Solutions for Sheet 2 of ISem 26

Team Hagen[1]

Exercise 1 (Positivity improvement of the inverse operator). Let X be a finite set and let L be an injective operator on $C(X)$. Show that the following assertions are equivalent:

(i) The inverse operator L^{-1} is positivity improving, i.e. for all $f \in C(X)$ such that $f \geq 0$ and $f \neq 0$, we have $L^{-1}f > 0$.

(ii) For each function $u \in C(X)$ satisfying the inequalities $\max_{x\in X} u(x) \geq 0$ and $Lu \leq 0$, we have $u = 0$.

Proof. (i) $\Rightarrow$ (ii): Let $u \in C(X)$ such that $\max_{x\in X} u(x) \geq 0$ and $Lu \leq 0$. If $Lu = 0$ identically, then by the injectivity it follows that $u = 0$. So, suppose that there exists atleast one element $y \in X$ such that $Lu(y) < 0$, i.e. $-Lu \geq 0$ with $-Lu \neq 0$. By the fact that L^{-1} is positivity improving, we obtain that $L^{-1}(-Lu) = -u > 0$, equivalently $u < 0$. But this contradicts $\max_{x\in X} u(x) \geq 0$. Thus we have $Lu = 0$, i.e. $u = 0$ by injectivity.
(ii) $\Rightarrow$ (i): Let $f \in C(X)$ with $f \geq 0$ and $f \neq 0$. We have to prove that $L^{-1}f > 0$. Suppose that there exists a $x \in X$ such that $L^{-1}f(x) \leq 0$. Put $u := -L^{-1}f$. Then $Lu = -f \leq 0$ and $\max_{x\in X} u(x) \geq -L^{-1}f(x) \geq 0$. Thus, by assumption, it follows that $u = -L^{-1}f = 0$, i.e. $f = 0$, a contradiction. Thus $L^{-1}f > 0$ and the proof is complete. □

Exercise 2 (Cauchy problem / Heat equation). Let (X, m) be a finite measure space and let L be a self-adjoint operator on $\ell^2(X, m)$ and for $t \geq 0$ let e^{-tL} be defined via spectral calculus.

(a) Show that for all $t \geq 0$,

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{1}{n!}(-tL)^n$$

In particular, show that the sum is absolutely convergent with respect to the operator norm.

(b) Show that $\{e^{-tL} \mid t \geq 0\}$ equipped with the composition of operators is an operator semigroup, i.e., $e^{0L} = I$ and $e^{(t+s)L} = e^{tL}e^{sL}$ for all $t, s \geq 0$ and $t \mapsto e^{-tL}f$ is continuously differentiable at $t = 0$ for all $f \in \ell^2(X, m)$. Moreover, show that (in this finite dimensional case)

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-tL} = -Le^{-tL} = -e^{-tL}L.$$

(c) Show that for all $f \in \ell^2(X, m)$, the function $t \mapsto \varphi_t := e^{-tL}f$ is the unique solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = -L\varphi_t, \quad \varphi_0 = f,$$

for all $t \geq 0$.

Proof. (a) Recalling the functional calculus $\Phi(L)$ for a (continuous) function $\Phi \in C(X)$, we have

$$\Phi(L) = \sum_{\lambda\in\sigma(L)} \Phi(\lambda)E_\lambda.$$

So considering $\Phi_t : 0e^{-t\cdot}$, we obtain

$$e^{-tL} = \sum_{\lambda\in\sigma(L)} e^{-t\lambda}E_\lambda = \sum_{\lambda\in\sigma(L)} \sum_{n=0}^{\infty} \frac{1}{n!}(-t\lambda)^n E_\lambda = \sum_{n=0}^{\infty} \frac{1}{n!}(-t)^n \underbrace{\sum_{\lambda\in\sigma(L)} \lambda^n E_\lambda}_{=L^n} = \sum_{n=0}^{\infty} \frac{1}{n!}(-tL)^n.$$

noting that the sum converges absolutely as

$$\sum_{n=0}^{\infty} \left\| \frac{1}{n!}(-tL)^n \right\| \leq \sum_{n=0}^{\infty} \frac{1}{n!}t^n\|L\|^n = e^{t\|L\|} < \infty.$$

