**26th Internet Seminar on Evolution Equations**

# Graphs and Discrete Dirichlet Spaces

Matthias Keller, Daniel Lenz, Marcel Schmidt and Christian Seifert

## Lecture 10

CHAPTER 8

# Agmon–Allegretto–Piepenbrink Theorem

In Chapter 7 we have seen a geometric approach to the infimum $\lambda_0(L)$ of the spectrum of a graph Laplacian $L = L^{(D)}$. Here, we consider a functional analytic characterization. Specifically, we show that $\lambda_0(L)$ can be characterized by the following property: There exists a strictly positive $u \in \mathcal{F}$ with

$$(\mathcal{L} - \lambda)u \geq 0$$

if and only if $\lambda \leq \lambda_0(L)$. This theorem is known as Agmon–Allgretto–Piepenbrink theorem.

Let $X$ be a countable set and $m$ a measure on $X$ with full support.

## 8.1. Local Harnack inequality

Given a graph $(b, c)$ over $(X, m)$ and $\lambda \in \mathbb{R}$, our aim is to study $u \in \mathcal{F}$ with

$$(\mathcal{L} - \lambda)u \geq 0.$$

Such a $u$ is known as a *supersolution* to $\mathcal{L} - \lambda$. Here, we first show that any such solution satisfies certain bounds, known as local Harnack inequality.

THEOREM 8.1 (Strict positivity and local Harnack inequality). *Let $(b, c)$ be a connected graph over $(X, m)$, $\lambda \in \mathbb{R}$. Then, any positive supersolution $u \in \mathcal{F}$ to $(\mathcal{L} - \lambda)u \geq 0$ is strictly positive. Moreover, to any $x, x^* \in X$ there exists a decreasing function $C_{x,x^*} : \mathbb{R} \to [0, \infty)$ such that for every $\lambda \in \mathbb{R}$ and every $u \in \mathcal{F}$ with to $(\mathcal{L} - \lambda)u \geq 0$ the inequality*

$$u(x) \leq C_{x,x^*}(\lambda)u(x^*)$$

*holds.*

PROOF. Let $u$ be a positive supersolution to $\lambda$. Assume that $u$ is nontrivial (as for trivial $u$ there is nothing to show). By definition of $\mathcal{L}$ the inequality

$$(\mathcal{L} - \lambda)u \geq 0$$

reads as

$$\frac{1}{m(x)} \sum_{y \in X} b(x, y)(u(x) - u(y)) + \frac{c(x)}{m(x)} u(x) - \lambda u(x) \geq 0$$

for all $x \in X$. This implies

$$(\deg(x) - \lambda m(x))u(x) \geq \sum_{y \in X} b(x, y)u(y) \geq 0$$

with

$$\deg(x) = \sum_{y \in X} b(x,y) + c(x).$$

Since $u$ is positive, so must be $(\deg - \lambda m)$. Furthermore, if $(\deg - \lambda m)(x) = 0$ or $u(x) = 0$ then the above inequality gives $u(y) = 0$ for all $y \sim x$. By connectedness of the graph we then deduce by induction $u = 0$ which contradicts non-triviality of $u$. Hence, $(\deg - \lambda m)(x) > 0$ and and $u(x) > 0$ and

$$u(x) \geq \frac{\sum_{y \in X} b(x,y)}{(\deg(x) - \lambda m(x))} u(y)$$

hold for all $x \in X$.

Now, chose an arbitrary path $x_0, \ldots, x_n$ connecting $x$ and $x^*$. From what we have shown already we infer

$$\begin{aligned} u(x_0) &\geq \frac{\sum_{y \in X} b(x,y)}{(\deg(x) - \lambda m(x))} u(y) \geq \frac{b(x_0, x_1)}{(\deg(x_0) - \lambda m(x_0))} u(x_1) \\ &\geq \left( \prod_{j=0}^{n-1} \frac{b(x_j, x_{j+1})}{(\deg(x_j) - \lambda m(x_j))} \right) u(x_n). \end{aligned}$$

