26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 12

10.3. The Heat Equation Perspective

In this section we turn to stochastic completeness and present the heat equation viewpoint on it.

We first define the notion.

DEFINITION 10.15 (Stochastic completeness). A graph (b,c) over (X,m) is called *stochastically complete* or *conservative* if

$$e^{-tL}1 = 1$$

holds for all $t \geq 0$, where $L := L^{(D)}$ is the associated operator. Otherwise, (b,c) over (X,m) is called *stochastically incomplete.*

REMARK 10.16. We have defined the notion of stochastic completeness via the semigroup. We will see that stochastic completeness can equivalently be characterized via resolvents by the condition that

$$\alpha(L+\alpha)^{-1}1 = 1$$

holds $\alpha > 0$.

REMARK 10.17. Stochastic completeness is only possible for $c = 0$. Indeed, assume that there exist an $x \in X$ with $c(x) > 0$. Let $u_t := e^{-tL}1$ and assume $u_t = 1$ for all $t \geq 0$. Then, $t \mapsto u_t(x)$ is constant (to 1) and this gives (for any $t > 0$) the contradiction

$$0 = \partial_t u_t(x) = -\mathcal{L}u_t(x) = -\mathcal{L}1(x) = -\frac{c(x)}{m(x)} < 0.$$

To better understand the concept and its name we present the following lemma. Recall that $(\cdot,\cdot)$ given by

$$(g,f) := \sum_{y\in X} g(y)f(y)m(y)$$

denotes the dual pairing between $\ell^\infty(X,m)$ and $\ell^1(X,m)$.

LEMMA 10.18 (Stochastic completeness as preservation of heat). *Let (b,c) be a graph over (X,m) and $L := L^{(D)}$ the associated operator. Define for $f \in \ell^1(X,m)$ the amount of heat at time $t \geq 0$ by*

$$A_f(t) := \sum_{y\in X} e^{-tL}f(y)m(y).$$

Then, the following assertions are equivalent:

(i) *The graph is stochastically complete.*
(ii) *For any $x \in X$ the function A_{1_x} is constant on $[0,\infty)$.*
(iii) *For any $f \in \ell^1(X,m)$ the function A_f is constant on $[0,\infty)$.*

PROOF. This is basically a direct consequence of

$$A_f(t) = (1, e^{-tL}f) = (e^{-tL}1, f),$$

where the first equality follows directly from the definition of the dual pairing and the second equality follows from the symmetry. For the convenience of the reader we provide the details.

(iii)$\Longrightarrow$ (ii): This is clear.

(ii)$\Longrightarrow$ (i): We have

$$m(x) = A_{1_x}(0) = A_{1_x}(t) = (1, e^{-tL}1_x) = (e^{-tL}1, 1_x) = m(x)e^{-tL}1(x)$$

for all $x \in X$ and this gives (i).

(i)$\Longrightarrow$(iii): For all $t > 0$ we can compute

$$A_f(t) = (1, e^{-tL}f) = (e^{-tL}1, f) = (1, f) = A_f(0).$$

This gives (iii). □

Here is the main characterization of stochastic completeness in terms of the heat equation.

THEOREM 10.19 (Stochastic completeness and the heat equation). *Let b be a connected graph over (X, m) and $L := L^{(D)}$ the associated operator. Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$e^{-tL}1(x) = 1.$$

(*"Stochastic completeness"*)

(i.a) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*

$$(L + \alpha)^{-1}1(x) = 1.$$

(vi) *For every $f \in \ell^\infty(X)$ there exists a unique bounded solution u of the heat equation*

$$(\mathcal{L} + \partial_t)u = 0 \quad \text{with} \quad u_0 = f.$$

(*"Heat equation"*)

(vi.a) *Every bounded solution u of the heat equation $(\mathcal{L} + \partial_t)u = 0$ with $u_0 = 0$ is trivial.*

We start the proof of Theorem 10.19 by showing the equivalence of the "for some" and "for all" statements in (i). For this, the connectedness of the graph is essential. We need the following lemma. It shows that if the total amount of heat in the graph drops below 1 at some time, then it drops below 1 for all times.

