ISEM26 - SOLUTIONS TO THE EXERCISES OF LECTURE 3

WUPPERTAL TEAM[1]

Exercise 1 ($\mathcal{F} = C(X)$)**.** Let $\mathcal{F}$ be the domain of the formal Laplacian $\mathcal{L}$ associated to a graph. Show that the following statements are equivalent:

(1) The graph is locally finite.
(2) $\mathcal{L}C_c(X) \subseteq C_c(X)$
(3) $C(X) = \mathcal{F}$

Solution. "(1) ⇔ (2)" Since the characteristic functions form a basis of $C_c(X)$, it suffices to prove that (1) holds if and only if $\mathcal{L}1_x \in C_c(X)$ for all $x \in X$. It follows from the definition of the formal Laplacian that

$$(\mathcal{L}1_x)(z) = \begin{cases} \frac{1}{m(x)} \sum_{y\in X} b(x,y) + \frac{c(x)}{m(x)} & \text{if } z = x, \\ \frac{-1}{m(z)} b(x,z) & \text{if } z \neq x \end{cases}$$

for every $z \in X$. Since $b(x,\cdot)$ vanishes everywhere except on neighbours of x, we conclude

$$|\{y \in X : x \sim y\}| < \infty \iff \mathcal{L}1_x \in C_c(X)$$

and arrive at the desired equivalence.

"(1) ⇒ (3)" Let $f \in C(X)$ be a function and consider $x \in X$. If the graph is locally finite, we have

$$b(x,y)|f(y)| = 0 \qquad \text{for all but finitely many } y \in X.$$

It follows that $\sum_{y\in X} b(x,y)|f(y)|$ converges. Since our choice of x was arbitrary, this gives $f \in \mathcal{F}$.

"(3) ⇒ (1)" Assume by contradiction that the graph is not locally finite, i.e. there exists $x \in X$ such that

$$\{y \in X : x \sim y\} =: \{y_1, y_2, \dots\}$$

is infinite and $y_i \neq y_j$ whenever $i \neq j$. Consider the function $f \in C(X)$ defined by

$$f(y) := \begin{cases} \frac{1}{b(x,y_i)} & \text{if } y = y_i \text{ where } i \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

for $y \in X$. Since

$$\sum_{y\in X} b(x,y)|f(y)| = \sum_{i=1}^{\infty} \underbrace{b(x,y_i)|f(y_i)|}_{=1}$$

does not converge, we have $f \notin \mathcal{F}$ and the claim follows by contradiction. //

Exercise 2 (Maximum principle)**.** Let $\mathcal{A}\colon C_c(X) \to C_c(X)$ be a symmetric linear operator, i.e., $\mathcal{A}1_x(y) = \mathcal{A}1_y(x)$ for all $x, y \in X$. Show the following equivalence:

(i) $\mathcal{A} = \mathcal{L}_{b,c}$ on $C_c(X)$ for a locally finite graph (b,c) over X.
(ii) $\mathcal{A}$ satisfies a maximum principle, i.e., if $f \in C_c(X)$ has a non-negative local maximum in $x \in X$, then $\mathcal{A}f(x) \geq 0$.

Solution. For the whole proof we set without loss of generality $m(x) = 1$ for all $x \in X$.
(i) ⇒ (ii): Let $f \in C_c(X)$ be with non-negative local maximum in $x \in X$. Since $\mathcal{A} = \mathcal{L}_{b,c}$ and $C_c(X) \subseteq \mathcal{F}$, we have

$$\mathcal{A}f(x) = \sum_{y\in X} b(x,y)(f(x) - f(y)) + c(x)f(x).$$

Since f has a non-negative local maximum in $x \in X$, we conclude $f(x) \geq 0$ and $f(x) \geq f(y)$ for all $x \sim y$. Additionally we know, that $b(x,y) = 0$ for all $x \nsim y$. Thus, by the positivity of the functions b and c we obtain

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$\mathcal{A}f(x) \geq 0$.
(ii) $\Rightarrow$ (i): We set as weight function $b\colon X \times X \to [0,\infty)$

$$b(x,y) := \begin{cases} -\mathcal{A}1_x(y), & x \neq y, \\ 0, & x = y. \end{cases}$$