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(b) By multiplicativity of the functional calculus we have for $\Phi_t := e^{t\cdot}$

$$e^{(t+s)L} = \Phi_{t+s}(L) = (\Phi_t\Phi_s)(L) = \Phi_t(L)\Phi_s(L) = e^{tL}e^{sL}$$

as well as $e^{0L} = \Phi_0(L) = \mathbf{1}(L) = I$. One can also see this by using the series expression from (a) as well as the Cauchy product formular as follows:

$$\begin{aligned} e^{(t+s)L} &= \sum_{n=0}^{\infty} \frac{1}{n!}(t+s)^n L^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!}\binom{n}{k}(tL)^k(sL)^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!(n-k)!}(tL)^k(sL)^{n-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!}(tL)^k \frac{1}{(n-k)!}(sL)^{n-k} \overset{\text{Cauchy}}{\underset{\text{product}}{=}} \left(\sum_{n=0}^{\infty} \frac{1}{n!}(tL)^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!}(tS)^n\right) = e^{tL}e^{sL}. \end{aligned}$$

Furthermore, we have $e^{0L} = \sum_{n=0}^{\infty} \frac{1}{n!}(0L)^n = I$. Thus $(e^{-tL})_{t\geq 0}$ defines a operator semigroup. Moreover, we have

$$\|e^{-tL} - I\| = \left\|\sum_{n=1}^{\infty} \frac{1}{n!}(-tL)^n\right\| \leq \sum_{n=1}^{\infty} \frac{1}{n!}t\|L\|^n = e^{t\|L\|} - 1 \to 0 \text{ as } t \searrow 0,$$

which proves the continuity of the map $t \mapsto e^{-tL}$ and in particular the continuity of the derivatives whenever they exist. To compute the derivative of $t \mapsto e^{-tL}$ at first in $t_0 = 0$, we observe analogously:

$$\begin{aligned} \left\|\frac{e^{-tL} - I}{t} - (-L)\right\| &= \frac{1}{t}\left\|\sum_{n=2} \frac{1}{n!}(-tL)^n\right\| \leq \frac{1}{t}\sum_{n=2}^{\infty} \frac{1}{n!}t^n\|L\|^n \\ &= \frac{1}{t}(e^{t\|L\|} - 1 - t\|L\|) = \frac{e^{t\|L\|} - 1}{t} - \|L\| \to 0 \text{ as } t \searrow 0. \end{aligned}$$

But according to the semigroup property, this implies that

$$\lim_{h\downarrow 0} \frac{e^{-(t+h)L} - e^{-tL}}{h} = \lim_{h\downarrow 0} \frac{e^{-tL}e^{-hL} - e^{-tL}}{h} = e^{-tL}\lim_{h\downarrow 0} \frac{e^{-hL} - I}{h} = -e^{-tL}L$$

as well as

$$\lim_{h\downarrow 0} \frac{e^{-(t+h)L} - e^{-tL}}{h} = \lim_{h\downarrow 0} \frac{e^{-hL}e^{-tL} - e^{-tL}}{h} = \lim_{h\downarrow 0} \frac{e^{-hL} - I}{h}e^{-tL} = -Le^{-tL}.$$

Thus $t \mapsto e^{-tL}$ is differentiable with $\frac{\mathrm{d}}{\mathrm{d}t}e^{-tL} = -Le^{-tL} = -e^{-tL}L$.
(c) From part (b) we can see that $\phi_t := e^{-tL}f$ solves the initial value problem. Now let $x \in X$. We have

$$\frac{d}{dt}\varphi_t(x) = -L\varphi_t(x), \quad \varphi_0(x) = f(x).$$

This is an initial value problem for a first-order homogeneous linear differential equation with continuous L. Therefore, by the existence and uniqueness theorem for first-order homogeneous linear differential equations, it has a unique solution. Since this is true for all x, the solution ϕ_t is unique. ☐

Exercise 3 (Stochastic incompleteness). Let (b, c) be a connected graph over (X, m) and let $L = L_{b,c,m}$ denote the associated Laplacian.