LEMMA 10.20. *Let (b, c) be a graph over (X, m) and $L := L^{(D)}$ the associated operator. If $t \geq s \geq 0$, then*

$$e^{-tL}1 \leq e^{-sL}1.$$

PROOF. From Theorem 10.7, we get that the heat semigroup is both positivity preserving and contracting. Let $t = s + h$ with $s, h \geq 0$. As the semigroup is contracting we have

$$e^{-hL}1 \leq 1.$$

As the semigroup is positivity preserving, this gives, after we apply e^{-sL} to both sides,

$$e^{-tL}1 = e^{-sL}e^{-hL}1 \leq e^{-sL}1.$$

This is the desired statement. □

Lemma 10.21. *Let b be a connected graph over (X,m) and $L := L^{(D)}$ the associated operator. If $e^{-tL}1(x) < 1$ for some $t > 0$ and some $x \in X$, then $e^{-tL}1(x) < 1$ for all $t > 0$ and all $x \in X$.*

Proof. Assume that $e^{-tL}1(x) < 1$ for some $t > 0$ and some $x \in X$. By Lemma 10.20 we have $e^{-tL}1 \leq 1$. Thus, $(1 - e^{-tL}1)$ is positive and non-trivial. As the graph is connected and the semigroup is positivity improving we obtain that $e^{-sL}(1 - e^{-tL}1) > 0$. Hence, we find $e^{-(s+t)L}1 < 1$ for all $s > 0$. So, $e^{-rL}1 < 1$ for all $r > t$. It remains to show $e^{-rL}1 < 1$ also for $0 < r \leq t$. Assume not. Then, there exists a $y \in X$ and $0 < r \leq t$ such that $e^{-rL}1(y) = 1$. By what we have shown already this gives $e^{-uL}1 = 1$ for all $0 < u < r$. Fix any such $u > 0$. Let $n \in \mathbb{N}$ be such that $nu > t$. By the semigroup property, we get $e^{-nuL}1 = 1$. Thus, $e^{-tL}1 = 1$ as $nu > t$, which yields a contradiction to the assumption that $e^{-tL}1(x) < 1$. This completes the proof. □

Proof of Theorem 10.19. The equivalence of the "for some" and "for all" statements found in (i) has already been shown in Lemma 10.21.

(i) $\iff$ (i.a): This follows readily from Theorem 10.11 and Lemma 10.21. This also shows the equivalence of the "for some" and "for all" statements found in (i.a).

(vi) $\implies$ (vi.a): This is clear since if we let $f = 0$, then $u = 0$ is the unique bounded solution of the heat equation with initial condition 0.

(vi.a) $\implies$ (vi): Given $f \in \ell^\infty(X,m)$, the existence of a bounded solution of the heat equation with initial condition f has been shown in Theorem 10.11. Hence, we only need to establish uniqueness. Therefore, let u and v be two bounded solutions of the heat equation with initial condition f. Then, $w := u - v$ is a bounded solution of the heat equation with initial condition $w_0 = 0$. By (vi.a) we get that $w = 0$, so that $u = v$.

(vi.a) $\implies$ (i): We show this by contraposition. If $e^{-tL}1(x) < 1$ for some $x \in X$ and some $t > 0$, then $e^{-tL}1 < 1$ for all $t > 0$ by Lemma 10.21. Hence, suppose that $e^{-tL}1 < 1$ for $t > 0$. Moreover, $u_t := e^{-tL}1$ defines a bounded solution to the heat equation

$$(\mathcal{L} + \partial_t)u_t = 0$$

with $u_0 = 1$ by Theorem 10.11. Furthermore, it is clear that $(\mathcal{L} + \partial_t)1 = 0$. Therefore, it follows that

$$v_t := 1 - e^{-tL}1$$

is a bounded solution of the heat equation with initial condition 0, which is non-trivial since $e^{-tL}1 < 1$ for $t > 0$.

