Solution for Lecture 1 of ISEM 26

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Exercise 1 (Normal contractions)

(a) Show that the following maps from $\mathbb{R}$ to $\mathbb{R}$ are normal contractions:

- $C_+ : t \mapsto t \vee 0$,
- $C_- : t \mapsto (-t) \vee 0$,
- $C_{(-\infty,1]} : t \mapsto t \wedge 1$ and
- $C_{[0,1]} : t \mapsto 0 \vee (t \wedge 1)$.

For which $a \leq b$ is $C_{[a,b]} : t \mapsto a \vee (t \wedge b)$ a normal contraction?

Solution: We first show that the map $C_{[a,b]}$ is a normal contraction if and only if $a \leq 0 \leq b$. Indeed, if $b < 0$ then $C_{[a,b]}(0) = a \vee (b \wedge 0) = a \vee b = 0$ if and only if $a = 0$, which is a contradiction to $a \leq b$. Now let $b \geq 0$. Analogously, it follows that $C_{[a,b]}(0) = a \vee (b \wedge 0) = a \vee 0 = 0$ if only if $a \leq 0$. Thus, $C_{[a,b]}(0) = 0$ if and only if $a \leq 0 \leq b$. It is now convenient to define the map $C_{(-\infty,b]} : \mathbb{R} \to \mathbb{R}$, $t \mapsto t \wedge b$ for $b \geq 0$. Obviously, $C_{(-\infty,b]}(0) = 0$ holds. Now let $t, s \in \mathbb{R}$. Then we compute

$$|C_{(-\infty,b]}(t) - C_{(-\infty,b]}(s)| = |b \wedge t - b \wedge s| = \begin{cases} |t-s| & (t, s \leq b), \\ |b-s| \leq |t-s| & (s \leq b \leq t), \\ |t-b| \leq |t-s| & (t \leq b \leq s), \\ 0 \leq |t-s| & (b \leq t, s). \end{cases}$$

Therefore, $C_{(-\infty,b]}$ is a normal contraction for all $b \geq 0$. For $a \leq 0$ and $t \in \mathbb{R}$ we can write $a \vee t = -((-a) \wedge (-t)) = -C_{(-\infty,-a]}(-t)$. We can thus rewrite $C_{[a,b]}$ as a composition of the form $C_{[a,b]} = -C_{(-\infty,-a]} \circ (-C_{(-\infty,b]})$. Since for any normal contraction C the map defined by $\widetilde{C}(t) \coloneqq sC(t/s)$ for any $s \neq 0$ is also a normal contraction and the composition of normal contractions is a normal contraction, we find that $C_{[a,b]}$ is a normal contraction and also all afore mentioned maps.

(b) Let X be a finite set and let Q be a symmetric bilinear form on $C(X)$. Show that Q is compatible with normal contractions if it is compatible with the map $C_{(-\infty,1]}$.

Solution: Using Theorem 1.11 it is sufficient to show that Q is compatible with $C_{[0,1]}$. Using our results from (a), we get for any $s < 0$ that $C_{[s,1]} = sC_{(-\infty,1]} \circ (\frac{1}{s} C_{(-\infty,1]})$. It then follows that

$$Q(C_{[s,1]} \circ f) = Q\left(sC_{(-\infty,1]} \circ \left(\frac{1}{s} C_{(-\infty,1]}\right) \circ f\right) = s^2 Q\left(C_{(-\infty,1]} \circ \left(\frac{1}{s} C_{(-\infty,1]}\right) \circ f\right)$$
$$\leq s^2 Q\left(\frac{1}{s} C_{(-\infty,1]} \circ f\right) = Q(C_{(-\infty,1]} \circ f) \leq Q(f) \quad (f \in C(X)).$$

Since X is a finite set, Q is continuous. Using the pointwise convergence of $C_{[s,1]} \to C_{[0,1]}$ for $s \to 0$, we find $Q(C_{[0,1]} \circ f) \leq Q(f)$.

Exercise 2 (First Beurling-Deny Criterion)

Let X be a finite set and let Q be a symmetric bilinear form over X. For any $f \in C(X)$, let $f_+ = f \vee 0$ be the *positive part* and let $f_- = (-f) \vee 0$ be the *negative part* of f. Show the following equivalence:

(i) $Q(|f|) \leq Q(f)$ for all $f \in C(X)$,

(ii) $Q(f_+, f_-) \leq 0$ for all $f \in C(X)$,

(iii) $Q(f \vee g) + Q(f \wedge g) \leq Q(f) + Q(g)$ for all $f, g \in C(X)$,

and for Q positive show that this is also equivalent to:

(iv) $Q(f_+) \leq Q(f)$ for all $f \in C(X)$.