(a) Show that $e^{-tL}\mathbf{1} < \mathbf{1}$ for all $t > 0$ if and only if $c \neq 0$.

(b) Show that if $e^{-tL}\mathbf{1} < \mathbf{1}$ for some $t < 0$, then $e^{-tL}\mathbf{1} < \mathbf{1}$ for all $t > 0$.

Proof. (a)/(b) „$\Rightarrow$“: This direction is clear by Remark 1.22 because if we suppose that $c = 0$, then this implies that $e^{-tL}\mathbf{1} = \mathbf{1}$ for every $t \geq 0$, a contradiction. Hence $c \neq 0$
„$\Leftarrow$“: As $(e^{-tL})_{t\geq 0}$ is Markov by Corollary 1.21, we obtain that $e^{-tL}\mathbf{1} \leq \mathbf{1}$ for every $t \geq 0$. Again, as $c \neq 0$ and the fact that $e^{0L} = I$ we obtain by Remark 1.22 that there is at least one $t_0 > 0$ such that $e^{-t_0L}\mathbf{1} < \mathbf{1}$. We prove now that $e^{-tL}\mathbf{1} < \mathbf{1}$ for every $t > 0$. So let $t > 0$ be arbitrary. If $t \geq t_0$, then we have $e^{-(t-t_0)L}\mathbf{1} \leq \mathbf{1}$ and this implies $e^{-t_0L}e^{-(t-t_0)}\mathbf{1} \leq e^{-t_0L}\mathbf{1}$ as the semigroup is positive. Hence

$$e^{-tL}\mathbf{1} = e^{-t_0L}e^{-(t-t_0)L}\mathbf{1} \leq e^{-t_0L}\mathbf{1} < 1.$$

Thus, we have $e^{-tL}\mathbf{1} < \mathbf{1}$ for every $t \geq t_0$. Suppose now that $0 < t < t_0$ and assume that $e^{-tL}\mathbf{1} = \mathbf{1}$. Then

$$e^{-ntL}\mathbf{1} = (e^{-tL})^n\mathbf{1} = \mathbf{1} \text{ for every } n \in \mathbb{N}. \tag{1}$$

Now choose $n_0 \in \mathbb{N}$ such that $nt > t_0$. Then by the first part, it would follow that $e^{-n_0tL}\mathbf{1} < \mathbf{1}$, a contradiction to (1). Therefore, it follows that $e^{-tL}\mathbf{1} < \mathbf{1}$ and we have also proved part (b). ☐

Exercise 4 (Effective resistance). Let b be a graph over a finite set X, let $Q = Q_b$ be the associated form and let

$$W_{\text{eff}}(x,y) := \sup\left\{ \frac{1}{Q(h)} \;\middle|\; h \in C(X), h(x) - h(y) = 1 \right\}$$

be the *effective resistance* for $x \neq y$ and $W_{\text{eff}}(x,x) := 0$.

(a) Prove the following equation

$$W_{\text{eff}}(x,y) = \max\{|f(x) - f(y)|^2 \mid Q(f) \leq 1\}.$$

(b) Show that

$$\rho : X \times X \to [0,\infty),\ (x,y) \mapsto W_{\text{eff}}(x,y)^{\frac{1}{2}}$$

defines a metric on the graph.

