26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 13

CHAPTER 11

Spherically Symmetric Graphs

In this chapter we apply the theory developed in the preceding chapters to a certain class of graphs defined by a symmetry property. This class contains both trees (satisfying a natural symmetry condition) and anti-trees. Hence, it covers a wealth of (counter)examples. We can characterize this class by an operator theoretic condition. Specifically, the Laplacian of graphs in these class commutes with a certain averaging operator. For graphs in this class we can give an explicit lower bound on the infimum of the spectrum and characterize both recurrence and stochastic completeness by finiteness of certain sums.

Let X be a countable set and m a measure on X with full support.

11.1. Weakly Spherically Symmetric Graphs

Let (b, c) be a connected graph over (X, m). Denote the combinatorial distance on X by d. Let $O \subseteq X$ be nonempty and define the combinatorial graph distance to O by

$$d(O, x) := \min_{o \in O} d(o, x)$$

for $x \in X$. For most of our results below we will assume that O is a finite set.

We denote the *distance sphere of radius* $r \in \mathbb{N}_0$ *about* O by

$$S_r(O) := \{x \in X \mid d(O, x) = r\}.$$

For convenience, we let $S_{-1}(O) := \varnothing$. Moreover, we denote the *distance ball of radius* $r \in \mathbb{N}_0$ *about* O by

$$B_r(O) := \bigcup_{n=0}^{r} S_n(O) = \{x \in X \mid d(O, x) \leq r\}.$$

In case $O = \{o\}$ for some $o \in X$ then we just write $S_r(o) := S_r(O)$ and $B_r(o) := B_r(O)$. If (b, c) is locally finite and O is finite, then the sets $S_r(O)$ and $B_r(O)$ are finite for all $r \in \mathbb{N}_0$. In any case, connectedness of the graph means that

$$X = \bigcup_{r \in \mathbb{N}_0} B_r(O)$$

holds. So, we can exhaust the graph by balls.

We call a function $f \in C(X)$ *spherically symmetric (with respect to O)* if there exists a function $g \colon \mathbb{N}_0 \longrightarrow \mathbb{R}$ such that $f(x) = g(r)$ for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. With a slight abuse of notation, we then write

$$f(r) := f(x)$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Although all of our notions involving symmetry depend on O, we will mostly omit this dependence in our notation and statements. Clearly, any spherically symmetric functions is bounded on each distance ball. Hence, it can easily be seen to belong to the domain $\mathcal{F}$ of the formal Laplacian.

We next define the functions for which we will assume spherical symmetry. Let $O \subseteq X$ nonempty be given. By connectedness of the graph, we can define $k_\pm \colon X \longrightarrow [0, \infty)$ via

$$k_\pm(x) := \frac{1}{m(x)} \sum_{y \in S_{r\pm 1}(O)} b(x, y)$$

for $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Here, the sum over the empty set arising in the definition of $k_-(x)$ for $x \in S_0(O) = O$ is defined to be zero.

We call $k_\pm$ the *outer and inner degrees (with respect to O)*, which are functions. Furthermore, we define the *potential* $q \colon X \longrightarrow [0, \infty)$ by

$$q(x) := \frac{c(x)}{m(x)}$$

for $x \in X$.

With these preparations we can now define the class of graphs which will be studied in this chapter.

Definition 11.1 (Weakly spherically symmetric graphs). We call a connected graph (b, c) over (X, m) *weakly spherically symmetric* with respect to a nonempty set $O \subseteq X$ if the outer and inner degrees $k_\pm$ and the potential q are spherically symmetric with respect to O.

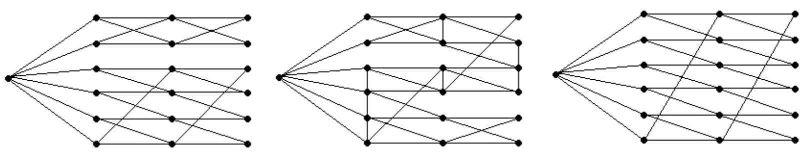

Figure 1. Examples of weakly spherically symmetric graphs.

We call a sequence of vertices $(x_0, \ldots, x_n)$ in X with $n \in \mathbb{N}$ a *cycle* if $x_j \sim x_{j+1}$ for all $j = 0, 1, \ldots, n-1$, $x_j \neq x_k$ for $0 \leq j, k \leq n$ with $j \neq k$, and $x_0 = x_n$. Put differently, a cycle is a path whose start and end vertex agrees but all other vertices are pairwise different. A connected graph with no cycles is called a *tree*.

Example 11.2 (Spherically symmetric trees). Let (b, c) be a connected graph over (X, m) with standard weights and counting measure, i.e., $b \colon X \times X \longrightarrow \{0, 1\}$, $c = 0$ and $m = 1$. Let $O = \{o\}$ for $o \in X$. We say that b is a *spherically symmetric tree with branching numbers k* if there exists a $k \colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ such that

$$k_+(x) = k(r)$$

for every vertex $x \in S_r(o)$ and $r \in \mathbb{N}_0$ and

$$k_-(x) = 1$$

for all $x \in S_r(o)$ for $r \in \mathbb{N}$ and $b|_{S_r(o)\times S_r(o)} = 0$ for all $r \in \mathbb{N}_0$.

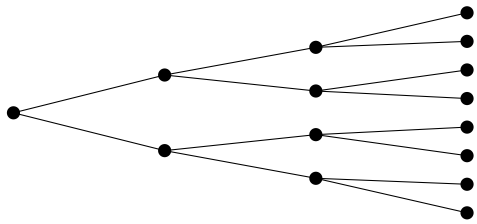

FIGURE 2. A tree with $k(r) = 2$.

