26th Internet Seminar on Evolution Equations

# Graphs and Discrete Dirichlet Spaces

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## Lecture 11

CHAPTER 10

# Stochastic Completeness

Graphs and their Laplacians can be used to model diffusion of heat on discrete sets. The corresponding equation is known as heat equation. The basic ingredient is the distribution of heat at a given point of time. This is modeled by a function (with values in $[0,\infty)$) on the set and the heat equation captures how this function develops in time. A particularly relevant aspect is whether the total amount of heat on the set is conserved over time. This is known as stochastic completeness or conservativeness or honesty. Details are discussed in this chapter. To put things in perspective we already provide an outline of the material here.

Let $X$ be a countable set and $m$ a measure on $X$ with full support. Let $(b,c)$ be a graph over $(X,m)$ and $\mathcal{L}$ the associated formal Laplacian. Then the equation

$$-\mathcal{L}v = \partial_t v$$

is known as heat equation (induced from $\mathcal{L}$)). A function

$$u\colon [0,\infty)\times X \longrightarrow \mathbb{R}$$

is said to be a *solution of the heat equation* with *initial condition* $f \in C(X)$ if the following holds:

- For each $x \in X$ the function $[0,\infty) \longrightarrow \mathbb{R}$, $t \mapsto u_t(x)$ is continuous and differentiable on $(0,\infty)$ with $u_0 = f$.
- For each $t > 0$ the map $X \longrightarrow \mathbb{R}, x \mapsto u_t(x)$ belongs to $\mathcal{F}$.
- The equality $-\mathcal{L}u_t(x) = \partial_t u_t(x)$ holds for all $t > 0$ and all $x \in X$.

By Theorem 4.9 the action of the self-adjoint operator $L$ associated to $(b,c)$ is given by $Lf = \mathcal{L}f$ for $f \in D(L)$. Hence, by Theorem 3.24 for any $f \in \ell^2(X,m)$ the function

$$[0,\infty) \ni t \mapsto u_t := e^{-tL}f \in \ell^2(X,m)$$

is a solution to the heat equation with initial condition $f$ (as had indeed been already discussed above). Now, the operators $S(t) := e^{-tL}$, $t \geq 0$, satisfy the Markov property and this can be used to extend them to operators $S^{(p)}(t)$ on $\ell^p(X,m)$ for any $1 \leq p \leq \infty$. Details are discussed in Section 10.1. This can be used to show that the heat equation can be solved for bounded initial condition $f$, i.e. for $f \in \ell^\infty(X,m)$. Specifically, as discussed in Section 10.2, for any $f \in \ell^\infty(X,m)$ the function

$$u\colon [0,\infty)\times X \longrightarrow \mathbb{R}, u_t(x) := S^{(\infty)}(t)f(x),$$

is a solution to the heat equation with initial condition $f$. Moreover,

$$\sum_{x\in X} g(x)(S^{(\infty)}(t)f)(x)m(x) = \sum_{x\in X}(S^{(1)}(t)g)(x)f(x)m(x)$$

holds for all $g \in \ell^1(X,m)$, $f \in \ell^\infty(X,m)$ and $t \geq 0$. Now, the question arises whether for any $g \in \ell^1(X,m)$ with $g \geq 0$ the total amount of heat at time $t$ given by

$$A(t) := \sum_{x\in X} (S^{(1)}(t)g)(x)m(x)$$

remains constant (in $t$). This is what is known as stochastic completeness and the second part of this chapter, starting in Section 10.3, is devoted to various equivalent characterizations.

## 10.1. The Semigroup and Resolvent on $\ell^p$

Here, we discuss how the operators $e^{-tL}$, $t \geq 0$, on $\ell^2(X,m)$ associated to the graph $(b,c)$ with induced Laplacian $L = L^{(D)}$ can be extended to operators on each $\ell^p(X,m)$, $1 \leq p \leq \infty$. The crucial point here is that these operators satisfy the Markov property.

We start with the definition of a strongly continuous contraction Markov semigroup. For $p \in [1,\infty]$, we denote the bounded linear operators on $\ell^p(X,m)$ by

$$B(\ell^p(X,m)) := \{L\colon \ell^p(X,m) \longrightarrow \ell^p(X,m) \mid L \text{ is linear and bounded}\}.$$

Note that we have $\ell^\infty(X,m) = \ell^\infty(X)$.

Definition 10.1 (Semigroup). Let $p \in [1,\infty]$. A map $S\colon [0,\infty) \longrightarrow B(\ell^p(X,m))$ is called a *semigroup* on $\ell^p(X,m)$ if

$$S(0) = I, \quad \text{and} \quad S(s+t) = S(s)S(t)$$

for all $s,t \geq 0$. A semigroup $S$ is called *strongly continuous* if

$$\lim_{t\to 0^+} S(t)f = f$$

for all $f \in \ell^p(X,m)$. A semigroup $S$ is called a *contraction semigroup* if

$$\|S(t)\|_p \leq 1$$

for all $t \geq 0$. Finally, a semigroup is called a *Markov semigroup* if

$$0 \leq S(t)f \leq 1$$

for all $t \geq 0$ and $f \in \ell^p(X,m)$ with $0 \leq f \leq 1$.

Example 10.2. If $A \in B(\ell^p(X,m))$, then $(e^{-tA})_{t\geq 0}$ defined by the absolutely convergent series

$$e^{-tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

for $t \geq 0$ gives a strongly continuous semigroup which yields a solution of the parabolic equation involving $A$ (Exercise). Furthermore, given an initial condition, this solution is unique (Exercise).

Any strongly continuous semigroup $S$ defines an operator called the generator of the semigroup.