In fact, for $x \in X$, the function -1_x has a maximum in all $x \neq y \in X$. Thus, by the maximum principle, we obtain $\mathcal{A}(-1_x)(y) = -\mathcal{A}1_x(y) \geq 0$ for all $x \neq y$. The symmetry of b follows directly from the symmetry of $\mathcal{A}$, as

$$b(x,y) = -\mathcal{A}1_x(y) = -\mathcal{A}1_y(x) = b(y,x),$$

for all $x \neq y \in X$. Furthermore, we have for all $x \in X$

$$\sum_{y \in X} b(x,y) = -\sum_{\substack{y \in X \\ y \neq x}} \mathcal{A}1_x(y).$$

As $\mathcal{A}C_c(X) \subseteq C_c(X)$ by assumption, the sum on the right-hand side has only finitely many entries unequal to zero, hence it is finite. Moreover, we set $c\colon X \to [0,\infty)$,

$$c(x) := \sum_{y \in X} \mathcal{A}1_x(y),$$

which is in fact well defined. To be more precise, we have

$$c(x) = -\sum_{y \in X} b(x,y) + \mathcal{A}1_x(x) < \infty.$$

Let $y_1, \ldots, y_n \in X$, such that $\operatorname{supp}(\mathcal{A}1_x) \subseteq \{x, y_1, \ldots, y_n\}$. Since

$$\sum_{i=1}^{n} 1_{y_i} + 1_x \in C_c(X)$$

has a maximum in x we deduce

$$c(x) = \sum_{y \in X} \mathcal{A}1_x(y) = \sum_{i=1}^{n} \mathcal{A}1_{y_i}(x) + \mathcal{A}1_x(x) = \mathcal{A}\left(\sum_{i=1}^{n} 1_{y_i} + 1_x\right)(x) \geq 0.$$

Thus, (b,c) defines a graph over X. Next, we calculate for all $x, y \in X$

$$\begin{aligned}
\mathcal{L}_{b,c}1_x(y) &= \sum_{z \in X} b(y,z)(1_x(y) - 1_x(z)) + c(y)1_x(y) \\
&= \sum_{\substack{z \in X \\ z \neq y}} \mathcal{A}1_y(z)(1_x(z) - 1_x(y)) + \sum_{z \in X} \mathcal{A}1_y(z)1_x(y) \\
&= \sum_{\substack{z \in X \\ z \neq y}} \mathcal{A}1_y(z)1_x(z) + \mathcal{A}1_y(y)1_x(y) \\
&= \begin{cases} \sum_{z \in X, z \neq y} \mathcal{A}1_y(z)1_y(z) + \mathcal{A}1_y(y) = \mathcal{A}1_y(y), & \text{for } x = y, \\ \mathcal{A}1_y(x), & \text{for } x \neq y \end{cases} \\
&= \mathcal{A}1_x(y).
\end{aligned}$$

This shows $\mathcal{A} = \mathcal{L}_{b,c}$ on the set of all characteristic functions $\{1_x \mid x \in X\}$. Since this is a basis of $C_c(X)$ we obtain the equality on $C_c(X)$. It remains to show that (b,c) is locally finite. But this follows directly from Exercise 1, since we know $\mathcal{A}C_c(X) \subseteq C_c(X)$. //

Exercise 3 (Uncountable graphs)**.** Let X be an arbitrary set and assume that $b : X \times X \longrightarrow [0,\infty)$ satisfies $b(x,y) = b(y,x),\ b(x,x) = 0$ and

$$\sum_{z \in X} b(x,z) = \sup_{U \subseteq X \text{finite}} \sum_{y \in U} b(x,y) < \infty$$

for all $x \in X$. Call a subset Y of X connected if for arbitrary $x, y \in Y$ there exists $n \in \mathbb{N}$ and $x_0, \ldots, x_n \in Y$ with $x_0 = x,\ x_n = y$ and $b(x_k, x_{k+1}) > 0$ for all $k = 0, \ldots, n-1$. Show that any connected subset of X is countable.