Proof. First of all, we have:

$$\begin{aligned} W_{\text{eff}}(x,y) &= \sup\left\{ \frac{1}{Q(h)} \;\middle|\; h \in C(X), h(x) - h(y) = 1 \right\} \\ &= \frac{1}{\inf\{Q(h) \mid h \in C(X),\ h(x) - h(y) = 1\}} \\ &= \frac{1}{\min\{Q(h) \mid h \in C(X),\ h(x) - h(y) = 1\}} \\ &\overset{(1)}{=} \frac{1}{\inf\limits_{f(x)\neq f(y)} \frac{Q(f)}{(f(x)-f(y))^2}} = \sup_{f(x)\neq f(y)} \frac{(f(x)-f(y))^2}{Q(f)} \\ &= \sup\{|f(x) - f(y)|^2 \mid f \in C(X),\ Q(f) \leq 1\} \\ &= \max\{|f(x) - f(y)|^2 \mid f \in C(X),\ Q(f) \leq 1\}. \end{aligned}$$

First of all, note that $Q(f) = 0$ if and only if $c\mathbf{1}$, because if $Q(f) = Q_b(f) = 0$, then

$$0 = \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))^2 = 0,$$

i.e. $f(x) = f(y)$ whenever $b(x,y) > 0$. So if $x_0 \in X$ is arbitrary but fixed and $y \in X$, then as X is connected there exists a path $(x_0, x_1, \ldots, x_n, y)$ such that $x_j \tilde{x}_{j+1}$ for $j = 0, \ldots, n-1$ as well as $x_n \tilde{y}$. Thus, it follows that $f(x_0) = f(x_1) = \cdots = f(y)$, in other words $f = f(x_0)\mathbf{1}$. We now discuss the equations (1) and (2).
(1) Let $h \in C(X)$ with $h(x) - h(y) = 1$, hence $h \notin \mathbb{C}\mathbf{1}$ in particular $h(x) \neq h(y)$ and we have

$$Q(h) = \frac{Q(h)}{(h(x) - h(y))^2} \geq \inf_{f(x)\neq f(y)} \frac{Q(f)}{(f(x)-f(y))^2} \quad \Rightarrow \quad \inf_{h(x)-h(y)=1} Q(h) \geq \inf_{f(x)\neq f(y)} \frac{Q(f)}{(f(x)-f(y))^2}.$$

Conversely, let $f \in C(X)$ with $f(x) \neq f(y)$ and put $h := \frac{f}{f(x)-f(y)} \in C(X)$. Then clearly $h(x) - h(y) = 1$ and

$$\inf_{h(x)-h(y)=1} Q(h) \leq Q(h) = Q\left(\frac{f}{f(x)-f(y)}\right) = \frac{Q(f)}{f(x)-f(y)}.$$

But this implies

$$\inf_{h(x)-h(y)=1} Q(h) \leq \inf_{f(x)\neq f(y)} \frac{Q(f)}{(f(x)-f(y))^2} \leq \inf_{h(x)-h(y)=1} Q(h), \text{ hence (1) follows.}$$

(2) Let $f \in C(X)$ with $Q(f) \leq 1$. If $Q(f) = 0$, then $f = c\mathbf{1}$ for some $c \in \mathbb{R}$ and therefore clearly

$$0 = |f(x) - f(y)| \leq \sup_{f(x)\neq f(y)} \frac{|f(x)-f(y)|^2}{Q(f)}.$$

Otherwise if $Q(f) > 0$, then we can also estimate

$$|f(x) - f(y)|^2 = \frac{|f(x)-f(y)|^2}{Q(f)} \underbrace{Q(f)}_{\leq 1} \leq \sup_{f(x)\leq f(y)} \frac{|f(x)-f(y)|^2}{Q(f)},$$

which implies

$$\sup_{Q(f)\le 1} |f(x)-f(y)|^2 \le \sup_{f(x)\le f(y)} \frac{|f(x)-f(y)|^2}{Q(f)}.$$

Conversely, if $f \in C(X)$ with $f(x) \neq f(y)$, then in particular $Q(f) > 0$ and we can define $h := \frac{f}{Q(f)^{\frac{1}{2}}}$. Then, $Q(h) = 1$ and we obtain

$$\frac{|f(x)-f(y)|^2}{Q(f)} = \left| \frac{f(x)}{Q(f)^{\frac{1}{2}}} - \frac{f(y)}{Q(f)^{\frac{1}{2}}} \right|^2 = |h(x)-h(y)|^2 \le \sup_{Q(f)\le 1} |f(x)-f(y)|,$$

and therefore, we finally have that

$$\sup_{f(x)\neq f(y)} \frac{|f(x)-f(y)|^2}{Q(f)} \le \sup_{Q(f)\le 1} |f(x)-f(y)|^2 \le \sup_{f(x)\neq f(y)} \frac{|f(x)-f(y)|^2}{Q(f)}, \text{ hence (2) follows.} \qquad \square$$