Definition 10.3 (Generator of a semigroup). Let $p \in [1,\infty]$. If $S$ is a strongly continuous semigroup on $\ell^p(X,m)$, then the operator $A$ with

$$D(A) := \{f \in \ell^p(X,m) \mid \lim_{t\to 0^+} \frac{1}{t}(f - S(t)f) \text{ exists in } \ell^p(X,m)\}$$

and

$$Af := \lim_{t\to 0^+} \frac{1}{t}(f - S(t)f)$$

for $f \in D(A)$ is called the *generator* of $S$.

It follows from general considerations (which we omit here as we do not need them) that for a strongly continuous semigroup $S$ on $\ell^p(X,m)$ with generator $A$ the domain $D(A)$ of $A$ is dense in $\ell^p(X,m)$ and that $A$ is a closed operator. If $S$ is additionally a contraction semigroup, then it can be shown that $A+\alpha$ is a bijection for $\alpha > 0$ with inverse given by

$$(A+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} S(t)\mathrm{d}t.$$

We now introduce resolvents and point out their connections to semigroups.

Definition 10.4 (Resolvents). Let $p \in [1,\infty]$. A map $G\colon (0,\infty) \longrightarrow B(\ell^p(X,m))$ is called a *resolvent* on $\ell^p(X,m)$ if $G$ satisfies the *resolvent identity*

$$G(\alpha) - G(\beta) = -(\alpha-\beta)G(\alpha)G(\beta)$$

for all $\alpha, \beta > 0$. A resolvent $G$ is called *strongly continuous* if

$$\lim_{\alpha\to\infty} \alpha G(\alpha) f = f$$

for all $f \in \ell^p(X,m)$. A resolvent $G$ is called a *contraction resolvent* if

$$\|\alpha G(\alpha)\|_p \le 1$$

for all $\alpha > 0$. Finally, a resolvent $G$ is called a *Markov resolvent* if

$$0 \le \alpha G(\alpha) f \le 1$$

for all $\alpha > 0$ and $f \in \ell^p(X,m)$ with $0 \le f \le 1$.

Of course, our main concern are semigroups and resolvents arising from the self-adjoint operator $L = L^{(D)}$ associated to a graph $(b,c)$ over $(X,m)$ and these have the Markov property.

The Markov property of a semigroup and a resolvent will allow us to extend $S = (e^{-tL})_{t\ge 0}$ to all $\ell^p(X,m)$ spaces for $p \in [1,\infty]$. A crucial ingredient in this extension process is the fact that a Markov matrix defines a bounded operator on $\ell^p(X,m)$ for $p \in [1,\infty]$.

Definition 10.5 (Markov matrix). A function $a\colon X\times X \longrightarrow \mathbb{R}$ is called a *Markov matrix* if $a$ satisfies the following properties:

- $a(x,y) = a(y,x)$
- $a(x,y) \ge 0$
- $\sum_{z\in X} a(x,z)m(z) \le 1$

for all $x,y \in X$.

With this notion we now show that a Markov matrix can be used to define a bounded operator on $\ell^p(X,m)$ for all $p \in [1,\infty]$.

Lemma 10.6 (General bound for a Markov matrix). *Let $a$ be a Markov matrix. Then, for all $f \in C(X)$,*

$$\sup_{x\in X}\sum_{y\in X}|a(x,y)f(y)|m(y) \leq \sup_{x\in X}|f(x)|,$$

*and for $p \in [1,\infty)$,*

$$\sum_{x\in X}\left(\sum_{y\in X}|a(x,y)f(y)|m(y)\right)^p m(x) \leq \sum_{x\in X}|f(x)|^p m(x).$$

*Here, the value $\infty$ is allowed to occur.*

*In particular, for $p \in [1,\infty]$ the matrix $a$ induces a bounded operator $A^{(p)}$ with norm not exceeding $1$ on each $\ell^p(X,m)$ by*

$$A^{(p)}f(x) := \sum_{y\in X} a(x,y)f(y)m(y).$$

Proof. The "in particular" statement is a direct consequence of the inequalities. Thus, it suffices to show these inequalities. The case $p=\infty$ is clear and the case $p=1$ follows easily from Fubini's theorem.

Consider now $p \in (1,\infty)$. Let $q \in (1,\infty)$ satisfy $1/p+1/q=1$. Then, we can estimate

$$\begin{aligned}
&\sum_{x\in X}\left(\sum_{y\in X}a(x,y)|f(y)|m(y)\right)^p m(x)\\
&\quad=\sum_{x\in X}\left(\sum_{y\in X}|a(x,y)m(y)|^{1/q}|a(x,y)m(y)|^{1/p}|f(y)|\right)^p m(x)\\
&\quad\leq\sum_{x\in X}\left(\sum_{y\in X}a(x,y)m(y)\right)^{p/q}\left(\sum_{y\in X}a(x,y)|f(y)|^p m(y)\right)m(x)\\
&\quad\leq\sum_{x\in X}\sum_{y\in X}a(x,y)|f(y)|^p m(y)m(x)\\
&\quad=\sum_{y\in X}|f(y)|^p m(y)\sum_{x\in X}a(x,y)m(x)\\
&\quad\leq\sum_{y\in X}|f(y)|^p m(y),
\end{aligned}$$

where we used Hölder's inequality in the third line, $\sum_{y\in X}a(x,y)m(y) \leq 1$ in the fourth line, Fubini's theorem in the fifth line, and $a(x,y)=a(y,x)$ and $\sum_{x\in X}a(y,x)m(x)\leq 1$ in the last line. The case of $p=1$ follows in a similar manner by using Fubini's theorem. This finishes the proof. $\square$