We note that these graphs are indeed trees. Furthermore, we note that removing a single edge between spheres will disconnect any tree. This contrasts with anti-trees, which we now define.

EXAMPLE 11.3 (Anti-trees). Let (b, c) be a connected graph over (X, m) with standard weights and counting measure, i.e., $b\colon X \times X \longrightarrow \{0, 1\}$, $c = 0$ and $m = 1$. Let $O = \{o\}$ for $o \in X$. Let $s\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ be given by $s(r) := \#S_r(o)$ for all $r \in \mathbb{N}_0$. We then say that b is an *anti-tree with sphere size* s if

$$k_\pm(x) = s(r) \qquad \text{for all } x \in S_{r\mp 1}(o) \text{ and } r \in \mathbb{N}_0.$$

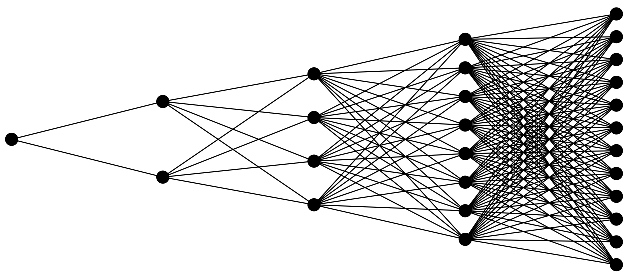

FIGURE 3. An anti-tree with $s(r) = 2^r$.

We note that the definition of an anti-tree implies

$$b|_{S_r(o)\times S_{r\pm 1}(o)} = 1$$

for all $r \in \mathbb{N}_0$. In other words, every vertex in $S_r(o)$ is connected to all vertices in $S_{r+1}(o)$ for all $r \in \mathbb{N}_0$ (and every vertex in $S_{r+1}(o)$ is connected to every vertex in $S_r(o)$ for all $r \in \mathbb{N}_0$). Hence, to disconnect such a graph, we must remove all vertices between spheres. Furthermore, we note that we impose no restrictions on $b|_{S_r(o)\times S_r(o)}$.

For a given sequence $s\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ of natural numbers with $s(0) = 1$, we can easily construct an anti-tree with sphere size s. Indeed, we can just partition the (infinite) set X into disjoint subsets U_r with $\#U_r = s(r)$ and let $b|_{U_r\times U_{r\pm 1}} = 1$ for $r \in \mathbb{N}_0$ with $b = 0$ otherwise, $m = 1$ and $c = 0$.

We will regularly revisit spherically symmetric trees and anti-trees to illustrate our results in this chapter.

The following formulas will play a crucial role in the proofs of several results below. Hence we gather them together into one statement.

Lemma 11.4. *Let (b,c) be a weakly spherically symmetric graph over (X,m) with respect to a nonempty $O \subseteq X$. Then,*

$$k_+(r)m(S_r(O)) = k_-(r+1)m(S_{r+1}(O))$$

for all $r \in \mathbb{N}_0$, where the value ∞ is allowed on both sides. In particular, $m(S_r(O)) < \infty$ holds for all $r \in \mathbb{N}_0$ if and only if $m(O) < \infty$.

If f is a spherically symmetric function, then $f \in \mathcal{F}$ and $\mathcal{L}f$ is spherically symmetric with

$$\mathcal{L}f(x) = k_+(r)\big(f(r) - f(r+1)\big) + k_-(r)\big(f(r) - f(r-1)\big) + q(r)f(r)$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$.

Proof. The first formula follows by a simple computation using $k_+(r) = k_+(x)$ for all $x \in S_r(O)$, Fubini's theorem and the symmetry of b. Specifically, we have

$$\begin{aligned} k_+(r)m(S_r(O)) &= \sum_{x\in S_r(O)} k_+(x)m(x) = \sum_{x\in S_r(O)} \sum_{y\in S_{r+1}(O)} b(x,y) \\ &= \sum_{y\in S_{r+1}(O)} \sum_{x\in S_r(O)} b(y,x) = \sum_{y\in S_{r+1}(O)} k_-(y)m(y) \\ &= k_-(r+1)m(S_{r+1}(O)). \end{aligned}$$

The "in particular" statement now follows by the formula and induction.

A spherically symmetric function f is clearly a bounded function on the neighbors of any vertex and, therefore, $f \in \mathcal{F}$. The second formula follows immediately from the definition of $\mathcal{L}$ and the assumption that f is spherically symmetric. □

11.2. Symmetry of the Kernel and Green Function

In this section we establish the symmetry of the heat kernel and the Green function on a weakly spherically symmetric graph. Here and subsequently we will often make the additional assumption that the graph is locally finite. This will simplify our dealing with the averaging operator below.

Let (b,c) be a graph over (X,m). We first recall the definition of the heat kernel. The semigroup $(e^{-tL})_{t\geq 0}$ of the Laplacian $L := L^{(D)}$ on $\ell^2(X,m)$ for $t \geq 0$ gives rise to the heat kernel $p\colon [0,\infty) \times X \times X \longrightarrow \mathbb{R}$ with

$$e^{-tL}f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $t \geq 0$.

By the fact that the semigroup is positivity preserving, established in Corollary 5.6, we have $p_t(x,y) \geq 0$ for all $x,y \in X$ and $t \geq 0$ as $p_t(x,y) = e^{-tL}1_y(x)/m(y)$.

For a nonempty finite set $O \subseteq X$, we now define

$$p_t(x,O) := \frac{1}{m(x)m(O)}\langle 1_x, e^{-tL}1_O\rangle = \frac{1}{m(O)}e^{-tL}1_O(x) = \frac{1}{m(O)}\sum_{o\in O} p_t(x,o)m(o)$$

for $x \in X$ and $t \geq 0$. Thus, whenever $O = \{o\}$ for $o \in X$, we recover the heat kernel

$$p_t(x,o) = p_t(x,\{o\})$$

for $x \in X$ and $t \geq 0$.