Solution: Let $f, g \in C(X)$.
$(i) \Leftrightarrow (ii)$: We have $f_+ = \frac{1}{2}(|f| + f)$ and $f_- = \frac{1}{2}(|f| - f)$. Using this identity and a binomial formula yields the following:

$$Q(f_+, f_-) = \frac{1}{4}Q(|f| + f, |f| - f) = \frac{1}{4}\Big(Q(|f|) - Q(f)\Big).$$

This establishes the equivalence between (i) and (ii).
$(ii) \Rightarrow (iii)$: We use the following identities:

$$\begin{aligned} f \vee g &= (g - f)_+ + f = (g - f)_- + g. \\ f \wedge g &= g - (g - f)_+ = f - (g - f)_-. \end{aligned}$$

Together with (ii) and the binomial formula these identities allows us to estimate

$$\begin{aligned} Q(f \vee g) &= Q((g - f)_+ + f, (g - f)_- + g) \\ &= Q(f, g) + Q((g - f)_+, (g - f)_-) + Q((g - f)_+, g) + Q((g - f)_-, f) \\ &\underset{(ii)}{\leq} Q(f, g) + Q((g - f)_+, g) + Q((g - f)_-, f). \end{aligned}$$

And similarly

$$\begin{aligned} Q(f \wedge g) &= Q(g - (g - f)_+, f - (g - f)_-) \\ &\leq Q(f, g) - Q((g - f)_+, f) - Q((g - f)_-, g). \end{aligned}$$

Adding the left- and right-hand side of the inequalities gives (iii):

$$\begin{aligned} Q(f \vee g) + Q(f \wedge g) &\leq 2Q(f, g) + Q((g - f)_+ - (g - f)_-, g) + Q((g - f)_- - (g - f)_+, f) \\ &= 2Q(f, g) + Q(g - f, g) - Q(g - f, f) \\ &= Q(f) + Q(g). \end{aligned}$$

$(iii) \Rightarrow (ii)$: For $|f|$ we have the two identities

$$|f| = f_+ + f_- = f_+ \vee f_-.$$

It follows that

$$\begin{aligned} 0 = Q(f_+ \vee f_-) - Q(f_+ + f_-) &= Q(f_+ \vee f_-) + Q(\underbrace{f_+ \wedge f_-}_{=0}) - Q(f_+ + f_-) \\ &\underset{(iii)}{\leq} Q(f_+) + Q(f_-) - Q(f_+ + f_-) \\ &= -2Q(f_+, f_-). \end{aligned}$$

Therefore $Q(f_+, f_-) \leq 0$ and (ii) holds.
Let now Q be a positive form, i.e. $Q(h) \geq 0$ for all $h \in C(X)$. Then we show that (iv) is equivalent to (ii).
$(ii) \Rightarrow (iv)$: We have

$$Q(f) = Q(f_+) - 2Q(f_+, f_-) + Q(f_-) \geq Q(f_+),$$

since $Q(f_+, f_-) \leq 0$ by (ii) and $Q(f_-) \geq 0$ by positivity of the form.
$(iv) \Rightarrow (ii)$: Let $s > 0$. We employ the strategy of Lemma 1.10 to show that $2Q(f_+, f_-) \leq sQ(f_-)$ from which (ii) follows by letting s tend to 0. Define $f_s := f_+ - sf_-$. Then $(f_s)_+ = f_+$. From (iv) it follows that

$$Q(f_+) = Q((f_s)_+) \leq Q(f_s) = Q(f_+) - 2sQ(f_+, f_-) + s^2 Q(f_-)$$

and thus, $2Q(f_+, f_-) \leq sQ(f_-)$.

Exercise 3 (Harmonic functions and connected components)

Let (b, c) be a graph over a finite measure space (X, m) with associated Laplacian $L = L_{b,c,m}$ and let

$$H = \{f \in C(X) \mid Lf = 0\}$$

be the subspace of *harmonic funtions*. Show that $\dim H$ is equal to the number of connected components of (b, c) on which c vanishes.
Solution: Let $Z_1, ..., Z_k$ be the connected components of (b, c) on which c vanishes and $Z_{k+1}, ..., Z_l$ the connected components of (b, c) on which c does not vanish. We show that

$$H = \{f \in C(X) \mid f|_{Z_i} \text{ is constant } (i \in \{1, \dots, k\}) \text{ and } f|_{Z_j} = 0 \ (j \in \{k+1, \dots, l\})\} =: G,$$

from which the assertion follows.