Letting

$$C_{x,x^*}(\lambda) := \prod_{j=0}^{n-1} \frac{(\deg(x_j) - \lambda m(x_j))}{b(x_j, x_{j+1})},$$

we find the desired inequality for any $\lambda$ with $\lambda < \deg(x)/m(x)$ for all $x \in X$. For $\lambda \geq \inf_{x \in X} \frac{\deg(x)}{m(x)}$ there is no supersolution and we can choose $C_{x,x^*}(\lambda) := 0$. Clearly, the expression for $C_{x,x^*}(\lambda)$ is decreasing in $\lambda$. □

Remark 8.2. As noted in the proof there are no positive non-trivial supersolutions for $\lambda \geq \inf_{x \in X} \deg(x)/m(x)$.

Remark 8.3. If there is a positive supersolution $u$ for some $\lambda$ then for all $\lambda' \leq \lambda$ we have

$$(\mathcal{L} - \lambda')u \geq (\mathcal{L} - \lambda)u \geq 0.$$

Hence, if there exists a non-trivial positive supersolution $u$ for $\lambda$ then $u$ is a supersolution for every $\lambda' \in (-\infty, \lambda)$. In the next proposition we show that if there are supersolutions for all $\lambda' \in (-\infty, \lambda)$ then there is also a supersolution for $\lambda$.

Proposition 8.4 (Harnack principle). *Let $(b, c)$ be a connected graph over $(X, m)$. Let $\lambda \in \mathbb{R}$, $(\lambda_n)$ in $\mathbb{R}$ such that $\lambda_n \to \lambda$ and assume there are positive non-trivial supersolutions $u_n$ to $\lambda_n$ with $u_n(o) = 1$ for some $o \in X$ and all $n \in \mathbb{N}$. Then, there is a subsequences $(u_{n_k})$ of $(u_n)$ which converges pointwise to a positive non-trivial supersolution $u$ for $\lambda$.*

Proof. Set $\widetilde{\lambda} := \inf_{n\in\mathbb{N}} \lambda_n$. Since $u_n(o) = 1$ for all $n \in \mathbb{N}$ we infer from the local Harnack inequality

$$0 \leq u_n(x) \leq C_{x,o}(\lambda_n) \leq C_{x,o}(\widetilde{\lambda})$$

for any $x \in X$ and all $n \in \mathbb{N}$. Thus, for any $x \in X$ the sequence $(u_n(x))_n$ is contained in $[0, C_{x,o}(\widetilde{\lambda})]$. As $X$ is countable, $(u_n)$ has a subsequence $(u_{n_k})$ that converges pointwise to some $u$. Furthermore, we have by Fatou's lemma applied to $(\sum_{y\in X} b(x,y)u_{n_k}(y))_k$

$$0 \leq \lim_{k\to\infty} (\mathcal{L} - \lambda_{n_k})u_{n_k}(x) \leq (\mathcal{L} - \lambda)u(x)$$

for all $x \in X$. Thus, $u$ is a supersolution to $\lambda$. Since the $u_{n_k}$ are positive and $u_{n_k}(o) = 1$ for all $k \in \mathbb{N}$, we have that $u$ is positive and $u(o) = 1$. □

## 8.2. The Ground State Transform

On the intuitive level the ground state transform is a tool to convert a graph $(b, c)$ into a new graph $(b_u, 0)$ (with $b_u(x, y) = b(x, y)u(x)u(y)$) provided $u$ is a strictly positive (super)solution to $(\mathcal{L} - \lambda)u = 0$ for some $\lambda \in \mathbb{R}$. Precise versions (which give even more general statements) can be given both on the level of operators and on the level of forms. This is useful in order to obtain lower bounds.

We start with some notation. Let $u \in C(X)$ with $u > 0$ be given. Then, we denote $\mathcal{T}_u : C(X) \to C(X)$ by

$$\mathcal{T}_u f := uf.$$

Also, for $b: X \times X \longrightarrow [0, \infty)$ we define

$$b_u : X \times X \longrightarrow [0, \infty), \quad b_u(x, y) := b(x, y)u(x)u(y).$$

We next present the ground state transform on the level of operators.