The first theorem of this section states that the functions $p_t(\cdot, O)$ for $t \geq 0$ are spherically symmetric whenever the graph is weakly spherically symmetric with respect to the subset O.

THEOREM 11.5 (Spherical symmetry of the heat kernel). *Let (b,c) be a locally finite graph over (X,m). If (b,c) is weakly spherically with respect to a nonempty finite set $O \subseteq X$, then $p_t(\cdot,O)$ is a spherically symmetric function for all $t \geq 0$.*

To start the proof we introduce the *averaging operator*

$$\mathcal{A}\colon C(X) \longrightarrow C(X)$$

on a locally finite graph with respect to a nonempty finite set $O \subseteq X$ by

$$\mathcal{A}f(x) := \frac{1}{m(S_r(O))}\sum_{y\in S_r(O)} f(y)m(y)$$

for $f \in C(X)$ $x \in S_r(O)$ and $r \in \mathbb{N}_0$. As the graph is locally finite, the operator $\mathcal{A}$ is indeed defined on the whole of $C(X)$. Moreover, we note that $\mathcal{A}f$ is spherically symmetric for any $f \in C(X)$ and a function $f \in C(X)$ is spherically symmetric if and only if $\mathcal{A}f = f$. In particular, $\mathcal{A}$ maps $C(X)$ onto the spherically symmetric functions. We will denote the restriction of $\mathcal{A}$ to $\ell^2(X,m)$ by A, i.e.,

$$A = \mathcal{A}|_{\ell^2(X,m)}.$$

We now collect some basic facts about A.

LEMMA 11.6 (Basic facts about A). *Let (b,c) be a locally finite connected graph over (X,m) and let $O \subseteq X$ be a nonempty finite set. Let $\mathcal{A}$ be the averaging operator with respect to O and let A be the restriction of $\mathcal{A}$ to $\ell^2(X,m)$. Then, A is a bounded self-adjoint operator on $\ell^2(X,m)$. More specifically, A is an orthogonal projection of $\ell^2(X,m)$ onto the subspace of spherically symmetric functions in $\ell^2(X,m)$.*

PROOF. Let $f \in \ell^2(X,m)$. We note that $X = \bigcup_{r\in\mathbb{N}_0} S_r(O)$ from the assumption that (b,c) is connected. To show that A maps $\ell^2(X,m)$ to

$\ell^2(X,m)$ and that A is bounded, we use the Cauchy–Schwarz inequality to obtain for $f \in \ell^2(X,m)$

$$\begin{aligned}\sum_{x\in X}(Af)(x)^2 m(x) &= \sum_{r=0}^{\infty}\sum_{x\in S_r(O)}\left(\frac{1}{m(S_r(O))}\sum_{y\in S_r(O)} f(y)m(y)\right)^2 m(x)\\ &= \sum_{r=0}^{\infty}\frac{1}{m(S_r(O))}\left(\sum_{y\in S_r(O)} f(y)m(y)\right)^2\\ &\leq \sum_{r=0}^{\infty}\frac{1}{m(S_r(O))}\left(\sum_{y\in S_r(O)} m(y)\right)\left(\sum_{y\in S_r(O)} f(y)^2 m(y)\right)\\ &= \|f\|^2.\end{aligned}$$

Hence, A is a bounded operator of norm 1 since $Af = f$ for any spherically symmetric function $f \in \ell^2(X,m)$.

Moreover, A is symmetric, and thus self-adjoint, by a direct calculation. As the range of A is included in the spherically symmetric functions and A is the identity on the spherically symmetric functions, we have $A^2 = A$ and the range of A is just the set of spherically symmetric functions in $\ell^2(X,m)$. Hence, A is a bounded self-adjoint operator with $A^2 = A$. Hence, it is the projection on its range. □

REMARK 11.7. The operator A can easily be seen to be a Markovian operator, i.e. to satisfy $0 \leq Af \leq 1$ for all $0 \leq f \leq 1$. Hence, the considerations on Markovian operators apply and would also give boundedness of A.

The next lemma shows that weak spherical symmetry is equivalent to $\mathcal{A}$ and A commuting with the Laplacians $\mathcal{L}$ and $L := L^{(D)}$ on suitable spaces. Since $O \subseteq X$ is assumed to be finite and (b,c) locally finite below, $S_r(O)$ is finite for all $r \in \mathbb{N}_0$ and A and L both map $C_c(X)$ to $C_c(X)$, i.e., $C_c(X)$ is invariant under both A and L.

LEMMA 11.8 (Characterization of weak spherical symmetry). *Let (b,c) be a connected locally finite graph over (X,m) and let $O \subseteq X$ be nonempty and finite. Then, the following statements are equivalent:*

(i) *The graph (b,c) is weakly spherically symmetric.*
(ii) *The operator $\mathcal{A}$ commutes with $\mathcal{L}$ on $C(X)$, i.e.,*
$$\mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{A} \qquad \text{on } C(X).$$
(iii) *The operator A commutes with L on $C_c(X)$, i.e.,*
$$AL = LA \qquad \text{on } C_c(X).$$

PROOF. We denote $S_r := S_r(O)$ for $r \in \mathbb{N}_0$.

(i) $\Longrightarrow$ (ii): Obviously, multiplication by the spherically symmetric function q commutes with $\mathcal{A}$. Hence, we may assume that $q = 0$.

