UNIVERSITÀ DEGLI STUDI DI SALERNO

Internet Seminar 26

Solutions to Exercises of Lecture 4

Team Salerno

Exercise 1 (Resolvents are continuous)**.** *Show that the resolvent map of an operator A on a Hilbert space H*

$$\varrho(A) \to B(H), \qquad z \mapsto (A-z)^{-1},$$

is continuous.

Solution. Let (z_n) be a sequence such that $z_n \in \varrho(A)$ for all $n \in \mathbb{N}$ and $z_n \to z \in \varrho(A)$. The resolvent map of A is continuous if and only if

$$\lim_{n\to\infty} \left\|(A-z_n)^{-1} - (A-z)^{-1}\right\| = 0.$$

As z_n, $z \in \varrho(A)$, we have, using the resolvent identity,

$$(A-z_n)^{-1} - (A-z)^{-1} = (z_n - z)(A-z_n)^{-1}(A-z)^{-1},$$

and then

$$\left\|(A-z_n)^{-1} - (A-z)^{-1}\right\| \le |z_n - z| \left\|(A-z_n)^{-1}\right\| \left\|(A-z)^{-1}\right\|.$$

We can estimate $\|(A-z)^{-1}\| \le C$, with C not depending on n, since $z \in \varrho(A)$.

Observe now that

$$A - z_n = A - z + z - z_n = [I + (z-z_n)(A-z)^{-1}](A-z).$$

As $z_n \to z$, we can assume that $|z_n - z| \le \frac{1}{2\|(A-z)^{-1}\|}$.
It follows that $\|(z-z_n)(A-z)^{-1}\| \le \frac{1}{2}$ and, by applying [1, Theorem A.22], we conclude that $I + (z-z_n)(A-z)^{-1}$ is bijective and its inverse is given by the Neumann Series as follows

$$[I + (z-z_n)(A-z)^{-1}]^{-1} = \sum_{j=0}^{\infty} (z_n - z)^j (A-z)^{-j}.$$

This implies that $A - z_n$ is bijective too with bounded inverse. Moreover, we have

$$\begin{aligned}(A-z_n)^{-1} &= (A-z)^{-1}[I + (z-z_n)(A-z)^{-1}]^{-1} \\ &= (A-z)^{-1}\sum_{j=0}^{\infty}(z_n - z)^j (A-z)^{-j} \\ &= \sum_{j=0}^{\infty}(z_n - z)^j (A-z)^{-j-1}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\left\|(A-z_n)^{-1}\right\| &\le \sum_{j=0}^{\infty} |z-z_n|^j \left\|(A-z)^{-1}\right\|^{j+1} \\ &\le \sum_{j=0}^{\infty} \frac{1}{2^j \left\|(A-z)^{-1}\right\|^j} \left\|(A-z)^{-1}\right\|^{j+1} \\ &= \sum_{j=0}^{\infty} \frac{1}{2^j} \left\|(A-z)^{-1}\right\| = 2\left\|(A-z)^{-1}\right\| \le 2\,C.\end{aligned}$$

In conclusion, we get

$$\begin{aligned}\left\|(A-z_n)^{-1}-(A-z)^{-1}\right\| &\le |z_n-z| \left\|(A-z_n)^{-1}\right\| \left\|(A-z)^{-1}\right\| \\ &\le 2\,C^2|z_n-z| =: C_2|z_n-z|\end{aligned}$$

for some constant C_2 not depending on n, so

$$\lim_{n\to\infty} \left\|(A-z_n)^{-1}-(A-z)^{-1}\right\| = 0$$

as $\lim_{n\to\infty}|z_n-z| = 0$. □

Exercise 2 (Multiplication operators I). *Let (X,μ) be a measure space and let $u\colon X\to\mathbb{C}$ be measurable. The operator M_u of multiplication by u has domain*

$$D(M_u) = \{f \in L^2(X,\mu) \mid uf \in L^2(X,\mu)\}$$

and acts as

$$M_u f = uf$$

for all $f \in D(M_u)$. Show the following statements:

(a) The operator M_u is densely defined.

(b) The operator M_u is closed.