So why is the supremum even a maximum? In fact, this follows by the unique solution of the Poisson equation. Again, consider the set $\{Q(h) \mid h \in C(X),\ h(x)-h(y) = 1\}$ for $x, y \in X$ arbitrary but fixed. Consider $g := \mathbf{1} + g(y)\mathbf{1}_x$. Then $g(x) = 1 + h(y) = h(x)$. So if we consider $B = \{x\}$ as well as

$$\mathcal{A}_g := \{f \in C(X) \mid f = g \text{ on } B\}.$$

Then clearly $\{Q(h) \mid h \in C(X),\ h(x)-h(y) = 1\} = \{f \in C(X) \mid f \in \mathcal{A}_g\}$. Using Theorem 1.31. there exists a unique solution of the Poisson equation $L_b f = 0$ on $X \setminus B$ and $f = g$ on B. such that in particular

$$Q(f) = \min_{f\in\mathcal{A}_g} Q(f) = \min_{h(x)-h(y)=1} Q(h).$$

Thus, in the above equality, we have in fact a maximum instead of just a supremum.
(b) By definition we have that $\rho(x,x) = W_{\text{eff}}(x,x)^{\frac{1}{2}} = 0$ as well as $\rho(x,y) \ge 0$ for every $x, y \in X$. Also, we have that $\rho(x,y) < \infty$ for every $x, y \in X$. Indeed, if $x, y \in X$ (with $x \neq y$), we find a path $p = (x, x_1, \ldots, x_n, y)$ such that $x_0 := x\tilde{x}_1$, $x_j\tilde{x}_{j+1}$ for $j = 1, \ldots, n-1$, $x_n\tilde{y} =: x_{n+1}$. For $f \in C(X)$ with $Q(f) \le 1$ we have

$$\begin{aligned}|f(x)-f(y)| &\le \sum_{j=0}^n |f(x_j)-f(x_{j+1})| \le \left(\sum_{j=0}^n b(x_j,x_{j+1})(f(x_j)-f(x_{j+1}))^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^n b(x_j,x_{j+1})^{-1}\right)^{\frac{1}{2}} \\ &\le Q(f)^{\frac{1}{2}} C(\gamma) < \infty,\end{aligned}$$

with $C(\gamma) := (\sum_{j=0}^n b(x_j,x_{j+1})^{-1})^{\frac{1}{2}}$ (this does not depend on f). Therefore, we conclude that $W_{\text{eff}}(x,y)^{\frac{1}{2}} < \infty$ for every $x, y \in X$. Conversely, let $\rho(x,y) = 0$, in other words $W_{\text{eff}}(x,y) = 0$. Choose $f = \mathbf{1}_x$, then in particular $Q(\mathbf{1}_x) > 0$ and we immediately observe

$$0 \le \frac{|\mathbf{1}_x(x) - \mathbf{1}_y(x)|}{Q(\mathbf{1}_x)} \le W_{\text{eff}}(x,y) = 0.$$

Hence, $1 - \mathbf{1}_x(y) = 0$, equivalently $x = y$ follows. It is left to show the triangle inequality. Let $x, y, z \in X$, then:

$$\begin{aligned}\rho(x,y)^2 &= \max_{Q(f)\le 1} |f(x)-f(y)|^2 \le \max_{Q(f)\le 1} \big(|f(x)-f(z)| + |f(z)-f(y)|\big)^2 \\ &= \max_{Q(f)\le 1} \big(|f(x)-f(z)|^2 + 2|f(x)-f(z)||f(z)-f(y)| + |f(z)-f(y)|^2\big) \\ &\le \max_{Q(f)\le 1} |f(x)-f(z)|^2 + 2 \underbrace{\max_{Q(f)\le 1} \big(|f(x)-f(z)||f(z)-f(y)|\big)}_{\le \rho(x,z)\rho(z,y)} + \max_{Q(f)\le 1} |f(z)-f(y)|^2 \\ &\le \rho(x,z)^2 + 2\rho(x,z)\rho(z,y) + \rho(z,y)^2 = (\rho(x,z)+\rho(z,y))^2 \ \Rightarrow\ \rho(x,y) \le \rho(x,z) + \rho(z,y),\end{aligned}$$

and therefore (X, ρ) is indeed a metric space.

Bonus Exercise (Effective resistance II) 1. Given the assumptions of Exercise 4, show that

$$\rho : X \times X \to [0,\infty),\ (x,y) \mapsto W_{\text{eff}}(x,y)$$

defines a metric on the graph, as well.

Proof. We only have to show the triangle inequality. We specify the effective resistance now with respect to the given set X, i.e.

$$\begin{aligned} W_{\text{eff}}^X(x,y) &= \sup\left\{ \frac{1}{Q(h)} \;\Big|\; h \in C(X),\ h(x)-h(y)=1 \right\} \\ &= \frac{1}{\inf\{Q(h) \mid h \in C(X),\ h(x)-h(y)=1\}} \\ &= \frac{1}{\inf\{Q(h) \mid h \in C(X),\ h(x)=1,\ h(y)=0\}} \\ &= \frac{1}{\min\{Q(h) \mid h \in C(X),\ h(x)=1,\ h(y)=0\}}. \end{aligned}$$

Let $x,y,z \in X$ and write $X = \{x_1,\dots,x_n\}$. We take two copies $X' = \{x_1',\dots,x_n'\}$ and $X'' = \{x_1'',\dots,x_n''\}$ and we identify y, y' and y'', so that if $k \in \{1,\dots,n\}$ such that $x_k = y$, then $x_k = x_k' = x_k''$. Define

$$\begin{aligned} \widetilde{X} &:= \{x_1',\dots,x_k' = x_k'' = x_k,\dots,x_n',x_1'',\dots,x_{k-1}'',x_{k+1}'',\dots,x_n''\} \\ &= \{x_1',\dots,x_n',x_1'',\dots,x_{k-1}'',x_{k+1}'',\dots,x_n''\} \end{aligned}$$

In the following, we write x' and z'' to emphasize that we take an element of the first resp. second copy of X. In fact, one has

$$\begin{aligned} W_{\text{eff}}^{\widetilde{X}}(x',z'') &= \frac{1}{\min\{Q(h) \mid h \in C(\widetilde{X}),\ h(x')=1, h(z'')=0\}} \\ &= \Big[\min_{\mu\in\mathbb{R}} \Big(\min\{Q(h) \mid h \in C(X'),\ h(x')=1, h(y')=\mu\} \\ &\quad + \min\{Q(h) \mid h \in C(X''),\ h(y'')=\mu, h(z'')=0\}\Big)\Big]^{-1} \\ &= \left[\min_{\mu\in\mathbb{R}} \left(\frac{|1-\mu|^2}{W_{\text{eff}}^X(x,y)} + \frac{|\mu|^2}{W_{\text{eff}}^X(y,z)} \right)\right]^{-1} \end{aligned}$$

and this indeed minimized by

$$\lambda = \frac{W_{\text{eff}}^X(y,z)}{W_{\text{eff}}^X(x,y) + W_{\text{eff}}^X(y,z)}.$$

Thus, we obtain

$$W_{\text{eff}}^{\widetilde{X}}(x',z'') = \left[\left(\frac{|1-\lambda|^2}{W_{\text{eff}}^X(x,y)} + \frac{|\lambda|^2}{W_{\text{eff}}^X(y,z)} \right)\right]^{-1} = W_{\text{eff}}^X(x,y) + W_{\text{eff}}^X(y,z).$$