We need a further piece of notation. Whenever $p,q \in [1,\infty]$ satisfy $1/p+1/q=1$ (where the cases $p=1$, $q=\infty$ and $p=\infty$, $q=1$ are allowed) we can appeal to the Hölder inequality to infer that

$$(f,g) := \sum_{x\in X} f(x)g(x)m(x)$$

exists as an absolutely convergent sum for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. Then, $(\cdot,\cdot)$ is called the *dual pairing* between $\ell^p(X,m)$ and $\ell^q(X,m)$. Of course, for $p = q = 2$ and $f, g \in \ell^2(X,m)$, we just have

$$(f,g) = \sum_{x\in X} f(x)g(x)m(x) = \langle f,g\rangle.$$

THEOREM 10.7 (Extension theorem – semigroups). *Let $(b,c)$ be a graph over $(X,m)$ and $L := L^{(D)}$ the induced self-adjoint operator. Let $S := (e^{-tL})_{t\geq 0}$ be the associated semigroup. Then, there exists a unique family of contraction Markov semigroups $S^{(p)}$ on $\ell^p(X,m)$ for $p \in [1,\infty]$ satisfying the following properties:*

- *$S^{(2)} = S$.* (*"Extension"*)
- *For all $t \geq 0$ and all $p, q \in [1,\infty]$ with $1/p + 1/q = 1$*
$$(S^{(p)}(t)f, g) = (f, S^{(q)}(t)g)$$
*for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$.* (*"Symmetry"*)
- *For all $t \geq 0$ and all $p, q \in [1,\infty]$*
$$S^{(p)}(t)f = S^{(q)}(t)f$$
*for all $f \in \ell^p(X,m) \cap \ell^q(X,m)$.* (*"Consistency"*)

*For $p \in [1,\infty)$, the semigroup $S^{(p)}$ is strongly continuous. For $p = \infty$, the semigroup $S^{(\infty)}$ is pointwise continuous, i.e. the map*

$$t \mapsto S^{(\infty)}(t)f(x)$$

*is continuous for all $f \in \ell^\infty(X,m)$ and $x \in X$.*

REMARK 10.8. The pointwise continuity of $S^{(\infty)}$ can be seen to be equivalent to weak* continuity in the discrete setting (Exercise).

PROOF. We first deal with the *uniqueness statement*. By consistency and the extension property, the semigroups are defined on $C_c(X)$. As $C_c(X)$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$ and all $S^{(p)}$ are bounded, this shows that the semigroups are uniquely determined on $\ell^p(X,m)$ for $p \in [1,\infty)$. For $p = \infty$, we note that the semigroup on $\ell^\infty(X,m)$ is uniquely determined by the semigroup on $\ell^1(X,m)$ by the symmetry condition.

We now turn to proving *existence*. By Corollary 5.6, $S = (e^{-tL})_{t\geq 0}$ is a Markov semigroup of self-adjoint operators on $\ell^2(X,m)$. Now, for every $t \geq 0$, there exists a $p_t\colon X \times X \longrightarrow \mathbb{R}$ with

$$S(t)f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y)$$

for all $f \in \ell^2(X,m)$. This $p$ is called the *heat kernel* of the semigroup of $S$. We will now show that $p_t$ is a Markov matrix for every $t \geq 0$.

First, note that by direct calculation $S(t)1_y(x) = p_t(x,y)m(y)$. As $S(t)$ is self-adjoint, we get

$$p_t(x,y)m(y)m(x) = \langle S(t)1_y, 1_x\rangle = \langle 1_y, S(t)1_x\rangle = p_t(y,x)m(x)m(y)$$

so that $p_t(x,y) = p_t(y,x)$ for all $x,y \in X$ and $t \geq 0$.

Since $S(t)$ is Markov, it follows that

$$0 \leq S(t)1_y(x) = p_t(x,y)m(y).$$

Therefore, $p_t(x,y) \geq 0$ for all $x,y \in X$ and $t \geq 0$.

Finally, for $n \in \mathbb{N}$ let $K_n \subseteq X$ be finite such that $K_n \subseteq K_{n+1}$ and $X = \bigcup_{n\in\mathbb{N}} K_n$. It follows by the Markov property that $0 \leq S(t)1_{K_n} \leq 1$ and thus

$$0 \leq \sum_{y\in X} p_t(x,y)1_{K_n}(y)m(y) = \sum_{y\in K_n} p_t(x,y)m(y) \leq 1$$

for all $n \in \mathbb{N}$, $x \in X$ and $t \geq 0$. By the monotone convergence theorem

$$\sum_{y\in X} p_t(x,y)m(y) \leq 1$$

for every $x \in X$ and $t \geq 0$.

Hence, for every $t \geq 0$, $p_t$ is a Markov matrix and Lemma 10.6 gives for any $p \in [1,\infty]$ that the operator $S^{(p)}(t)\colon \ell^p(X,m) \longrightarrow \ell^p(X,m)$ given by

$$S^{(p)}(t)f(x) := \sum_{y\in X} p_t(x,y)f(y)m(y)$$

is bounded with norm not exceeding 1. We now show that these operators have the desired properties.

*Markov property.* As $p_t$ is a Markov matrix for every $t \geq 0$, each operator $S^{(p)}(t)$ satisfies

$$0 \leq S^{(p)}(t)f \leq 1$$

whenever $0 \leq f \leq 1$ for $f \in \ell^p(X,m)$.

*Consistency.* By definition, we have

$$S^{(p)}(t)f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y) = S^{(q)}(t)f(x)$$

for all $t \geq 0$ whenever $f \in \ell^p(X,m) \cap \ell^q(X,m)$.

*$S^{(2)} = S$.* This is clear from the definition of $S^{(p)}$.