(i) $\implies$ (vi.a): We show this by contraposition as well. Let u be a non-zero bounded solution of the heat equation with $u_0 = 0$. Without loss of generality, we may assume that $u_{t_0}(x_0) > 0$ for some $t_0 > 0$ and some $x_0 \in X$ as otherwise we work with $-u$. Furthermore, by rescaling, we may assume $|u| \leq 1$.

Let $w := 1 - u$. Then, w is positive, bounded and $w_{t_0}(x_0) < 1$. Furthermore, since $u_0 = 0$, we get $w_0 = 1$ and since u solves the heat equation, we get

$$(\mathcal{L} + \partial_t)\, w = 0$$

for all $t > 0$. Since $v_t := e^{-tL}1$ defines the smallest positive function with these properties by Theorem 10.11, we infer $e^{t_0 L}1(x_0) \leq w_{t_0}(x_0) < 1$. Therefore, $e^{-tL}1(x) < 1$ for all $t > 0$ and all $x \in X$ by Lemma 10.21. This completes the proof. □

10.4. The Poisson Equation Perspective

It is possible to characterize stochastic completeness by means of Poisson equation. Details are discussed in this section.

THEOREM 10.22 (Stochastic completeness and the Poisson equation). *Let b be a connected graph over (X, m) and $L = L^{(D)}$ the associated operator. Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$e^{-tL}1(x) = 1.$$

(*"Stochastic completeness"*)

(v) *For some (all) $\alpha > 0$ and every $f \in \ell^\infty(X)$ there exists a unique $u \in \ell^\infty(X)$ satisfying*

$$(\mathcal{L} + \alpha)u = f.$$

(*"Poisson equation"*)

(v.a) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u \leq 0$ is trivial.*

(v.b) *For some (all) $\alpha > 0$ every $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u = 0$ is trivial.*

(v.c) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u = 0$ is trivial.*

The following lemma is the key to proving Theorem 10.22. It connects $(e^{-tL}1)_{t \geq 0}$ with bounded α-harmonic functions for $\alpha > 0$.

LEMMA 10.23 (Largest α-subharmonic function). *Let b be a graph over (X, m) and $L := L^{(D)}$ the associated operator. For $\alpha > 0$, the function*

$$w_\alpha := \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t = 1 - \alpha(L + \alpha)^{-1}1$$

satisfies $0 \leq w_\alpha \leq 1$, solves $(\mathcal{L} + \alpha)w_\alpha = 0$ and is the largest function $u \in \mathcal{F}$ with $0 \leq u \leq 1$ such that $(\mathcal{L} + \alpha)u \leq 0$.

PROOF. The Laplace transform formula proven in Theorem 10.9 gives that for every $\alpha > 0$ the function

$$v_\alpha := \int_0^\infty \alpha e^{-t\alpha} e^{-tL}1\mathrm{d}t = \alpha(L + \alpha)^{-1}1$$

satisfies $(\mathcal{L} + \alpha)v_\alpha = \alpha 1$ and is the minimal positive $v \in \mathcal{F}$ such that $(\mathcal{L} + \alpha)v \geq \alpha 1$ by Lemma 10.10. Furthermore, as $0 \leq e^{-tL}1 \leq 1$ for all $t \geq 0$

by Lemma 10.20, we get $0 \le v_\alpha \le 1$. Therefore,

$$\begin{aligned} w_\alpha = 1 - v_\alpha = 1 - \alpha(L+\alpha)^{-1}1 &= 1 - \int_0^\infty \alpha e^{-t\alpha} e^{-tL} 1 \mathrm{d}t \\ &= \int_0^\infty \alpha e^{-t\alpha} \mathrm{d}t - \int_0^\infty \alpha e^{-t\alpha} e^{-tL} 1 \mathrm{d}t \\ &= \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t, \end{aligned}$$