Since $\mathcal{A}f$ is spherically symmetric for $f \in C(X)$, by Lemma 11.4 we get, for $r \in \mathbb{N}_0$ and $x \in S_r$,

$$\mathcal{L}\mathcal{A}f(x) = k_+(r)(\mathcal{A}f(r) - \mathcal{A}f(r+1)) + k_-(r)(\mathcal{A}f(r) - \mathcal{A}f(r-1)).$$

On the other hand, a direct computation combined with the recursion of Lemma 11.4 and the formula for $\mathcal{L}$ given in Lemma 11.4 for symmetric functions gives for $r \in \mathbb{N}_0$ and $x \in S_r$,

$$
\begin{aligned}
\mathcal{A}\mathcal{L}f(x) &= \frac{1}{m(S_r)} \sum_{y \in S_r} \mathcal{L}f(y)m(y) \\
&= \frac{1}{m(S_r)} \sum_{y \in S_r} \sum_{z \in S_{r-1} \cup S_{r+1}} b(y,z)(f(y) - f(z)) \\
&= \frac{1}{m(S_r)} \sum_{y \in S_r} f(y) \sum_{z \in S_{r-1} \cup S_{r+1}} b(y,z) - \frac{1}{m(S_r)} \sum_{z \in S_{r-1} \cup S_{r+1}} f(z) \sum_{y \in S_r} b(y,z) \\
&= (k_+(r) + k_-(r))\mathcal{A}f(r) \\
&\quad - \frac{k_+(r-1)}{m(S_r)} \sum_{z \in S_{r-1}} f(z)m(z) - \frac{k_-(r+1)}{m(S_r)} \sum_{z \in S_{r+1}} f(z)m(z) \\
&= (k_+(r) + k_-(r))\mathcal{A}f(r) \\
&\quad - \frac{k_-(r)}{m(S_{r-1})} \sum_{z \in S_{r-1}} f(z)m(z) - \frac{k_+(r)}{m(S_{r+1})} \sum_{z \in S_{r+1}} f(z)m(z) \\
&= k_+(r)(\mathcal{A}f(r) - \mathcal{A}f(r+1)) + k_-(r)(\mathcal{A}f(r) - \mathcal{A}f(r-1)) \\
&= \mathcal{L}\mathcal{A}f(x).
\end{aligned}
$$

Thus we see $\mathcal{L}\mathcal{A}f = \mathcal{A}\mathcal{L}f$.

(ii) $\Longrightarrow$ (iii): This is clear as A and L are restrictions of $\mathcal{A}$ and $\mathcal{L}$.

(iii) $\Longrightarrow$ (i): Let $r \in \mathbb{N}_0$. Obviously, $A1_{S_r} = 1_{S_r}$ as 1_{S_r} is a spherically symmetric function. Furthermore, for $x \in S_{r\pm1}$, we have

$$L1_{S_r}(x) = -k_{\mp}(x)$$

by direct calculations. Thus, for $x \in S_{r\pm1}$, we have

$$
\begin{aligned}
-k_{\mp}(x) = LA1_{S_r}(x) = AL1_{S_r}(x) &= -\frac{1}{m(S_{r\pm1})} \sum_{y \in S_{r\pm1}} k_{\mp}(y)m(y) \\
&= -Ak_{\mp}(r \pm 1).
\end{aligned}
$$

Therefore, $k_\pm$ are spherically symmetric.

Similarly, for $x \in S_r$, we calculate $L1_{S_r}(x) = k_+(x) + k_-(x) + q(x)$. Therefore, for $x \in S_r$, we have

$$
\begin{aligned}
k_+(x) + k_-(x) + q(x) = L1_{S_r}(x) = LA1_{S_r}(x) &= AL1_{S_r}(x) \\
&= \frac{1}{m(S_r)} \sum_{y \in S_r} (k_+(y) + k_-(y) + q(y))m(y) \\
&= A(k_+ + k_- + q)(r).
\end{aligned}
$$

As we have already shown that $k_\pm$ are spherically symmetric, this shows that the potential q and, thus, the graph is weakly spherically symmetric. □

LEMMA 11.9. *Let (b,c) be a connected locally finite graph over (X,m) and let $O \subseteq X$ be nonempty and finite. Then, the following statements are equivalent:*

(i) *$AL = LA$ on $C_c(X)$.*

(ii) *A maps $D(L)$ into $D(L)$ and $AL = LA$ on $D(L)$.*
(iii) *$Ae^{-tL} = e^{-tL}A$ on $\ell^2(X,m)$ for all $t \geq 0$.*

Proof. The equivalence between (ii) and (iii) is a general result on self-adjoint operators commuting with orthogonal projections. We only sketch the proof. Indeed, (ii) just says that L operates — so to speak —- independently on the range of A and on its orthogonal complement. This then easily gives that the same applies to functions of L and this gives (iii). Conversely, from (iii) and the fact that L arises from the semigroup by differentiation we can infer that (ii) holds.

Furthermore, (ii) implies (i) as $C_c(X) \subseteq D(L)$ for locally finite graphs. Finally, from (i) and local finiteness of the graph and finiteness of O we infer that both $\mathcal{L}$ and $\mathcal{A}$ are defined on the whole of $C(X)$. The symmetry properties of A and L combined with (i) then give $\mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{A}$. From this we infer (ii) by restriction. □

Proof of Theorem 11.5. Let $t \geq 0$. From Lemmas 11.8 and 11.9 we see that if (b,c) is a locally finite and weakly spherically symmetric graph with respect to a nonempty finite set O, then A and e^{-tL} commute. Hence, as $p_t(x,O) = e^{-tL}1_O(x)/m(O)$ for $x \in X$, we get

$$Ap_t(x,O) = \frac{1}{m(O)}Ae^{-tL}1_O(x) = \frac{1}{m(O)}e^{-tL}A1_O(x) = p_t(x,O)$$

for $x \in X$. Hence, $p_t(\cdot,O)$ is spherically symmetric. □

We now extend the definition of the Green function from Section 9.1 by letting

$$G(x,O) := \frac{1}{m(O)}\sum_{o\in O} G(x,o)$$

whenever $O \subseteq X$ is nonempty and finite, and $x \in X$.