So we have to prove that $W_{\text{eff}}^{\widetilde{X}}(x',z'') \geq W_{\text{eff}}^X(x,z)$ to obtain the triangle inequality. To this end, let $f \in C(X)$ such that $f(x) = 1,\ f(z) = 0$ and

$$Q(f) = \min\{Q(h) \mid h \in C(X),\ h(x)=1,\ h(z)=0\}.$$

In particular, one has that $f(y) \in [0,1]$. Indeed, if for instance $f(y) > 1$, i.e.

$$f(w) = \max_{q\in X} f(q) > 1,$$

then by fixing $M := \max_{q\in X\setminus\{w\}} f(q)$ (the second largest value of f and defining

$$f'(q) := \begin{cases} M, & \text{if } q = w, \\ f(q), & \text{else.} \end{cases}$$

implies that $f'(x) = 1$ as well as $f'(z) = 0$ but also $Q(f') < Q(f)$ which is a contradiction (the case that $f(y) < 0$ cannot happen follows analogously). Now define $g \in C(\widetilde{X})$ through (noting that $y = x_k$)

$$g(x_i') := \begin{cases} f(x_i), & \text{if } f(x_i) > f(x_k), \\ f(x_k), & \text{if } f(x_i) \leq f(x_k). \end{cases} \quad \text{and} \quad g(x_j'') := \begin{cases} f(x_j), & \text{if } f(x_j) < f(x_k), \\ f(x_k), & \text{if } f(x_j) \geq f(x_k). \end{cases}$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\} \setminus \{k\}$. In particular, we have

$$\begin{aligned} g(x') &= f(x) = 1 \text{ as } f(x) \geq 1 \geq f(y) \text{ and} \\ g(z'') &= f(z) = 0 \text{ as } f(z) = 0 \leq f(y). \end{aligned}$$

Thus, g belongs to the set $\{Q(h) \mid h(x') = 1,\ h(z'') = 0\}$. If we can prove now that $Q(g) \leq Q(f) = \frac{1}{W_{\mathrm{eff}}^{X}(x,z)}$, then we obtain

$$W_{\mathrm{eff}}^{X}(x,z) \leq \frac{1}{Q(g)} \leq W_{\mathrm{eff}}^{\widetilde{X}}(x', z'').$$

So it is left to show that $Q(g) \leq Q(f)$. We denote

$$\begin{aligned} L &:= \{i \in \{1, \dots, n\} \mid f(x_i) < f(x_k)\}, \\ E &:= \{i \in \{1, \dots, n\} \mid f(x_i) = f(x_k)\}, \\ G &:= \{i \in \{1, \dots, n\} \mid f(x_i) > f(x_k)\} \end{aligned}$$

and obtain (by decomposing the sum over $\widetilde{X}$ into all possible combinations of the sets L, E, G)