*Semigroup property for $S^{(p)}$.* By the consistency of the family the space

$$\mathcal{C} := \bigcap_{p\in[1,\infty]} \ell^p(X,m)$$

is invariant under any $S^{(p)}(t)$ for $t \geq 0$ and $p \in [1,\infty]$. Moreover, the action of $S^{(p)}$ on $\mathcal{C}$ agrees with the action of $S$. As $S$ satisfies $S(0) = I$ and $S(s)S(t) = S(s+t)$ for all $s,t \geq 0$, the same will hold for $S^{(p)}$ on $\mathcal{C}$. As $\mathcal{C}$ contains $C_c(X)$, the space $\mathcal{C}$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$. Then, the semigroup property follows on $\ell^p(X,m)$ for $p \in [1,\infty)$ as each $S^{(p)}(t)$ is a bounded operator. To deal with the case $p = \infty$, it suffices to consider $f \geq 0$. Any such function can be written as a monotone limit of functions in $C_c(X)$. By the Markov property, the operators $S^{(\infty)}(t)$ on $\ell^\infty(X,m)$ are compatible with monotone limits and the desired statement follows.

*Symmetry.* The symmetry property is clear for $f,g \in C_c(X)$. It then follows in the generality stated by approximating $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$, where $1/p + 1/q = 1$, by sequences $(f_n)$ and $(g_n)$ in $C_c(X)$.

*Strong continuity for $p \in [1,\infty)$.* From the strong continuity of $S$ on $\ell^2(X,m)$ we infer pointwise continuity of $p_t$ for $t \to 0^+$ in the sense that we have

$$p_t(x,y) = \frac{1}{m(y)} e^{-tL} 1_y(x) \to \frac{1}{m(y)} 1_y(x)$$

as $t \to 0^+$ for every $x, y \in X$.

We now treat the general case and let $f \in \ell^p(X,m)$ for $p \in [1,\infty)$. In order to simplify the notation we set

$$u_t(x) := S^{(p)}(t)f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y).$$

Let $\varepsilon > 0$. Since $C_c(X)$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$ we can choose as finite subset $K \subseteq X$ with

$$\|(1-1_K)f\|_p^p < \varepsilon$$

so that

$$\|1_K f\|_p^p = \|f\|_p^p - \|(1-1_K)f\|_p^p > \|f\|_p^p - \varepsilon.$$

For $t$ sufficiently close to 0, we then infer from the pointwise continuity and the finiteness of $K$ that

$$\|1_K(u_t - f)\|_p^p < \varepsilon \text{ and } \|1_K u_t\|_p^p \geq \|f\|_p^p - \varepsilon.$$

Therefore, combining with the above, we obtain

$$\|1_K u_t\|_p^p > \|f\|_p^p - 2\varepsilon.$$

Moreover, as each $S^{(p)}(t)$ has norm not exceeding 1 we also have

$$\|u_t\|_p^p \leq \|f\|_p^p.$$

Therefore, for small enough $t > 0$, we infer from the last two inequalities

$$\|f\|_p^p \geq \|u_t\|_p^p = \|1_K u_t\|_p^p + \|(1-1_K)u_t\|_p^p > \|f\|_p^p - 2\varepsilon + \|(1-1_K)u_t\|_p^p.$$

Hence,

$$\|(1-1_K)u_t\|_p^p < 2\varepsilon.$$

This gives the desired continuity at 0 as

$$\|u_t - f\|_p \leq \|(1-1_K)u_t\|_p + \|1_K(u_t - f)\|_p + \|(1-1_K)f\|_p.$$

*Pointwise continuity for $p = \infty$.* This follows from the strong continuity for $p = 1$ and the symmetry of the family. □

From the preceding discussion we infer the following: Whenever $(b,c)$ is a graph over $(X,m)$ and $L := L^{(D)}$ is the associated operator and $p$ is the heat kernel then

$$S^{(p)}(t)f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y)$$

holds for all $p \in [1,\infty]$ and $f \in \ell^p(X,m)$, $t \geq 0$ and $x \in X$. For this reason we will subsequently — by a slight abuse of notation — write

$$e^{-tL}f(x) := \sum_{y \in X} p_t(x,y)f(y)m(y) = S^{(p)}(t)f(x)$$

whenever $f$ belongs to $\ell^p(X,m)$ for some $p \in [1,\infty]$, $t \geq 0$ and $x \in X$.

As one can extend the semigroups to all $\ell^p(X,m)$ one can also extend the resolvents to all $\ell^p(x,m)$. We omit the details but rather state the result and give some indication of the proof.

THEOREM 10.9 (Extension theorem – resolvents). *Let $(b,c)$ be a graph over $(X,m)$ and $L := L^{(D)}$ the associated self-adjoint operator. Let $G(\alpha) := (L+\alpha)^{-1}$ be the resolvent of $L$ for $\alpha > 0$. Then, there exists a unique family of contraction Markov resolvents $G^{(p)}$ on $\ell^p(X,m)$ for $p \in [1,\infty]$ satisfying the following properties:*

- *$G^{(2)} = G$.* *("Extension")*
- *For all $\alpha > 0$ and all $p,q \in [1,\infty]$ with $1/p + 1/q = 1$*
$$(G^{(p)}(\alpha)f, g) = (f, G^{(q)}(\alpha)g)$$
*for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$.* *("Symmetry")*
- *For all $\alpha > 0$ and all $p,q \in [1,\infty]$*
$$G^{(p)}(\alpha)f = G^{(q)}(\alpha)f$$
*for $f \in \ell^p(X,m) \cap \ell^q(X,m)$.* *("Consistency")*