Corollary 11.10 (Spherical symmetry of the Green function). *Let b be a locally finite weakly spherically symmetric graph over (X,m) with respect to a nonempty finite $O \subseteq X$. Assume that b is transient. Then, the Green function $G(\cdot,O)$ is spherically symmetric.*

Proof. We calculate by the definitions above and Fubini's theorem

$$\begin{aligned} G(x,O) &= \frac{1}{m(O)}\sum_{o\in O} G(x,o) = \frac{1}{m(O)}\sum_{o\in O}\int_0^\infty p_t(x,o)m(o)\mathrm{d}t \\ &= \int_0^\infty \frac{1}{m(O)}\sum_{o\in O} p_t(x,o)m(o)\mathrm{d}t = \int_0^\infty p_t(x,O)\mathrm{d}t \end{aligned}$$

for $x \in X$. By Theorem 11.5, $p_t(\cdot,O)$ is spherically symmetric for $t \geq 0$. This completes the proof. □

11.3. The Spectral Gap

Having introduced and studied spherically symmetric graphs, we begin in this section to apply the theory developed earlier. Here, we deal with a lower bound on the infimum of the spectrum.

Let (b, c) be a graph over (X, m). We recall the notion of the *boundary* ∂W of a set $W \subseteq X$ as

$$\partial W := (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

We remark that there are many notions of boundaries when considering subsets of a graph. Here, we take a notion that is symmetric.

We will be particularly interested in the boundary of balls around a set and the total edge weight of this boundary. We therefore introduce the *area of the boundary of* W for a set $W \subseteq X$ as

$$b(\partial W) := \sum_{(x,y) \in \partial W} b(x, y)$$

for $r \in \mathbb{N}_0$. We will be interested in the situation where W has the form $B_r(O)$ for a set $O \subseteq X$ and $r \in \mathbb{N}_0$. We note that in this case

$$b(\partial B_r(O)) = 2 \sum_{x \in S_r(O)} k_+(x) m(x)$$

by a direct computation. So if the graph is weakly spherically symmetric, we obtain

$$b(\partial B_r(O)) = 2k_+(r) m(S_r(O))$$

for $r \in \mathbb{N}_0$.

With these notions, we will prove the following summability criterion for positivity of the bottom of the spectrum for weakly spherically symmetric graphs.

Theorem 11.11 (Area-volume ratio and spectrum). *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. If*

$$a := \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then

$$\lambda_0(L) \geq \frac{1}{2a}.$$

The proof uses the Agmon–Allegretto–Piepenbrink theorem in Theorem 8.10. In order to apply it we need the following lemma on solutions. The lemma basically gives a recursion for solutions on spherically symmetric graphs. It will be used in subsequent sections as well.

Lemma 11.12 (Recursion formula for spherically symmetric solutions). *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$ and let $f \in C(X)$ be spherically symmetric. Then, a spherically symmetric function $u \in C(X)$ satisfies $(\mathcal{L} + f)u = 0$ if and only if*

$$u(r + 1) - u(r) = \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} (q(n) + f(n)) m(S_n(O)) u(n)$$

for all $r \in \mathbb{N}_0$. In particular, u is uniquely determined by the choice of $u(0)$. Furthermore, if $u(0) > 0$ and $f > 0$, then $u(r + 1) > u(r)$ for all $r \in \mathbb{N}_0$.

PROOF. We will prove the recursion formula by induction. The uniqueness and monotonicity statements are then obvious from the recursion formula.

We will omit O from our notation below, writing $B_r := B_r(O)$ and $S_r := S_r(O)$. We recall that $b(\partial B_r) = 2k_+(r)m(S_r)$ for $r \in \mathbb{N}_0$.

For $r = 0$, from $(\mathcal{L} + f)u(0) = 0$ we obtain

$$\begin{aligned} 0 &= k_+(0)(u(0) - u(1)) + (q(0) + f(0))u(0) \\ &= \frac{b(\partial B_0)}{2m(S_0)}(u(0) - u(1)) + (q(0) + f(0))u(0), \end{aligned}$$

which yields the desired formula after rearranging the terms.

Now, we assume that the recursion formula holds for $r-1$, where $r \in \mathbb{N}$. From $(\mathcal{L} + f)u(r) = 0$ we obtain

$$k_+(r)(u(r) - u(r+1)) + k_-(r)(u(r) - u(r-1)) + (q(r) + f(r))u(r) = 0.$$

Therefore, by the induction hypothesis, $b(\partial B_r) = 2k_+(r)m(S_r)$ and the formula $k_+(r-1)m(S_{r-1}) = k_-(r)m(S_r)$ proven in Lemma 11.4, we obtain

$$\begin{aligned} u(r+1) - u(r) &= \frac{k_-(r)}{k_+(r)}(u(r) - u(r-1)) + \frac{1}{k_+(r)}(q(r) + f(r))u(r) \\ &= \frac{k_-(r)}{k_+(r)}\left(\frac{2}{b(\partial B_{r-1})}\sum_{n=0}^{r-1}(q(n) + f(n))m(S_n)u(n)\right) \\ &\quad + \frac{2}{b(\partial B_r)}(q(r) + f(r))m(S_r)u(r) \\ &= \frac{k_-(r)}{k_+(r)}\left(\frac{1}{k_-(r)m(S_r)}\sum_{n=0}^{r-1}(q(n) + f(n))m(S_n)u(n)\right) \\ &\quad + \frac{2}{b(\partial B_r)}(q(r) + f(r))m(S_r)u(r) \\ &= \frac{2}{b(\partial B_r)}\sum_{n=0}^{r}(q(n) + f(n))m(S_n)u(n). \end{aligned}$$

This proves the recursion formula and thus completes the proof. □

We now use the recursion formula above to show that under the summability assumption found in Theorem 11.11, there exists a strictly positive α-harmonic function for $\alpha < 0$. This will prove the spectral gap via the Agmon–Allegretto–Piepenbrink theorem.