$$\begin{aligned}
Q(g) &= \frac{1}{2} \sum_{x,y \in \widetilde{X}} b(x,y)(g(x) - g(y))^2 \\
&= \underbrace{\frac{1}{2} \sum_{i,j \in L} b(x_i', x_j')(g(x_i') - g(x_j'))^2}_{=0} + \underbrace{\frac{1}{2} \sum_{i,j \in E} b(x_i', x_j')(g(x_i') - g(x_j'))^2}_{=0} + \frac{1}{2} \sum_{i,j \in G} b(x_i', x_j')(g(x_i') - g(x_j'))^2 \\
&\quad + \underbrace{\frac{1}{2} \sum_{i \in L,\, j \in E} b(x_i', x_j')(g(x_i') - g(x_j'))^2}_{=0} + \underbrace{\frac{1}{2} \sum_{i \in E,\, j \in L} b(x_i', x_j')(g(x_i') - g(x_j'))^2}_{=0} + \frac{1}{2} \sum_{i \in G,\, j \in L} b(x_i', x_j')(g(x_i') - g(x_j'))^2 \\
&\quad + \frac{1}{2} \sum_{i \in L,\, j \in G} b(x_i', x_j')(g(x_i') - g(x_j'))^2 + \frac{1}{2} \sum_{i \in E,\, j \in G} b(x_i', x_j')(g(x_i') - g(x_j'))^2 + \frac{1}{2} \sum_{i \in G,\, j \in E} b(x_i', x_j')(g(x_i') - g(x_j'))^2 \\
&\quad + \frac{1}{2} \sum_{i,j \in L} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 + \underbrace{\frac{1}{2} \sum_{i,j \in E} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2}_{=0} + \underbrace{\frac{1}{2} \sum_{i,j \in G} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2}_{=0} \\
&\quad + \frac{1}{2} \sum_{i \in L,\, j \in E} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 + \frac{1}{2} \sum_{i \in E,\, j \in L} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 + \frac{1}{2} \sum_{i \in G,\, j \in L} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 \\
&\quad + \frac{1}{2} \sum_{i \in L,\, j \in G} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 + \underbrace{\frac{1}{2} \sum_{i \in E,\, j \in G} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2}_{=0} + \underbrace{\frac{1}{2} \sum_{i \in G,\, j \in E} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2}_{=0} \\
&= \frac{1}{2} \sum_{i,j \in G} b(x_i', x_j')(g(x_i') - g(x_j'))^2 + \sum_{i \in L, j \in G} b(x_i', x_j')(g(x_i') - g(x_j'))^2 + \sum_{i \in E, j \in G} b(x_i', x_j')(g(x_i') - g(x_j'))^2 \\
&\quad + \frac{1}{2} \sum_{i,j \in L} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 + \sum_{i \in L, j \in G} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 + \sum_{i \in E, j \in L} b(x_i'', x_j'')(g(x_i'') - g(x_j''))^2 \\
&\overset{\text{Def.}}{=} \frac{1}{2} \sum_{i,j \in G} b(x_i, x_j)(f(x_i) - g(x_j))^2 + \sum_{i \in L, j \in G} b(x_i, x_j)(f(x_k) - f(x_j))^2 + \sum_{i \in E, j \in G} b(x_i, x_j)(f(x_k) - f(x_j))^2 \\
&\quad + \frac{1}{2} \sum_{i,j \in L} b(x_i, x_j)(f(x_i) - f(x_j))^2 + \sum_{i \in L, j \in G} b(x_i, x_j)(f(x_i) - f(x_k))^2 + \sum_{i \in E, j \in L} b(x_i, x_j)(f(x_k) - f(x_j))^2 \\
&= \frac{1}{2} \sum_{i,j \in G} b(x_i, x_j)(f(x_i) - g(x_j))^2 + \sum_{i \in L, j \in G} b(x_i, x_j) \underbrace{\left[(f(x_k) - f(x_j))^2 + (f(x_i) - f(x_k))^2\right]}_{\leq (f(x_k) - f(x_j) + f(x_i) - f(x_k))^2 = (f(x_i) - f(x_j))^2} \\
&\quad + \frac{1}{2} \sum_{i,j \in L} b(x_i, x_j)(f(x_i) - f(x_j))^2 + \sum_{i \in E, j \in G} b(x_i, x_j)(f(x_k) - f(x_j))^2 + \sum_{i \in E, j \in L} b(x_i, x_j)(f(x_k) - f(x_j))^2 \\
&\leq \frac{1}{2} \sum_{i,j \in G} b(x_i, x_j)(f(x_i) - g(x_j))^2 + \sum_{i \in L, j \in G} b(x_i, x_j)(f(x_i) - f(x_j))^2 + \frac{1}{2} \sum_{i,j \in L} b(x_i, x_j)(f(x_i) - f(x_j))^2 \\
&\quad + \sum_{i \in E, j \in G} b(x_i, x_j)(f(x_k) - f(x_j))^2 + \sum_{i \in E, j \in L} b(x_i, x_j)(f(x_k) - f(x_j))^2 = Q(f).
\end{aligned}$$

Hence, we have shown the triangle inequality, in other words $X \times X \ni (x, y) \mapsto W_{\text{eff}}(x, y)$ indeed defines a metric on X. $\square$