*For $p \in [1,\infty)$, the resolvent $G^{(p)}$ is strongly continous. For $p = \infty$, the resolvent $G^{(\infty)}$ ispointwise continuous, i.e., the map*
$$\alpha \mapsto \alpha G^{(\infty)}(\alpha)f(x)$$
*is continuous for all $f \in \ell^\infty(X,m)$ and $x \in X$.*

*If $S^{(p)}$ is the contraction Markov semigroup with generator $L^{(p)}$, then $G^{(p)}$ satisfies*
$$G^{(p)}(\alpha)f(x) = \int_0^\infty e^{-t\alpha} S^{(p)}(t)f(x)\mathrm{d}t$$
*for all $\alpha > 0$, $p \in [1,\infty]$, and, $f \in \ell^p(X,m)$ and $x \in X$.*
*( "Laplace transform" )*

IDEA OF THE PROOF. Uniqueness is clear from the given properties. For existence we define the resolvent, for $\alpha > 0$, $p \in [1,\infty]$, $f \in \ell^p(X,m)$ and $x \in X$ by
$$G^{(p)}(\alpha)f(x) = \int_0^\infty e^{-t\alpha} S^{(p)}(t)f(x)\mathrm{d}t$$
and show the claimed properties. □

THEOREM 10.10 (Resolvents as minimal solutions to $(\mathcal{L}+\alpha)u = f$). *Let $(b,c)$ be a graph over $(X,m)$ and let $L = L^{(D)}$ be the associated self-adjoint operator. Let $p \in [1,\infty]$ and let $G^{(p)}$ be the resolvent on $\ell^p(X,m)$ associated to $L$. If $f \in \ell^p(X,m)$, $\alpha > 0$ and*
$$u := G^{(p)}(\alpha)f,$$
*then $u$ belongs to $\mathcal{F}$ and satisfies the Poisson equation*

*("Poisson equation")* $$(\mathcal{L}+\alpha)u := f.$$

*Furthermore, if additionally $f \geq 0$, then $u \geq 0$ and for the resolvent $G_{b,c,m}$ associated to $L$ we have that*
$$u = G^{(p)}_{b,c,m}(\alpha)f$$

*is the smallest* $v \in \mathcal{F}$ *with* $v \geq 0$ *and* $(\mathcal{L}+\alpha)v \geq f$.

PROOF. Let $f \in \ell^p(X,m)$ for $p \in [1,\infty]$. Without loss of generality, we assume $f \geq 0$. For a general $f \in \ell^p(X,m)$, we can decompose $f = f_+ - f_-$ into its positive and negative part.

Let $(K_n)$ be an increasing sequence of finite subsets of $X$ with $X = \bigcup_{n\in\mathbb{N}} K_n$ and let $f_n := f 1_{K_n}$ so that $f_n \in C_c(X)$ for all $n \in \mathbb{N}$. Let $\alpha > 0$ and

$$u_n := G^{(p)}(\alpha) f_n, \quad n \in \mathbb{N}.$$

As $G^{(p)}$ is Markov by Theorem 10.9, $u_n \geq 0$ for all $n \in \mathbb{N}$ and the sequence $(u_n)$ is monotonically increasing and converges to $u := G^{(p)}(\alpha) f \in \ell^p(X,m)$ by the monotone convergence theorem.

As the resolvents agree on their common domain due to the consistency statement in Theorem 10.9 and $f_n \in C_c(X) \subseteq \ell^p(X,m)$ for all $p \in [1,\infty]$, we have

$$u_n = G^{(p)}(\alpha) f_n = G^{(2)}(\alpha) f_n$$

for all $n \in \mathbb{N}$. By definition we have $G^{(2)}(\alpha) = (L+\alpha)^{-1}$ and since $L = \mathcal{L}$ on $D(L) = G^{(2)}(\alpha)\ell^2(X,m) \supseteq G^{(2)} C_c(X)$ we get

$$(\mathcal{L}+\alpha)u_n = (L+\alpha)G^{(2)}(\alpha) f_n = f_n$$

for all $n \in \mathbb{N}$. We conclude that $u \in \mathcal{F}$ solves the Poisson equation by taking monotone limits, Lemma 2.11. Furthermore, $u = G^{(p)}(\alpha) f \geq 0$ for $f \geq 0$ since $G^{(p)}$ is Markov and, therefore, positivity preserving by Theorem 10.9.

Now, let $v \in \mathcal{F}$ with $v \geq 0$ satisfy $(\mathcal{L}+\alpha)v \geq f$. Then, $(\mathcal{L}+\alpha)v \geq f_n$ for all $n \in \mathbb{N}$ and as $u_n := G^{(2)}_{b,c,m}(\alpha) f_n$ is the minimum positive solution of this inequality by Lemma 5.8 we obtain $u_n \leq v$ for all $n \in \mathbb{N}$. Taking the limit gives $u \leq v$. □

## 10.2. The Heat Equation on $\ell^\infty$

In this section we show that $t \mapsto e^{-tL} f$ provides a solution to the heat equation for bounded $f$.

Let $(b,c)$ be a graph over $(X,m)$. A function $v\colon [0,\infty) \longrightarrow \mathbb{R}$ is called a *supersolution* to the heat equation with initial condition $f$ if $t \mapsto v_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ for all $x \in X$, and

$$-\mathcal{L} v_t \geq \partial_t v$$

holds for all $t > 0$ and $v_0 = f$ is valid.

THEOREM 10.11 (Existence of bounded solutions of the heat equation). *Let* $(b,c)$ *be a graph over* $(X,m)$ *with associated self-adjoint operator* $L := L^{(D)}$ *and let* $f \in \ell^\infty(X)$ *be given. Then,*

$$u\colon [0,\infty) \times X \longrightarrow \mathbb{R}, \quad u_t(x) := e^{-tL} f(x),$$

*is a bounded solution of the heat equation with initial condition* $f$.