LEMMA 11.13. *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. If*

$$a := \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then there exists a strictly positive monotonically decreasing spherically symmetric function u which satisfies $u(0) = 1$ and

$$\left(\mathcal{L} - \frac{1}{2a}\right) u = 0.$$

PROOF. We will define a spherically symmetric function u with the required properties. We start by letting $u(0) := 1$. Then, by Lemma 11.12, u will satisfy $\left(\mathcal{L} - \frac{1}{2a}\right) u = 0$ if and only if u satisfies the recursion formula

$$u(r+1) - u(r) = -\frac{1}{a \cdot b(\partial B_r(O))} \sum_{n=0}^{r} m(S_n(O))u(n)$$

for $r \in \mathbb{N}_0$, as we assume $c = 0$ and, thus, $q = 0$.

We will show that u is strictly monotonically decreasing and remains positive by using strong induction. More specifically, we will show that

$$0 < 1 - \frac{1}{a} \sum_{n=0}^{r} \frac{m(B_n(O))}{b(\partial B_n(O))} \leq u(r+1) < u(r)$$

for all $r \in \mathbb{N}_0$. The first inequality above is clear from the definition of a. For $r = 0$, the remaining inequalities follow directly from the recursion formula and $u(0) = 1$ as

$$u(1) - u(0) = -\frac{1}{a \cdot b(\partial B_0(O))} m(S_0(O)) = -\frac{m(O)}{a \cdot b(\partial O)} < 0$$

gives

$$1 - \frac{m(O)}{a \cdot b(\partial O)} = u(1) < 1 = u(0).$$

Now, assume that the inequalities hold up to $r - 1$, that is,

$$0 < 1 - \frac{1}{a} \sum_{n=0}^{k} \frac{m(B_n(O))}{b(\partial B_n(O))} \leq u(k+1) < u(k)$$

for $k = 0, 1, \ldots, r-1$. Therefore, $u(k) > 0$ for $k = 0, 1, \ldots, r$ and the recursion formula gives $u(r+1) - u(r) < 0$. Moreover, as u is then strictly decreasing up to r, we get $u(n) < u(0) = 1$ for all $n = 1, 2, \ldots, r$. Hence, from the recursion formula and the inductive hypotheses we obtain

$$\begin{aligned} u(r+1) &= u(r) - \frac{1}{a \cdot b(\partial B_r(O))} \sum_{n=0}^{r} m(S_n(O))u(n) \\ &> u(r) - \frac{m(B_r(O))}{a \cdot b(\partial B_r(O))} \\ &\geq 1 - \frac{1}{a} \sum_{n=0}^{r-1} \frac{m(B_n(O))}{b(\partial B_n(O))} - \frac{m(B_r(O))}{a \cdot b(\partial B_r(O))} \\ &= 1 - \frac{1}{a} \sum_{n=0}^{r} \frac{m(B_n(O))}{b(\partial B_n(O))}. \end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 11.11. By Lemma 11.13 for

$$a := \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty$$

there exists a strictly positive function u which satisfies

$$\left(\mathcal{L} - \frac{1}{2a}\right) u = 0.$$

Thus, $\lambda_0(L) \geq 1/2a$ follows from the Agmon–Allegretto–Piepenbrink theorem in Theorem 8.10. □

We next illustrate Theorem 11.11 for spherically symmetric trees and anti-trees, i.e., for Examples 11.2 and 11.3. For trees, we get the following criterion for the spectral gap and discreteness of the spectrum.

EXAMPLE 11.14 (Spherically symmetric trees and spectrum). Let b be a spherically symmetric tree with branching number k. If

$$a := \sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{2 \prod_{j=0}^{r} k(j)} < \infty,$$

then $\lambda_0(L) \geq 1/2a$ (Exercise).

For anti-trees we obtain the following criterion. We note, in particular, that this can be used to construct examples of graphs with strictly positive bottom of the spectrum and whose distance balls grow polynomially.

EXAMPLE 11.15 (Anti-trees and spectrum). Let b be an anti-tree with sphere size s. If

$$a := \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{2s(r)s(r+1)} < \infty,$$

then $\lambda_0(L) \geq 1/2a$ (Exercise).

REMARK 11.16 (Discrete spectrum for the graphs in question). The argument to bound the infimum of the spectrum above can be iterated after truncating the original graph by chopping of larger and larger balls around o. This shows that the infima of the spectrum of the truncated graphs goes to ∞. Combined with some standard spectral theory this then yields that the graphs in questions have pure point spectrum with eigenvalues that are of finite multiplicity and tend to ∞.

11.4. Recurrence

We continue our investigation of spherically symmetric graphs by characterizing recurrence.

THEOREM 11.17 (Area ratio and recurrence). *Let b be a locally finite weakly spherically symmetric graph over X with respect to a nonempty finite set $O \subseteq X$. Then, b is recurrent if and only if*

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} = \infty$$

holds.

Proof. As noted in Section 9.2, the measure plays no role in recurrence. Hence, we let $m = 1$ be the counting measure on X. We also let $a > 0$ be a constant. By Lemma 11.12, the unique spherically symmetric function u with $u(0) = 1$ and

$$\left(\mathcal{L} - \frac{1}{2a \cdot m(O)} 1_O\right) u = 0$$

satisfies

$$\begin{aligned} u(r+1) - u(r) &= \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} \frac{-1}{2a \cdot m(O)} 1_O(n) m(S_n(O)) u(n) \\ &= \frac{-1}{a \cdot b(\partial B_r(O))} \end{aligned}$$

for all $r \in \mathbb{N}_0$. Iterating this and using $u(0) = 1$, we obtain

$$u(r+1) = 1 - \frac{1}{a} \sum_{k=0}^{r} \frac{1}{b(\partial B_k(O))}$$

for $r \in \mathbb{N}_0$. We will use this equality with different constants a for both implications in the proof.