and $0 \le w_\alpha \le 1$. Furthermore, as v_α satisfies $(\mathcal{L}+\alpha)v_\alpha = \alpha 1$ and since $(\mathcal{L}+\alpha)1 = \alpha 1$ by a direct calculation, we get

$$(\mathcal{L}+\alpha)\, w_\alpha = 0.$$

We now show the maximality of w_α. Hence, let u satisfy $(\mathcal{L}+\alpha)u \le 0$ with $0 \le u \le 1$. Then, $1-u \ge 0$ satisfies $(\mathcal{L}+\alpha)(1-u) \ge \alpha 1$. As v_α is the minimal such positive function by Theorem 10.10, we get $v_\alpha \le 1-u$. As $w_\alpha = 1 - v_\alpha$, $w_\alpha \ge u$ follows. This completes the proof. □

The equivalence of the "for some" and "for all" statements in (v.a), (v.b) and (v.c) is shown in the next lemma.

Lemma 10.24. *Let (b,c) be a graph over (X,m). If there exists a bounded non-trivial $v \ge 0$ such that $(\mathcal{L}+\alpha)v \le 0$ for some $\alpha > 0$, then for every $\alpha > 0$ there exists a bounded non-trivial $v \ge 0$ such that $(\mathcal{L}+\alpha)v = 0$.*

Proof. Let $\alpha > 0$ and let v be a bounded non-trivial positive function on X satisfying $(\mathcal{L}+\alpha)v \le 0$. By rescaling, we may assume that $0 \le v \le 1$. By Lemma 10.23, $w_\alpha := \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t$ is the maximal function $u \in \mathcal{F}$ with $0 \le u \le 1$ such that $(\mathcal{L}+\alpha)u \le 0$. Therefore, $v \le w_\alpha$. As v is non-trivial, w_α is non-trivial and we conclude that $e^{-tL}1 < 1$ for some t. Therefore, $e^{-tL}1 < 1$ for all $t > 0$ by Lemma 10.21. Hence, for all $\beta > 0$, the function $w_\beta = \int_0^\infty \beta e^{-t\beta}(1 - e^{-tL}1)\mathrm{d}t$ is non-trivial. Furthermore, by Lemma 10.23 we have $0 \le w_\beta \le 1$ and $(\mathcal{L}+\beta)w_\beta = 0$ for $\beta > 0$. This completes the proof. □

Proof of Theorem 10.22. The equivalence of the "for some" and "for all" statements in (v.a) and (v.c) follows from Lemma 10.24. The equivalence of the "for some" and "for all" statements in (v) and (v.b) will follow from the arguments given below.

For the rest of the proof recall that

$$w_\alpha := \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t$$

solves $(\mathcal{L}+\alpha)w_\alpha = 0$ and is the largest function u with $0 \le u \le 1$ and $(\mathcal{L}+\alpha)u \le 0$ by Lemma 10.23. Obviously, $w_\alpha = 0$ for some (all) $\alpha > 0$ if and only if $e^{-tL}1 = 1$ for some (all) $t > 0$, i.e., if and only if the graph is stochastically complete.

We first show (i) $\Longrightarrow$ (v.a) $\Longrightarrow$ (v.b) $\Longrightarrow$ (v.c) $\Longrightarrow$ (i) for fixed $\alpha > 0$.

(i) $\Longrightarrow$ (v.a): Let $u \ge 0$ be a bounded solution of $(\mathcal{L}+\alpha)u \le 0$. By rescaling, we may assume that $u \le 1$. Then, $0 \le u \le w_\alpha$ since w_α is the largest such solution. If $e^{-tL}1 = 1$, then $w_\alpha = 0$ and, therefore, $u = 0$.

(v.a) $\Longrightarrow$ (v.b): This follows immediately from Lemma 2.12, which states that if $u \in \mathcal{F}$ is α-harmonic, then $|u|$ is α-subharmonic.