*Furthermore, if additionally* $f \geq 0$, *then* $u$ *is the smallest positive supersolution of the heat equation with initial condition greater than or equal to* $f$.

PROOF. We start by showing the continuity and boundedness of $u$. We denote the dual pairing between $\ell^1(X,m)$ and $\ell^\infty(X,m)$ by $(\cdot,\cdot)$ and for $x \in X$ let $\eta_x \in \ell^1(X,m)$ be given by $\eta_x := \frac{1}{m(x)}1_x$. Since

$$u_t(x) = (\eta_x, e^{-tL}f)$$

for $x \in X$ and $t \geq 0$, continuity of the function $t \mapsto u_t(x)$ for $t \geq 0$ and $x \in X$ follows from the weak$^*$ continuity of the semigroup on $\ell^\infty(X,m)$ established in Theorem 10.7. Furthermore, as the semigroup on $\ell^1(X,m)$ is strongly continuous, we have $u_0 = f$. Finally, as $(e^{-tL})_{t\geq 0}$ is a contraction semigroup on $\ell^\infty(X,m)$ by Theorem 10.7, it follows that $u_t$ is bounded by $\|f\|_\infty$ for every $t \geq 0$. In particular, $u_t \in \mathcal{F}$ for every $t \geq 0$.

As an intermediate step, we next show the continuity of

$$t \mapsto \mathcal{L}u_t(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u_t(x) - u_t(y)) + c(x)u_t(x)\right)$$

on $[0,\infty)$ for every $x \in X$. Indeed, this is immediate from the continuity of $t \mapsto u_t(y)$ for each $y \in X$, the uniform boundedness of $u$ in both variables, and the summability of $b(x,\cdot)$ for every $x \in X$.

We will now show differentiability and the fact that $u$ satisfies the heat equation. We will do so by approximating $f$ by functions with finite support and using that the heat equation holds for functions in $\ell^2(X,m)$ and, hence, for functions with finite support.

We note that the preceding considerations hold for any bounded function $f$ and, in particular, for the elements of the approximating sequence $(f_n)$ and $u_t^{(n)} := e^{-tL}f_n$ which we introduce below. Hence, the functions $t \mapsto u_t(x), t \mapsto u_t^{(n)}(x)$, $t \mapsto \mathcal{L}u_t(x)$ and $t \mapsto \mathcal{L}u_t^{(n)}(x)$ will be continuous for all $x \in X$ and all $n \in \mathbb{N}$.

Let $t > 0$. By decomposing $f$ into positive and negative parts, we can assume without loss of generality that $f$ is positive. Let $(K_n)$ be a sequence of finite increasing subsets of $X$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$. Furthermore, let $f_n := f1_{K_n}$ and let

$$u_t^{(n)} := e^{-tL}f_n, \quad n \in \mathbb{N}.$$

Since $(e^{-tL})_{t\geq 0}$ on $\ell^\infty(X,m)$ is a bounded Markov semigroup by Theorem 10.7, $(e^{-tL})_{t\geq 0}$ admits a positive kernel $p$. That is,

$$e^{-tL}f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y), \quad x,y \in X, t \geq 0,$$

where $p_t(x,y) \geq 0$ for all $x,y \in X$ and $t \geq 0$. Thus, $u_t^{(n)}(x) \nearrow u_t(x)$ as $n \to \infty$ for all $x \in X$ and $t > 0$. Moreover, the convergence is uniform on compact subintervals of $(0,\infty)$ by Dini's theorem as $(t \mapsto u_t^{(n)}(x))_n$ in $C([0,\infty))$ and $t \mapsto u_t(x)$ is continuous.

Since $f_n \in C_c(X) \subseteq \ell^2(X,m)\cap\ell^\infty(X,m)$ and the semigroup on $\ell^\infty(X,m)$ agrees with the semigroup on $\ell^2(X,m)$ for functions in $\ell^2(X,m)\cap\ell^\infty(X,m)$ by Theorem 10.7, it follows that $u_t^{(n)} \in \ell^2(X,m)$. Therefore, for all $x \in X$,

$t \geq 0$ and $n \in \mathbb{N}$, we infer by Lemma 5.9 that

$$\begin{aligned}\partial_t u_t^{(n)}(x) &= -Lu_t^{(n)}(x) = -\mathcal{L}u_t^{(n)}(x)\\ &= -\frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u_t^{(n)}(x) - u_t^{(n)}(y)) + c(x)u_t^{(n)}(x)\right).\end{aligned}$$

Monotone convergence of $(u_t^{(n)}(y))_n$ to $u_t(y)$ for all $y \in X$ and $t \geq 0$, and the fact that $u_t \in \ell^\infty(X) \subseteq \mathcal{F}$, yields the convergence of the right-hand side to $-\mathcal{L}u_t(x)$ as $n \to \infty$ for each $x \in X$ and $t \geq 0$. Therefore, we obtain the convergence of $(\partial_t u_t^{(n)}(x))_n$ to $\mathcal{L}u_t(x)$ for each $x \in X$ and $t > 0$. In fact, this convergence is uniform in $t$ on compact subintervals of $(0,\infty)$ as the convergence of $(t \mapsto u_t^{(n)}(y))_n$ to $t \mapsto u_t(y)$ is uniform on compact subintervals of $(0,\infty)$ for each $y \in X$ and $b(x,\cdot)$ is summable for each $x \in X$.