First, if we assume that

$$a := \sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} < \infty,$$

then u is a non-constant strictly positive superharmonic function. Thus, b is transient by Theorem 9.7.

Conversely, assume that b is transient. Then, by Theorem 9.7, the Green function is finite, i.e., $G(x, y) < \infty$ for all $x, y \in X$. Furthermore, by Theorem 9.2, $G(\cdot, o)$ for $o \in O$ satisfies

$$\mathcal{L} G(\cdot, o) = 1_o.$$

Furthermore, $G(\cdot, o)$ is strictly positive as the graph is connected. By Corollary 11.10, the function g_O given by

$$g_O(x) := \frac{1}{m(O)} \sum_{o \in O} G(x, o)$$

for $x \in X$ is spherically symmetric and from the above satisfies

$$\mathcal{L} g_O = \frac{1}{m(O)} \sum_{o \in O} \mathcal{L} G(\cdot, o) = \frac{1}{m(O)} 1_O = \frac{1}{g_O(o') m(O)} 1_O g_O,$$

where the last equality holds for all $o' \in O$ since g_O is spherically symmetric. Hence, if we let $u := g_O / g_O(o')$ for $o' \in O$, then u is strictly positive spherically symmetric and satisfies $u(0) = 1$ with

$$\left(\mathcal{L} - \frac{1}{2a \cdot m(O)} 1_O\right) u = 0$$

for $a := g_O(o')/2$. Now, by the consideration in the beginning of the proof, u must also satisfy

$$u(r+1) = 1 - \frac{1}{a}\sum_{k=0}^{r} \frac{1}{b(\partial B_k(O))}$$

for all $r \in \mathbb{N}_0$. As u is positive we conclude

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} < \infty.$$

This completes the proof. $\square$

REMARK 11.18. Another viewpoint on Theorem 11.17 is that the Green function for weakly spherically symmetric graphs can be calculated explicitly as

$$G(x,o) = m(o)\sum_{n=r}^{\infty} \frac{1}{\frac{1}{2}b(\partial B_n(O))}$$

for $x \in S_r(O)$, $r \in \mathbb{N}_0$ and $o \in O$. In particular,

$$G(x,O) = \sum_{n=r}^{\infty} \frac{1}{\frac{1}{2}b(\partial B_n(O))}$$

for all $x \in S_r(O)$, $r \in \mathbb{N}_0$, from which Theorem 11.17 follows (Exercise).

We now illustrate the theorem above for our two main classes of examples, namely spherically symmetric trees and anti-trees from Examples 11.2 and 11.3. For trees the characterization of recurrence reads as follows.

EXAMPLE 11.19 (Spherically symmetric trees and recurrence). Let b be a spherically symmetric tree with branching number k. Then b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\prod_{n=0}^{r} k(n)} = \infty$$

(Exercise).

For anti-trees, rephrasing everything in terms of the sphere growth gives the following characterization.

EXAMPLE 11.20 (Anti-trees and recurrence). Let b be an anti-tree with sphere size s. Then, b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{s(r)s(r+1)} = \infty$$

(Exercise).

11.5. Stochastic Completeness

We end our investigation of spherically symmetric graphs by characterizing stochastic completeness.

THEOREM 11.21 (Volume-area ratio and stochastic completeness). *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. Then, b is stochastically complete if and only if*

$$\sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} = \infty$$

holds.

In order to prove the theorem above, we first investigate the boundedness of α-harmonic functions for $\alpha > 0$ by using the recursion formula for solutions found in Lemma 11.12.

LEMMA 11.22. *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. Then, the following statements are equivalent:*

(i) *There exists $\alpha > 0$ and a non-trivial spherically symmetric α-harmonic function that is bounded.*
(ii) *For all $\alpha > 0$, all spherically symmetric α-harmonic functions are bounded.*
(iii) *We have*

$$\sum_{r=0}^{\infty} \frac{c(B_r(O)) + m(B_r(O))}{b(\partial B_r(O))} < \infty.$$

PROOF. First of all, Lemma 11.12 gives that a spherically symmetric α-harmonic function u is uniquely determined by its value at 0. Thus, for a given $\alpha > 0$, all spherically symmetric α-harmonic functions are bounded if there exists a non-trivial α-harmonic function that is bounded.

Furthermore, for $\alpha > 0$, we let

$$a(\alpha) := \sum_{r=0}^{\infty} \frac{c(B_r(O)) + \alpha m(B_r(O))}{b(\partial B_r(O))}.$$

Obviously, the finiteness of $a(\alpha)$ for some $\alpha > 0$ is equivalent to the finiteness of $a(\alpha)$ for all $\alpha > 0$.

Thus, it remains to show that $a(\alpha) < \infty$ is equivalent to the existence of a non-trivial bounded spherically symmetric α-harmonic function.

By Lemma 11.12, any spherically symmetric u with $(\mathcal{L}+\alpha)u = 0$ satisfies the recursion formula

$$u(r+1) - u(r) = \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} \big(c(S_n(O)) + \alpha m(S_n(O))\big) u(n)$$

for all $r \in \mathbb{N}_0$, where we used $q(n)m(S_n(O)) = c(S_n(O))$ for $n \in \mathbb{N}_0$, which follows from the spherical symmetry of q. Now, if $u(0) = 0$, then u is trivial, hence, we may assume that $u(0) \neq 0$ as we are interested in non-trivial solutions.