Solution. For $x,\ y \in X$ define

$$x \sim_c y \ :\Longleftrightarrow\ \exists\, n \in \mathbb{N}, x_0, \ldots, x_n \in X : x_0 = x,\ x_n = y,\ \forall\, k = 0, \ldots, n-1 : b(x_k, x_{k+1}) > 0$$
$$\Longleftrightarrow:\ x \text{ is connected to } y \text{ by the path } x_0, \ldots, x_{n-1}.$$

This is an equivalence relation on X. Indeed, $x \sim_c x$ for all $x \in X$ since every element of X is connected to itself by the empty path. For $x, y \in X$ such that $x \sim_c y$ it follows, by definition of connectedness, that there exists $n \in \mathbb{N}$ and $x_0, \ldots, x_n \in X$ with $x_0 = x,\ x_n = y$ and $b(x_k, x_{k+1}) > 0$ for all $k = 0, \ldots, n-1$. Symmetry of $\sim_c$ follows by reversing the order of $x_0, \ldots, x_n \in X$ and $b(x_{k+1}, x_k) = b(x_k, x_{k+1}) > 0$ for all $k = 0, \ldots, n-1$. To show transitivity of $\sim_c$ let $x, y, z \in X$ be fixed with $x \sim_c y$ and $y \sim_c z$. Now, there are $n,\ m \in \mathbb{N}$ and $x_0, \ldots, x_n \in X$, $y_0, \ldots, y_m \in X$ with $x_0 = x,\ x_n = y = y_0,\ z = y_m$ and $b(x_k, x_{k+1}) > 0,\ b(y_j, y_{j+1}) > 0$ for all $k = 0, \ldots, n-1,\ j = 0, \ldots, m-1$. Hence x is connected to z by the path $x = x_0, \ldots, x_n, y_1, \ldots, y_m = z$ and (after renumbering) transitivity of $\sim_c$ follows.

The equivalence classes of this relations are called the (connected) components of X. For any $x \in X$ and $k \in \mathbb{N}$ define

$$[x] := \{y \in X : x \sim_c y\},$$
$$N_k(x) := \{y \in X : \exists x_0, \ldots, x_{k-1},\ x = x_0,\ y = x_{k-1}, \forall\, j = 0, \ldots, k-1 : b(x_j, x_{j+1}) > 0\},$$

i.e. [x] is the equivalence class containing x and $N_k(x)$ denotes set of elements of X which are connected to x by a path of length k. Note that $N_1(x)$ is countable for all $x \in X$. Indeed, for fixed $x \in X$ it holds that

$$N_1(x) = \{y \in X : b(x,y) > 0\} = \bigcup_{n \in \mathbb{N}} \left\{ y \in X : b(x,y) > \frac{1}{n} \right\}$$

and $A_n(x) := \left\{ y \in X : b(x,y) > n^{-1} \right\}$ is finite for all $n \in \mathbb{N}$, $n \geq 1$. Suppose, by contradiction, there exists $n \in \mathbb{N}$ such that $A_n(x)$ is infinite. Then

$$\sum_{z \in X} b(x,z) = \sup_{U \subseteq X \text{finite}} \sum_{y \in U} b(x,y) \geq \sup_{U \subseteq A_n(x) \text{ finite}} \sum_{y \in U} b(x,y) \geq \sup_{U \subseteq A_n(x) \text{ finite}} \sum_{y \in U} \frac{1}{n} = \infty,$$

which contradicts the assumption $\sum_{z \in X} b(x,z) < \infty$ for all $x \in X$. Therefore $N_1(x)$ is countable for every $x \in X$ as a countable union of finite sets.

Let $U \subseteq X$ be connected and non empty and let $x \in U$ be fixed. Any $y \in U$ is connected to x and therefore $U \subseteq [x]$ holds. Hence, it suffices to show that every connected component is countable. Let $x \in X$ be fixed. Since every element of $[x]$ is connected to x by some finite path it follows that $[x] = \bigcup_{k \in \mathbb{N}} N_k(x)$. Countability of $N_k(x)$ for all $k \in \mathbb{N}$ would imply that $[x]$ is countable as a countable union of countable sets. This can be proven by induction. Clearly, $N_0(x) = \{x\}$ is countable. For $k \geq 1$ it holds, by construction, that $N_k(x) = \bigcup_{y \in N_{k-1}(x)} N_1(y)$ and $N_k(x)$ is countable, again, as a countable union of countable sets since $N_{k-1}(x)$ is countable by induction hypothesis and $N_k(y)$ is countable for any $y \in X$ as discussed above. //

Exercise 4 (Summability)**.** Let X be a countable set, $b : X \times X \to [0, \infty)$ and $\mathcal{Q} : C(X) \to [0, \infty]$

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))^2.$$

Show that

$$\mathcal{Q}(\varphi) < \infty$$

for all $\varphi \in C_c(X)$ if and only if

$$\sum_{y \in X} b(x,y) < \infty$$

for all $x \in X$.