(c) The adjoint of M_u is gived by $(M_u)^* = M_{\bar{u}}$. In particular, M_u is self-adjoint if u is real-valued.

(d) The operator M_u is bounded if $u \in L^\infty(X,\mu)$.

Solution. Firstly we set $L^2 := L^2(X,\mu)$.

(a) Let $f \in L^2$ and let us consider the set

$$X_n \coloneqq \{x \in X\colon |u(x)| \le n\}.$$

If $x \in X_n$, then $|u(x)| \le n < n+1$. It follows that $x \in X_{n+1}$. This proves that $X_n \subset X_{n+1}$, for all $n \in \mathbb{N}$.

We now set $f_n \coloneqq 1_{X_n} f$. The function $f_n \in L^2$ since $f \in L^2$. Moreover, for a fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}\|M_u f_n\|_2^2 &= \int_X |u(x) f_n(x)|^2 \, d\mu(x) = \int_{X_n} |u(x) f(x)|^2 \, d\mu(x) \\ &\le n^2 \int_{X_n} |f(x)|^2 \, d\mu(x) \le n^2 \, \|f\|_2^2 < \infty.\end{aligned}$$

This shows that $M_u f_n \in L^2$ and so $f_n \in D(M_u)$.

Let us observe that, since $X = \cup_{n=1}^{\infty} X_n$, if $x \in X$, there exists $\bar{n} \in \mathbb{N}$ such that $x \in X_{\bar{n}}$. In addition, for all $n \in \mathbb{N}$ with $n \ge \bar{n}$, then $x \in X_n$ since the sequence (X_n) is increasing. As a consequence, for $n \to \infty$ and for almost all $x \in X$, we have that $1_{X_n}(x) \to 1_X(x) = 1$. This proves that $f_n \to f$ a.e. in X.
Moreover, $|f_n| \le |f|$ and $f \in L^2$. We can apply the Lebesgue's dominated convergence theorem to obtain that $\lim_{n\to\infty} f_n = f$ in L^2.

In conclusion, for all $f \in L^2$ there exists a sequence (f_n) in $D(M_u)$ such that $f_n \to f$ in L^2. Thus, M_u is densely defined.

(b) Let (f_n) be a sequence of $D(M_u)$ such that $f_n \to f$ in L^2 and $M_u f \to g$ in L^2. We prove that $f \in D(M_u)$ and $g = M_u f$.
Obviously, $f \in L^2$. Since $f_n \to f$ in L^2, there exists a subsequence (f_{n_k}) s.t.

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \text{ for a.e. } x \in X.$$

Hence, $\lim_{k\to\infty} u(x) f_{n_k}(x) = u(x) f(x)$ for a.e. $x \in X$. Since we assumed that $M_u f \to g$ in L^2, we conclude that $uf = g$ a.e. in X. This shows that $uf \in L^2$ and $M_u f = g$.

(c) Let $(M_u)^*$ be the adjoint of M_u. Then by definition

$$\begin{aligned}&D((M_u)^*) = \left\{ f \in L^2 s.t.\ \exists\, g \in L^2 \colon \langle f, M_u h\rangle = \langle g, h\rangle,\ \forall\, h \in D(M_u) \right\}, \\ &\quad (M_u)^* f = g.\end{aligned}$$

If $f \in D((M_u)^*)$ then $f \in L^2$, $(M_u)^* f \in L^2$ and we have

$$\langle f, M_u h\rangle = \langle (M_u)^* f, h\rangle, \quad \forall\, h \in D(M_u).$$

It follows that

$$\begin{aligned}\int_X (M_{\overline{u}}) f(x) \overline{h(x)} \, d\mu(x) &= \int_X \overline{u(x)} f(x) \overline{h(x)} \, d\mu(x) = \int_X f(x) \overline{u(x) h(x)} \, d\mu(x) \\ &= \int_X f(x) \overline{M_u h(x)} \, d\mu(x) = \langle f, M_u h\rangle = \langle (M_u)^* f, h\rangle \\ &= \int_X ((M_u)^* f)(x) \overline{h(x)} \, d\mu(x), \quad \forall\, h \in D(M_u).\end{aligned}$$

Hence we get

$$\int_X [(M_{\overline{u}})f(x) - ((M_u)^*f)(x)]\,\overline{h(x)}\,d\mu(x) = 0, \quad \forall\, h \in D(M_u).$$

Since $D(M_u)$ is dense in L^2, we obtain that $M_{\overline{u}}f = (M_u)^*f$ a.e., thus $f \in D(M_{\overline{u}})$. This proves that $D((M_u)^*) \subseteq D(M_{\overline{u}})$, and $(M_u)^* = M_{\overline{u}}$ on $D((M_u)^*)$.