On the other hand if we assume that $u(0) > 0$, then the recursion formula implies that u is monotonically increasing. In particular, $u(r) \geq u(0) > 0$

for all $r \in \mathbb{N}_0$. Thus, summation in the recursion formula over the spheres gives both the lower bound

$$u(r+1) - u(r) \geq \frac{2\,(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))} u(0)$$

for $r \in \mathbb{N}_0$ and the upper bound

$$u(r+1) \leq \left(1 + \frac{2(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))}\right) u(r)$$

for all $r \in \mathbb{N}_0$. The desired equivalence follows easily from these bounds.

Indeed, if $a(\alpha) = \infty$ holds, we find from the lower bound

$$\begin{aligned} u(r) = \sum_{n=0}^{r-1} (u(n+1) - u(n)) \geq \sum_{n=0}^{r-1} \frac{2\,(c(B_n(O)) + \alpha m(B_n(O)))}{b(\partial B_n(O))} u(0) \\ \to \infty \end{aligned}$$

as $r \to \infty$. So, u is unbounded in this case. An analogous argument shows that $u(r) \to -\infty$ as $r \to \infty$ if $u(0) < 0$ and $a(\alpha) = \infty$.

On the other hand, if $a(\alpha) < \infty$ holds we find from iteration of the upper bound

$$\begin{aligned} u(r+1) &\leq \left(1 + \frac{2(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))}\right) u(r) \\ &\leq \prod_{n=0}^{r} \left(1 + \frac{2(c(B_n(O)) + \alpha m(B_n(O)))}{b(\partial B_n(O))}\right). \end{aligned}$$

Now, by $a(\alpha) < \infty$ also

$$\prod_{n=0}^{\infty} \left(1 + \frac{c(B_n(O)) + \alpha m(B_n(O))}{b(\partial B_n(O))}\right) < \infty$$

is valid and the estimate above shows that u is bounded. A similar argument shows that u is strictly negative and bounded below if $u(0) < 0$ and $a(\alpha) < \infty$. This shows the equivalence between (i) and (iii).

Furthermore, since finiteness of $a(\alpha)$ for one $\alpha > 0$ is equivalent to finiteness of $a(\alpha)$ for all $\alpha > 0$, we get that (i) and (ii) are equivalent. This completes the proof. □

Proof of Theorem 11.21. If

$$\sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then there exists a non-trivial bounded α-harmonic function for $\alpha > 0$ by Lemma 11.22. Thus, the graph is stochastically incomplete by Theorem 10.22.

On the other hand, if the graph is stochastically incomplete, then there exists a positive non-trivial bounded function v which satisfies $(\mathcal{L} + \alpha)v = 0$ for $\alpha > 0$ by Theorem 10.22. We recall that $\mathcal{A}$ denotes the averaging operator given by

$$\mathcal{A}f(x) := \frac{1}{m(S_r(O))} \sum_{y \in S_r(O)} f(y)m(y)$$

for $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Applying this to v gives that $u := \mathcal{A}v$ is a spherically symmetric function with

$$(\mathcal{L} + \alpha)u = \mathcal{L}\mathcal{A}v + \alpha\mathcal{A}v = \mathcal{A}(\mathcal{L} + \alpha)v = 0$$

since $\mathcal{L}$ and $\mathcal{A}$ commute by Lemma 11.8. Therefore, there exists a non-trivial bounded spherically symmetric function u which satisfies $(\mathcal{L} + \alpha)u = 0$ for $\alpha > 0$ and, thus,

$$\sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty$$

by Lemma 11.22. This completes the proof. □

We again illustrate the characterization of stochastic completeness at infinity for our two main classes of examples, namely, spherically symmetric trees and anti-trees from Example 11.2 and 11.3.

Example 11.23 (Spherically symmetric trees and stochastic completeness). Let b be a spherically symmetric tree with branching number k. Then b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{\prod_{j=0}^{r} k(j)} = \infty$$

(Exercise).

For anti-trees we obtain the following characterization. We note, in particular, that we can use this to construct examples of stochastically incomplete graphs whose balls grow polynomially.

Example 11.24 (Anti-trees and stochastic completeness). Let b be an anti-tree with sphere size s. Then, b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r)s(r+1)} = \infty$$

(Exercise).

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Sheet 13

Trees and antitrees

Exercise 1 (Antitrees volume growth) 4 points

Let an antitree with $s(r) = [r^\gamma]$, $r \geq 1$, $\gamma > 0$ be given (where $[\cdot]$ is the integer function) and $m = 1$. Let ρ be the degree path metric (see Example 7.3 in chapter about intrinsic metrics). Show that the function $r \longmapsto \# B_r(o)$ grows polynomially for $0 < \gamma < 2$, exponentially for $\gamma = 2$ and the graph has finite diameter for $\gamma > 2$.

Exercise 2 (Antirees – spectrum) 4 points

Let an antitree with $s(r) = [r^\gamma]$, $r \geq 1$, $\gamma > 0$ be given and $m = 1$. Show that $\lambda_0(L) > 0$ if and only if $\gamma \geq 2$.

Exercise 3 (Antirees and trees – recurrence) 4 points

(a) Let b be a spherically symmetric tree with branching number k. Then b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\prod_{n=0}^{r} k(n)} = \infty.$$

(b) Let b be an anti-tree with sphere size s. Then, b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{s(r)s(r+1)} = \infty.$$

Exercise 4 (Antirees and trees – stochastic completeness) 4 points

(a) Let b be a spherically symmetric tree with branching number k. Then b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{\prod_{j=0}^{r} k(j)} = \infty.$$

(b) Let b be an anti-tree with sphere size s. Then, b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r)s(r+1)} = \infty.$$