Altogether we have established that $(t \mapsto u_t^{(n)}(x))_n$ converges uniformly on compact subintervals of $(0,\infty)$ to $t \mapsto u_t(x)$ and $(t \mapsto \partial_t u_t^{(n)}(x))_n$ converges uniformly on compact subintervals of $(0,\infty)$ to $t \mapsto -\mathcal{L}u_t(x)$ for each $x \in X$. As discussed above, all involved functions are continuous. Thus, this gives that $t \mapsto u_t(x)$ is differentiable for all $x \in X$ with the desired derivative.

It remains to show the last statement of the theorem. That $u$ is positive whenever $f$ is positive follows immediately from the fact that the semigroup on $\ell^\infty(X,m)$ is Markov and, in particular, positivity preserving by Theorem 10.7. We now show the minimality statement. Let $w$ be a supersolution of the heat equation with initial condition greater than or equal to $f$. From what we have shown above, $u^{(n)}$ satisfies

$$(\mathcal{L} + \partial_t)u_t^{(n)} = 0$$

for $t > 0$ and $u_0^{(n)} = f_n$ for $n \in \mathbb{N}$. Furthermore, the $u^{(n)}$ agree with the solution generated by the semigroup on $\ell^2(X,m)$ as the semigroups agree on their common domain by Theorem 10.7. As $w$ is a positive supersolution with initial condition greater than or equal to $f_n$ we obtain $u^{(n)} \leq w$ for all $n$ by Lemma 5.9. Letting $n \to \infty$ gives $u \leq w$ which completes the proof. □

PROPOSITION 10.12 (Heat equation at $t = 0$). *Let $(b,c)$ be a graph over $(X,m)$ and let $u$ be a bounded solution of the heat equation. Then, the function*

$$[0,\infty) \longrightarrow \mathbb{R}, t \mapsto \mathcal{L}u_t(x),$$

*is continuous for all $x \in X$, the limit*

$$\partial_t u_t(x)|_{t=0} = \lim_{h\to 0^+} \frac{u_h(x) - u_0(x)}{h}$$

*exists for all $x \in X$ and*

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

*for all $x \in X$ and $t \geq 0$.*

PROOF. Let $x \in X$. We first show the continuity of $t \mapsto \mathcal{L}u_t(x)$ on $[0,\infty)$: As $u$ is a solution of the heat equation the map $t \mapsto u_t(y)$ is continuous for each $y \in X$. As $b(x,\cdot)$ is summable and $u$ is bounded, we then obtain that

$$t \mapsto \mathcal{L}u_t(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u_t(x)-u_t(y)) + c(x)u_t(x)\right)$$

is continuous on $[0,\infty)$. Moreover, as $u$ is a solution of the heat equation, we have

$$\partial_t u_t(x) = -\mathcal{L}u_t(x)$$

for all $t > 0$.

Altogether, for each $x \in X$, the function $t \mapsto u_t(x)$ is a continuous function on $[0,\infty)$ which is differentiable on $(0,\infty)$ and $t \mapsto -\mathcal{L}u_t(x)$ is a continuous function on $[0,\infty)$ which agrees with the derivative of $t \mapsto u_t(x)$ on $(0,\infty)$. This implies that $t \mapsto u_t(x)$ is differentiable at $t = 0$ with derivative given by $-\mathcal{L}u_0(x)$: Indeed, fix $x \in X$. By the mean value theorem, for every $h > 0$, there exists a $\zeta(h) \in (0,h)$ such that

$$\frac{u_h(x)-u_0(x)}{h} = \partial_t u_t|_{t=\zeta(h)} = -\mathcal{L}u_{\zeta(h)}(x).$$

This implies

$$\lim_{h\to 0^+}\frac{u_h(x)-u_0(x)}{h} = -\mathcal{L}u_0(x)$$

by continuity. □

The next proposition connects bounded solutions of an inhomogeneous heat equation with solutions of the Poisson equation.

PROPOSITION 10.13 (Solutions of heat and Poisson equations). *Let $(b,c)$ be a graph over $(X,m)$ and $L := L^{(D)}$ the associated operator. Let $f,g \in \ell^\infty(X,m)$ and let $u\colon [0,\infty)\times X \longrightarrow \mathbb{R}$ be a bounded solution of*

$$(\mathcal{L}+\partial_t)\, u_t = f$$

*with initial condition $u_0 = g$. Then, for $\alpha > 0$ the function*

$$v := \int_0^\infty \alpha e^{-t\alpha} u_t \mathrm{d}t$$

*is bounded and satisfies*

$$(\mathcal{L}+\alpha)v = f + \alpha g.$$

*Moreover, if additionally $f,g \geq 0$, then*

$$w := \int_0^\infty e^{-t\alpha}e^{-tL}(f+\alpha g)\mathrm{d}t$$

*is the smallest positive function $w \in \mathcal{F}$ with $(\mathcal{L}+\alpha)w \geq f+\alpha g$. In particular,*

$$\int_0^\infty e^{-t\alpha}e^{-tL}(f+\alpha g)\mathrm{d}t \leq \int_0^\infty \alpha e^{-t\alpha}u_t\mathrm{d}t.$$

PROOF. The boundedness of $v$ follows since we assume that $u$ is bounded and since $[0,\infty) \ni t \mapsto \alpha e^{-t\alpha}$ is a probability density function.