Let now be $f \in D(M_{\overline{u}})$, then $f \in L^2$ and $\overline{u}f \in L^2$. Moreover we have

$$\langle \overline{u}f, h\rangle = \int_X \overline{u}f\overline{h}\,d\mu = \int_X f\overline{uh}\,d\mu = \int_X f\overline{M_u h}\,d\mu = \langle f, M_u h\rangle, \quad \forall\, h \in D(M_u).$$

This means that there exists $g := \overline{u}f \in L^2$ such that $\langle g, h\rangle = \langle f, M_u h\rangle$ for all $h \in D(M_u)$ and this implies by definition that $f \in D((M_u)^*)$. This proves that $D(M_{\overline{u}}) \subseteq D((M_u)^*)$.
In conclusion $D((M_u)^*) = D(M_{\overline{u}})$ and $(M_u)^* = M_{\overline{u}}$.

Moreover, if u is real-valued a.e., then $u = \bar{u}$. As a result, M_u is self-adjoint if and only if u is real-valued.

(d) We assume that $u \in L^\infty(X,\mu)$ and we claim that the operator M_u is bounded. Let $f \in D(M_u) \setminus \{0\}$. Since

$$\|M_u f\|_2^2 = \int_X |u(x)f(x)|^2\,d\mu(x) \le \|u\|_\infty^2\,\|f\|_2^2\,,$$

we obtain that

$$\|M_u\| \le \frac{\|M_u f\|_2}{\|f\|_2} \le \|u\|_\infty\,. \tag{1}$$

Hence, M_u is a bounded operator. □

Exercise 3 (Multiplication operators II)**.** *Let (X,μ) be a σ-finite measure space and M_u the multiplication operator for a measurable function $u\colon X \longrightarrow \mathbb{C}$.*

(a) The operator M_u is self-adjoint if and only if the essential range of u is contained in $\mathbb{R}$, which, in turn, holds if and only if u is real-valued almost everywhere.

(b) The operator M_u is bounded if and only if the essential range of u is bounded, which, in turn, holds if and only if $u \in L^\infty(X,\mu)$. In this case,

$$\|M_u\| = \|u\|_\infty = \sup\{|\lambda| \mid \lambda \text{ is in the essential range of } u\}. \tag{2}$$

(c) $M_u = 0$ holds if and only if the essential range of u is $\{0\}$, which, in turn, holds if and only if $u = 0$ holds almost everywhere.

Solution. (a) From Exercise 2 (c) we have only to prove that the essential range of u is contained in $\mathbb{R}$ if and only if u is real-valued almost everywhere.
Let us recall first the definition of the essential range of a measurable function $u : X \to \mathbb{C}$,

$$\operatorname{ess\,ran}(u) := \{\lambda \in \mathbb{C} : \mu(u^{-1}(B_\varepsilon(\lambda))) > 0, \forall \varepsilon > 0\}.$$

If $\operatorname{ess\,ran}(u) \subset \mathbb{R}$, then for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_\lambda > 0$ such that $B_{\varepsilon_\lambda}(\lambda) \subset \mathbb{C} \setminus \mathbb{R}$ and $\mu(u^{-1}(B_{\varepsilon_\lambda}(\lambda))) = 0$. The balls $B_{\varepsilon_\lambda}(\lambda)$ form an open cover of $\mathbb{C} \setminus \mathbb{R}$, which admits a countable subcover consisting of balls B_n such that $B_n \subset \mathbb{C} \setminus \mathbb{R}$ and $\mu(u^{-1}(B_n)) = 0$ for every $n \in \mathbb{N}$. By the sub-additivity of the measure μ we deduce that $\mu(u^{-1}(\mathbb{C} \setminus \mathbb{R})) = 0$. Hence, u is real-valued almost everywhere.