(v.b) $\Longrightarrow$ (v.c): This is clear.

(v.c) $\Longrightarrow$ (i): If there do not exist non-trivial positive functions $u \leq 1$ such that $(\mathcal{L} + \alpha)u = 0$, then the largest such function w_α satisfies $w_\alpha = 0$. Therefore, $e^{-tL}1 = 1$ for all $t > 0$.

Next, we show the implications (v)(for fixed α) $\Longrightarrow$ (i) $\Longrightarrow$ (v.b) $\Longrightarrow$ (v) (for all α). This completes the proof as it gives in particular the equivalence of the "for some" and "for all" statements in (v) and (v.b).

(v) $\Longrightarrow$ (i): We show this by contraposition. So, suppose that $e^{-tL}1 < 1$. Let $\alpha > 0$ and let $f \in \ell^\infty(X)$. Then, $u := (L + \alpha)^{-1} f \in \ell^\infty(X)$ solves $(\mathcal{L}+\alpha)u = f$ by Theorem 10.10. As we assume that $e^{-tL}1 < 1$, $w_\alpha > 0$ and, therefore, $v := u + w_\alpha > u$ also solves $(L + \alpha)v = f$ since $(\mathcal{L} + \alpha)w_\alpha = 0$. Therefore, there is no uniqueness of solutions to the Poisson equation for any $\alpha > 0$.

(i) $\Longrightarrow$ (v.b): We have already shown this in the first round of equivalences above.

(v.b) $\Longrightarrow$ (v): Let $f \in \ell^\infty(X)$ and let $\alpha > 0$. The existence of solutions to the Poisson equation for $\alpha > 0$ is given by $u := (L + \alpha)^{-1} f$. So, we have to show uniqueness. Therefore, assume that there exists $f \in \ell^\infty(X)$ and two bounded solutions u_1, u_2 such that $(\mathcal{L} + \alpha)u_1 = f = (\mathcal{L} + \alpha)u_2$. Then, $u := u_1 - u_2$ is bounded and satisfies $(\mathcal{L} + \alpha)u = 0$. From (v.b) we infer $u = 0$ and, therefore, $u_1 = u_2$. □

10.5. What we have shown (and more)

In the subsequent theorem we present the results of Section 10.4 in a combined version. The theorem contains some further aspects of stochastic completeness that we have not discussed.

THEOREM 10.25 (Characterization of stochastic completeness). *Let b be a connected graph over (X, m) and $L := L^{(D)}$ the associated operator. Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*
$$e^{-tL}1(x) = 1.$$

(i.a) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*
$$\alpha(L + \alpha)^{-1}1(x) = 1$$

(ii) *There exists a sequence of functions (e_n) in $D(Q)$ (equivalently, (e_n) in $C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*
$$\lim_{n\to\infty} Q(e_n, v) = 0$$
for all $v \in D(Q) \cap \ell^1(X, m)$.

(ii.a) *There exists a sequence of functions (e_n) in $D(Q)$ (equivalently, (e_n) in $C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*
$$\lim_{n\to\infty} Q(e_n, (L + \alpha)^{-1} v) = 0$$

for one $v \in \ell^2(X,m) \cap \ell^1(X,m)$ *with* $v > 0$ *and some (all)* $\alpha > 0$.

(iii) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m)$, *then*

(*"Green's formula"*)
$$\sum_{x \in X} \mathcal{L}v(x)m(x) = 0.$$

(iii.a) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m) \cap \ell^2(X,m)$, *then*

$$\sum_{x \in X} \mathcal{L}v(x)m(x) = 0.$$

(iv) *If* $u \in \mathcal{F}$ *satisfies* $\sup u \in (0,\infty)$ *and* $\beta \in (0, \sup u)$, *then*

$$\sup_{x \in X_\beta} \mathcal{L}u(x) \geq 0,$$

where $X_\beta := \{x \in X \mid u(x) > \sup u - \beta\}$.