Solution. „$\Rightarrow$“ We assume that $\mathcal{Q}(\varphi) < \infty$ holds for all $\varphi \in C_c(X)$. Let $x_0 \in X$ be arbitrary. In particular, our assumption holds for the characteristic function 1_{x_0}. At first we observe that

$$(1_{x_0}(x) - 1_{x_0}(y))^2 = \begin{cases} 1, & (x = x_0 \wedge y \neq x_0) \vee (x \neq x_0 \wedge y = x_0), \\ 0, & (x = x_0 = y) \vee (x \neq x_0 \neq y). \end{cases}$$

Thus, we obtain

$$\begin{aligned}\mathcal{Q}(1_{x_0}) &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(1_{x_0}(x)-1_{x_0}(y))^2\\ &= \frac{1}{2}\sum_{y\in X\setminus\{x_0\}} b(x_0,y)(1-1_{x_0}(y))^2 + \frac{1}{2}\sum_{x\in X\setminus\{x_0\}} b(x,x_0)(1_{x_0}(x)-1)^2\\ &= \frac{1}{2}\sum_{y\in X\setminus\{x_0\}} b(x_0,y) + \frac{1}{2}\sum_{x\in X\setminus\{x_0\}} b(x,x_0)\end{aligned}$$

This implies $\frac{1}{2}\sum_{y\in X\setminus\{x_0\}} b(x_0,y) \le \mathcal{Q}(1_{x_0}) < \infty$. Since $b(x_0,x_0)<\infty$, we obtain from this inequality $\sum_{y\in X} b(x_0,y) < \infty$.

„$\Leftarrow$“ We assume that $\sum_{y\in X} b(x,y) < \infty$ holds for all $x\in X$. Let $\varphi\in C_c(X)$ be arbitrary. Then there exist $x_1,\ldots,x_n\in X$ such that $\varphi(x_i)\in\mathbb{R}\setminus\{0\}$ for $i=1,\ldots,n$ and $\varphi(x)=0$ for all $x\in X\setminus\{x_1,\ldots,x_n\}$. Hence, we have

$$\begin{aligned}\mathcal{Q}(\varphi) &= \frac{1}{2}\sum_{x,y\in X} b(x,y)\left(\varphi(x)-\varphi(y)\right)^2\\ &= \frac{1}{2}\sum_{x\in X}\varphi(x)^2\sum_{y\in X} b(x,y) - \sum_{x\in X}\varphi(x)\sum_{y\in X} b(x,y)\varphi(y) + \frac{1}{2}\sum_{x\in X}\sum_{y\in X} b(x,y)\varphi(y)^2\\ &= \frac{1}{2}\sum_{i=1}^{n}\varphi(x_i)^2\sum_{y\in X} b(x_i,y) - \sum_{i=1}^{n}\varphi(x_i)\sum_{i=1}^{n} b(x,x_i)\varphi(x_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{x\in X} b(x,x_i)\varphi(x_i)^2 < \infty.\end{aligned}$$

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Bonus Exercise 1. Let $\mathcal{A}: C(X)\to C(X)$ be a symmetric linear operator, i.e., $A1_x(y) = A1_y(x)$ for all $x,y\in X$. Does one have $\mathcal{A}C_c(X)\subseteq C_c(X)$?

Solution. In general no, (if X is infinite). We give a counterexample for the inclusion: Let $X=\mathbb{N}$ and define the operator

$$\mathcal{A}: C_c(X)\to C(X), \quad (\mathcal{A}f)(y) := \sum_{z\in X} 2^{-(y+z)} f(z).$$

Then

$$\mathcal{A}1_x(y) = \sum_{z\in X} 2^{-(y+z)}1_x(z) = 2^{-(y+x)} = 2^{-(x+y)} = \sum_{z\in X} 2^{-(x+z)}1_y(z) = \mathcal{A}1_y(x)$$

for all $x,y\in X$. We extend $\mathcal{A}$ to a linear operator on $C(X)$ by defining it as 0 on the algebraic complement of $C_c(X)$ in $C(X)$. This operator is, of course, still *symmetric*. However, it does not leave the compactly supported functions invariant, since $\mathcal{A}1_x\notin C_c(X)$. Indeed, it holds that $\mathcal{A}1_x(y) = 2^{-(x+y)}\neq 0$ for all $y\in X$. //