Vice versa, assume that u is real-valued almost everywhere and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $\mathbb{C} \setminus \mathbb{R}$ is an open subset of $\mathbb{C}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\lambda) \subset \mathbb{C} \setminus \mathbb{R}$. As $\mu(u^{-1}(B_\varepsilon(\lambda))) = 0$, we get that $\lambda \notin \operatorname{ess\,ran}(u)$. Since λ is an arbitrary element of $\mathbb{C} \setminus \mathbb{R}$, it follows that the essential range of u is contained in $\mathbb{R}$.

(b) If $u \in L^\infty(X, \mu)$, Exercise 2 (d) and inequality (1) show that the operator M_u is bounded and

$$\|M_u\| \leq \|u\|_\infty . \tag{3}$$

Conversely, assume that $u \notin L^\infty(X, \mu)$ and for every $n \in \mathbb{N}$ define the set

$$X_n = \{x \in X \mid |u(x)| > n\}.$$

Then, $\mu(X_n) > 0$. Moreover, since the measure μ is σ-finite, there exists a subset $\tilde{X}_n$ of X such that $0 < \mu(\tilde{X}_n) < \infty$. Consequently, $1_{\tilde{X}_n} \in L^2(X, \mu)$ and the following inequalities hold

$$\left\|M_u 1_{\tilde{X}_n}\right\|_2^2 = \int_X \left|u(x) 1_{\tilde{X}_n}(x)\right|^2 d\mu(x) \geq n^2 \left\|1_{\tilde{X}_n}\right\|_2^2 .$$

Hence, M_u is not bounded. This proves that the operator M_u is bounded if and only if $u \in L^\infty(X, \mu)$.

We now show that

$$\|M_u\| \geq \|u\|_\infty . \tag{4}$$

We fix $\varepsilon > 0$ and we consider the set $Y = \{x \in X \mid |u(x)| \geq \|u\|_\infty - \varepsilon\}$. We first observe that $\mu(Y) > 0$. Moreover, we have

$$\|M_u 1_Y\|_2^2 = \int_X |u(x) 1_Y(x)|^2 d\mu(x) \geq (\|u\|_\infty - \varepsilon)^2 \|1_Y\|_2^2 .$$

As a result, we find that $\|M_u\| \geq \|u\|_\infty - \varepsilon$. Letting $\varepsilon \to 0$ we obtain (4). Combining this with (3) yields the first equality in (2).

To end the proof of (b) it suffices to show the second equality in (2). Indeed, if it holds, then the boundedness of the essential range of u is equivalent to the fact that u belongs to $L^\infty(X, \mu)$.

We first show that

$$\|u\|_\infty \geq \sup\{|\lambda| \mid \lambda \in \operatorname{ess\,ran}(u)\}. \tag{5}$$

Let $\lambda \in \operatorname{ess\,ran}(u)$ and $\varepsilon > 0$. We get

$$0 < \mu(u^{-1}(B_\varepsilon(\lambda))) \leq \mu(\{x \in X : |u(x)| > |\lambda| - \varepsilon\}),$$

which implies that $|\lambda| - \varepsilon \leq \|u\|_\infty$. Passing to the supremum over λ and letting $\varepsilon \to 0^+$, we get (5).

We finally prove

$$\|u\|_\infty \leq \sup\{|\lambda| \mid \lambda \in \operatorname{ess\,ran}(u)\}.$$

Defining $s := \sup\{|\lambda| \mid \lambda \in \operatorname{ess\,ran}(u)\}$, it suffices to show that

$$\mu(u^{-1}(\mathbb{C} \setminus \overline{B_s(0)})) = 0. \tag{6}$$

We observe that

$$\mathbb{C} \setminus \overline{B_s(0)} = \bigcup_{n=1}^{\infty} \left\{\lambda \in \mathbb{C} : s + \frac{1}{n} \leq |\lambda| \leq s + n\right\} =: \bigcup_{n=1}^{\infty} C_n. \tag{7}$$