(*"Omori–Yau maximum principle"*)

(v) *For some (all)* $\alpha > 0$ *and every* $f \in \ell^\infty(X)$ *there exists a unique bounded solution* u *of the Poisson equation*

(*"Poisson equation"*)
$$(\mathcal{L} + \alpha)u = f.$$

(v.a) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u \leq 0$ *is trivial.*

(v.b) *For some (all)* $\alpha > 0$ *every* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u = 0$ *is trivial.*

(v.c) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u = 0$ *is trivial.*

(vi) *For every* $f \in \ell^\infty(X)$ *there exists a unique bounded solution* u *of the heat equation*

(*"Heat equation"*)
$$(\mathcal{L} + \partial_t)u = 0 \quad \text{with} \quad u_0 = f.$$

(vi.a) *Every bounded solution* u *of the heat equation* $(\mathcal{L} + \partial_t)u = 0$ *with* $u_0 = 0$ *is trivial.*

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Sheet 12

Stochastic completeness

Exercise 1 (Subgraphs I) 4 points

Let b be a stochastically incomplete graph over (X,m). Show that there exist X' with $X \subseteq X'$, b' and m' which extend b and m to X' such that b' over (X',m') is stochastically complete.

Exercise 2 (Subgraphs II) 4 points

Let b be a graph over (X,m). Let $Y \subseteq X$ and suppose that the associated subgraph b_Y over (Y,m_Y) is stochastically incomplete at infinity. Let

$$\mathrm{Deg}_{X\setminus Y}(x) = \frac{1}{m(x)} \sum_{y \in X\setminus Y} b(x,y)$$

for $x \in Y$. Suppose that $\mathrm{Deg}_{X\setminus Y}$ is bounded on the set

$$\{x \in Y \mid \text{there exists a } y \sim x, y \notin Y\}.$$

Show that b over (X,m) is stochastically incomplete.
Hint: Omori-Yau maximum principle.

Exercise 3 (Khasminskii criterion) 4 points

Let b be a connected graph over X. Show that if there exists a positive function $v \in \mathcal{F}$ such that $v(x) \to \infty$ as $x \to \infty$ and

$$(\mathcal{L} + \alpha)v \geq 0$$

for $\alpha > 0$, then b is stochastically complete.

Exercise 4 (Uniqueness of bounded solutions with $c \neq 0$) 4 points

Let (b,c) be a graph over (X,m) with formal Laplacian $\mathcal{L} = \mathcal{L}_{b,c,m}$. Assume that b (without c) is stochastically complete. Show that the following equations only have the trivial solution.

(a) $(\mathcal{L} + \alpha)f = 0$ with $\alpha > 0$ and $f \in \ell^\infty(X)$.

(b) $(\partial_t + \mathcal{L})u = 0$ with $u\colon [0,\infty) \times X \to \mathbb{R}$ bounded and $u(0,\cdot) = 0$.

Hint: Prove uniqueness of bounded nonnegative solutions for the inequality $(\mathcal{L} + \alpha)f \leq 0$. Use the Laplace transform to establish (b).

Bonus Exercise 1 (Adding potentials) 1 point

Let b be a graph over (X,m). Show that there exists $c\colon X \to [0,\infty)$ such that the inequality $(\mathcal{L} + \alpha)f \leq 0$ with $\alpha > 0$ and $f \in \ell^\infty(X)_+$ only has the trivial solution $f = 0$.

Bonus Exercise 2 (Generalized conservation property) 1 point

Let (b,c) be a graph over (X,m) with Laplacian $L = L^{(D)}_{b,c,m}$. Assume that b (without c) is stochastically complete and that c/m is bounded. Prove the formulae

$$1 = \alpha(L+\alpha)^{-1}1 + (L+\alpha)^{-1}\frac{c}{m} \quad \text{and} \quad 1 = e^{-tL}1 + \int_0^t e^{-sL}\frac{c}{m}ds,$$

where the integral is understood pointwise.