(x) &= \left[-\mathrm{e}^{-s}\cos\left(\lambda^{-1/2}sx^{1/2}\right)\right]_0^\infty - \lambda^{-1/2}x^{1/2}\int_0^\infty \mathrm{e}^{-s}\sin\left(\lambda^{-1/2}sx^{1/2}\right)\mathrm{d}s \\ &= 1 - \lambda^{-1/2}x^{1/2}\left(\left[-\mathrm{e}^{-s}\sin\left(\lambda^{-1/2}sx^{1/2}\right)\right]_0^\infty + \lambda^{-1/2}x^{1/2}\ I(x)\right) \\ &= 1 - x\lambda^{-1}I(x). \end{aligned}$$

Therefore, $I(x) = \lambda(\lambda + x)^{-1}$. The conclusion follows by applying the functional calculus. $\square$