"$H \supseteq G$": Set $f_i := \mathbb{1}_{Z_i}$ for every $i \in \{1, \dots, k\}$. Then $\{f_i \mid i \in \{1, \dots, k\}\}$ is a basis for G. Now for every $x \in X$ we have

$$Lf_i(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y)(f_i(x) - f_i(y)) + c(x) f_i(x) \right).$$

The term $c(x)f_i(x)$ equals zero as $x \in Z_i$ implies $c(x) = 0$. The term $b(x, y)(f_i(x) - f_i(y))$ also equals zero as $f_i(x) \neq f_i(y)$ implies that x and y are in different connected components of (b, c), i.e. $b(x, y) = 0$. Thus $Lf_i(x) = 0$ for every $i \in \{1, \dots, k\}$ and every $x \in X$. This means that $f_i \in H$ for every $i \in \{1, \dots, k\}$ and hence $G \subseteq H$.

"$H \subseteq G$": Let $f \in H$ and $i \in \{1, \dots, l\}$. Let $x_i \in Z_i$ be such that $|f(x_i)| \geq |f(y)|$ for each $y \in Z_i$. Then

$$0 = Lf(x_i) = \sum_{y \in X} b(x_i, y)(f(x_i) - f(y)) + c(x_i) f(x_i).$$

By definition of x_i every summand of the above expression has the same sign. Therefore every summand equals zero. Now for every $y \in X$ with $b(x_i, y) \neq 0$ we have $f(x_i) = f(y)$. Applying the same argument to y instead of x_i we get $f|_{Z_i} = f(x_i)$ by induction. Hence f is constant on every connected component of (b, c). If c does not vanish on a connected component of (b, c), then f has to vanish on that component due to the last summand of the above equation being zero. Therefore we have $f \in G$ and hence $H \subseteq G$.

Exercise 4 (Poisson equation for $\alpha = 0$)

Let b be a connected graph over a finite measure space (X, m) (that is, $c = 0$) and let $L = L_{b,0,m}$ be the associated Laplacian. Furthermore, let

$$V := \{f \in C(X) \mid \sum_{x \in X} f(x)m(x) = 0\}.$$

Show that for each $f \in V$, there is a unique function $u \in V$ such that

$$Lu = f.$$

Solution: We will make use of the following (well-known) fact.

Theorem. *Let H_0, H_1 be Hilbert spaces and $L\colon H_0 \to H_1$ a bounded linear operator, such that $\operatorname{ran}(L)$ is closed. Then for each $f \in \ker(L^*)^\perp$ there exists a unique $u \in \ker(L)^\perp$ such that*

$$Lu = f.$$

Proof. For $f \in H_1$ we have that

$$\begin{aligned} f \in \operatorname{ran}(L)^\perp &\Leftrightarrow \forall v \in H_0 : \langle Lv, f\rangle_{H_1} = 0 \\ &\Leftrightarrow \forall v \in H_0 : \langle v, L^* f\rangle_{H_0} = 0 \\ &\Leftrightarrow L^* f = 0 \\ &\Leftrightarrow f \in \ker(L^*). \end{aligned}$$

Hence, $\operatorname{ran}(L)^\perp = \ker(L^*)$ and by taking orthogonal complements on both sides and using that $\operatorname{ran}(L)$ is closed, we infer $\operatorname{ran}(L) = \ker(L^*)^\perp$. This shows that for each $f \in \ker(L^*)^\perp$ we find an element $v \in H_0$ such that

$$Lv = f.$$

Using that $H_0 = \ker(L) \oplus \ker(L)^\perp$, we can decompose $v = v_0 + u$ with $v_0 \in \ker(L)$ and $u \in \ker(L)^\perp$ and obtain

$$Lu = L(v_0 + u) = f,$$

which shows the existence of a solution $u \in \ker(L)^\perp$. The uniqueness follows, since for $u, \widetilde{u} \in \ker(L)^\perp$ with $Lu = L\widetilde{u}$, we derive $u - \widetilde{u} \in \ker(L) \cap \ker(L)^\perp = \{0\}$ and thus, $u = \widetilde{u}$. $\square$

We apply the above theorem to $L\colon \ell_2(X, m) \to \ell_2(X, m)$ and recall that $L = L^*$ and $\operatorname{ran}(L)$ is closed as it is finite dimensional. Then, we infer that for each $f \in \ker(L)^\perp$ there exists a unique $u \in \ker(L)^\perp$ with

$$Lu = f.$$

Noting that by Exercise 1.3 we have

$$\ker(L) = \operatorname{span}\{1\}$$

as the graph is conncected and since $V = \{1\}^\perp$, the assertion follows.