Furthermore, by the boundedness of $u$ and Fubini's theorem, we have for all $x \in X$

$$\mathcal{L}v(x) = \int_0^\infty \alpha e^{-t\alpha} \mathcal{L}u_t(x)\mathrm{d}t = \lim_{T\to\infty} \int_0^T \alpha e^{-t\alpha}\mathcal{L}u_t(x)\mathrm{d}t.$$

Since $u$ satisfies $\mathcal{L}u_t = -\partial_t u_t + f$, we infer

$$\begin{aligned}\mathcal{L}v(x) &= \lim_{T\to\infty} \int_0^T \alpha e^{-t\alpha}(-\partial_t u_t(x))\mathrm{d}t + \int_0^\infty \alpha e^{-t\alpha} f(x)\mathrm{d}t \\ &= \lim_{T\to\infty} \left( -\alpha e^{-t\alpha} u_t(x)|_0^T - \int_0^T \alpha^2 e^{-t\alpha} u_t(x)\mathrm{d}t \right) + f(x),\end{aligned}$$

where we used integration by parts and again the fact that $[0,\infty) \ni t \mapsto \alpha e^{-t\alpha}$ is a probability density function. Next, we conclude from the boundedness of $u$ and $u_0 = g$ that the first term tends to $\alpha g$ and, therefore,

$$\begin{aligned}\mathcal{L}v(x) &= \alpha g(x) - \alpha \int_0^\infty \alpha e^{-t\alpha} u_t(x)\mathrm{d}t + f(x) \\ &= \alpha g(x) - \alpha v(x) + f(x)\end{aligned}$$

by the definition of $v$. Therefore, $v$ is bounded and satisfies $(\mathcal{L}+\alpha)v = f + \alpha g$.

If $f, g \in \ell^\infty(X,m)$ and $x \in X$, then

$$\int_0^\infty e^{-t\alpha} e^{-tL}(f+\alpha g)(x)\mathrm{d}t = (L+\alpha)^{-1}(f+\alpha g)(x)$$

by the Laplace transform formula, see Theorem 10.9. Furthermore, if $f$ and $g$ are positive, Theorem 10.10 gives that $(L+\alpha)^{-1}(f+\alpha g)$ is the minimal positive function $w \in \mathcal{F}$ with $(\mathcal{L}+\alpha)w \geq f + \alpha g$. As $v$ is such a function by what we have shown above, $(L+\alpha)^{-1}(f+\alpha g) \leq v$ follows. □

COROLLARY 10.14 (Solutions of the heat equation and $\alpha$-harmonic functions). *Let $(b,c)$ be a graph over $(X,m)$ and let $u$ be a bounded solution of the heat equation with $u_0 = 0$. Then, for $\alpha > 0$ the function*

$$v := \int_0^\infty e^{-t\alpha} u_t \mathrm{d}t$$

*is bounded and satisfies*

$$(\mathcal{L}+\alpha)v = 0.$$

*In particular, if there exists a positive non-trivial bounded solution of the heat equation with trivial initial conditions, then there exists a positive non-trivial bounded $\alpha$-harmonic function for any $\alpha > 0$.*

PROOF. This follows immediately from Proposition 10.13 by letting $f$ and $g$ be 0. □

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# Sheet 11

## Stochastic completeness

**Exercise 1 (Semigroups with bounded generators on $\ell^p(X,m)$)**

Let $p \in [1,\infty]$ and $A \in B(\ell^p(X,m))$. For $t \in \mathbb{R}$ define

$$e^{-tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

(a) Show that $(e^{-tA})_{t\geq 0}$ is a strongly continuous semigroup.

(b) Show that $(e^{-tA})_{t\geq 0}$ is even norm continuous, i.e. $t \mapsto e^{-tA} \in B(\ell^p(X,m))$ is continuous.

(c) Show that $t \mapsto e^{-tA} \in B(\ell^p(X,m))$ is continuously differentiable with derivative $t \mapsto -Ae^{-tA}$.

(d) Show that for $t \in \mathbb{R}$ we have that $e^{-tA}$ is invertible with inverse $e^{tA}$.

**Exercise 2 (Norm continuous semigroups on $\ell^p(X,m)$)**

Let $p \in [1,\infty]$ and $(S(t))_{t\geq 0}$ a norm continuous semigroup on $\ell^p(X,m)$, i.e. $t \mapsto S(t) \in B(\ell^p(X,m))$ is continuous. Let $A$ be the generator of $(S(t))_{t\geq 0}$. Show that $A \in B(\ell^p(X,m))$.

Hint: You can use the following fact without proof: For $f \in \ell^p(X,m)$ and $t > 0$ we have $\int_0^t S(s)f \,\mathrm{d}s \in D(A)$ and

$$A \int_0^t S(s)f \,\mathrm{d}s = S(t)f - f.$$

**Exercise 3**

Let $(X,m)$ be a discrete measure space. Let $(S(t))_{t\geq 0}$ be a positivity preserving semigroup on $\ell^\infty(X)$, i.e., for all $t \geq 0$ we have that $f \geq 0$ implies $S(t)f \geq 0$. Show that the following assertions are equivalent:

(i) $(S(t))_{t\geq 0}$ is weak*-continuous, i.e., for all $g \in \ell^1(X,m)$ and all $f \in \ell^\infty(X)$ the map

$$[0,\infty) \to \mathbb{R}, \quad t \mapsto \sum_{x\in X} g(x)S(t)f(x)m(x)$$

is continuous.

(ii) For all $f \in \ell^\infty(X)$ and all $x \in X$ the map

$$[0,\infty) \to \mathbb{R}, \quad t \mapsto S(t)f(x)$$

is continuous.

**Exercise 4 (Graphs of finite measure)** 4 points

Let $b$ be a connected graph over $(X,m)$. Suppose that $m$ satisfies $m(X) < \infty$. Show that the following statements are equivalent:

(i) $b$ is recurrent.

(ii) $e^{-tL^{(D)}}1 = 1$.

(iii) $Q^{(D)} = Q^{(N)}$.