By the definition of s, for any $\lambda \in C_n$, there exists $\varepsilon_\lambda > 0$ such that $\mu(u^{-1}(B_{\varepsilon_\lambda}(\lambda))) = 0$. Since C_n is compact, it can be covered by a finite number of balls with center in itself, i.e. there exists $m \in \mathbb{N}$ such that

$$C_n \subset \bigcup_{k=1}^{m} B_{\varepsilon_{\lambda_k}}(\lambda_k),$$

for some $\lambda_1, \cdots, \lambda_m \in C_n$. Using the subadditivity of the measure μ, we deduce that

$$\mu(C_n) \leq \mu\left(\bigcup_{k=1}^{m} B_{\varepsilon_{\lambda_k}}(\lambda_k)\right) = 0. \tag{8}$$

Combining (7) and (8), we get (6).

(c) Since $\|M_u\| = \|u\|_\infty$ by (2), then $M_u = 0$ holds if and only if $u = 0$ almost everywhere. Similarly, the second indentity in (2) implies that $u = 0$ almost everywhere if and only if the essential range of u is $\{0\}$. □

Exercise 4 (Closure convergence)**.** *Let (L_n) be a sequence of self-adjoint operators on a Hilbert space H and let L be a self-adjoint operator. Assume that for a family $(\Phi_\alpha)_{\alpha \in I}$ of measurable bounded functions from $\mathbb{R}$ to $\mathbb{R}$ and some index set I we have*

$$\lim_{n\to\infty} \Phi_\alpha(L_n)f = \Phi_\alpha(L)f \tag{9}$$

for all $f \in H$ *and for all* $\alpha \in I$*. Let* $\mathcal{A}$ *be the closure of* $\{\Phi_\alpha \,|\, \alpha \in I\}$ *with respect to the supremum norm. Show that*

$$\lim_{n\to\infty} \Phi(L_n)f = \Phi(L)f$$

for all $\Phi \in \mathcal{A}$ *and* $f \in H$*.*

Solution. Let $f \in H$ and $\Phi \in \mathcal{A}$. Then for any $\varepsilon > 0$, there exist an index set $J \subset I$ and a sequence $(\Phi_h)_{h\in J}$ such that $\|\Phi_h - \Phi\|_\infty \leq \varepsilon$. We observe that, since Φ_h is bounded, for any $h \in J$, then also Φ is bounded. Furthermore, by Corollary 3.14, we infer

$$D(\Phi_h(L_n)) = D(\Phi(L_n)) = D(\Phi_h(L)) = D(\Phi(L)) = H.$$

We now show that

$$\|\Phi(L_n)f - \Phi(L)f\| \to 0, \quad \text{as } n \to \infty. \tag{10}$$

By using the triangle inequality, we split the norm in three terms:

$$\begin{aligned}
&\|\Phi(L_n)f - \Phi(L)f\| \\
&\leq \|\Phi(L_n)f - \Phi_h(L_n)f\| + \|\Phi_h(L_n)f - \Phi_h(L)f\| + \|\Phi_h(L)f - \Phi(L)f\| \\
&= (I) + (II) + (III).
\end{aligned}$$

Thanks to the assumption (9), $(II) \to 0$, as $n \to \infty$. We show that (III) is infinitesimal. With the same argument we also obtain the (I) tends to 0, as $n \to \infty$. By Proposition 3.9 (c), (e) and Corollary 3.14, we infer

$$\begin{aligned}
(III) &= \|(\Phi_h(L) - \Phi(L))f\| \\
&= \|(\Phi_h - \Phi)(L)(f)\| \\
&\leq \|(\Phi_h - \Phi)(L)\| \, \|f\| \\
&= \left\|(\Phi_h - \Phi)|_{\sigma(L)}\right\|_\infty \|f\| \\
&= \|\Phi_h - \Phi\|_\infty \, \|f\| \\
&\leq \varepsilon\|f\|.
\end{aligned}$$

Therefore, (10) is proved. □

Bibliography

[1] Bátkai, A. and Fijavž, M.K. and Rhandi, A. (2017), *Positive Operator Semigroups: From Finite to Infinite Dimensions*. Springer International Publishing.