Lemma 8.5 (Ground state transform – operator version). *Let $(b, c)$ be a graph over $(X, m)$. Let $u \in C(X)$ be strictly positive and $w \in C(X)$ such that $(\mathcal{L} - w)u = 0$. Then,*

$$\mathcal{L}_u := \mathcal{T}_u^{-1} \mathcal{L} T_u$$

*acts on $f \in C(X)$ such that $uf \in \mathcal{F}$ as*

$$\mathcal{L}_u f(x) = \frac{1}{u(x)^2 m(x)} \left( \sum_{y\in X} b(x, y)u(y)u(x)(f(x) - f(y)) \right) + w(x)f(x)$$

*for all $x \in X$. In particular, if $w$ is positive then $\mathcal{L}_u = \mathcal{L}_{b_u, u^2 w, u^2 m}$.*

Proof. This follows by direct computation. First we note that $(\mathcal{L} - w)u = 0$ implies

$$w(x)u(x) = \sum_{y\in X} b(x, y)(u(x) - u(y)) + c(x)u(x)$$

for all $x \in X$ and that

$$u(x)f(x) - u(y)f(y) = u(y)(f(x) - f(y)) + (u(x) - u(y))f(x)$$

holds for all $f \in C(X)$ and $x, y \in X$. Given this we compute for $f \in C(X)$ and $x \in X$

$$\begin{aligned}
&\mathcal{T}_u \mathcal{L}_u f(x) = \mathcal{L} \mathcal{T}_u f(x) \\
&= \frac{1}{m(x)} \left( \sum_{y \in X} b(x,y)(u(x)f(x) - u(x)f(y)) + c(x)u(x)f(x) \right) \\
&= \frac{1}{m(x)} \left( \sum_{y \in X} b(x,y) \left( u(y)(f(x) - f(y)) + (u(x) - u(y))f(x) \right) + c(x)u(x)f(x) \right) \\
&= \frac{1}{u(x)m(x)} \left( \sum_{y \in X} b(x,y)u(y)u(x)(f(x) - f(y)) \right) + u(x)w(x)f(x).
\end{aligned}$$

Dividing by $u$ then gives the desired statement. □

As a consequence we see how the operator $\mathcal{L} - w$, which can be seen as associated to the graph $(b, c)$ perturbed by $-w$, is converted into the graph $(b_u, 0)$ over $(X, u^2 m)$:

Corollary 8.6. *Let $(b, c)$ be a graph over $(X, m)$. Let $u \in C(X)$ be strictly positive and $w \in C(X)$ with $w \geq 0$ such that $(\mathcal{L} - w)u = 0$. Then,*

$$\frac{1}{u}(\mathcal{L} - w)(u\varphi) = \mathcal{L}_{b_u, u^2 m} \varphi$$

*for all $\varphi \in C_c(X)$.*

We define the quadratic form $\mathcal{Q}_u := \mathcal{Q}_{b_u, 0}$. Hence, $\mathcal{Q}_u$ acts on $C(X)$ by

$$\mathcal{Q}_u(f) = \frac{1}{2} \sum_{x, y \in X} b(x,y)u(x)u(y)(f(x) - f(y))^2.$$

Now, we can state the ground state transform on the level of forms.

Theorem 8.7 (Ground state transform – form version). *Let $(b, c)$ be a graph over $(X, m)$. Let $u \in C(X)$ be strictly positive and $w \in C(X)$ such that $(\mathcal{L} - w)u = 0$. Then, for $\varphi \in C_c(X)$ we have both*

$$\mathcal{Q}(u\varphi) = \mathcal{Q}_u(\varphi) + \sum_{x \in X} \varphi(x)^2 w(x) u(x)^2 m(x).$$

*and*

$$\mathcal{Q}_u(\varphi/u) = \mathcal{Q}(\varphi) - \sum_{x \in X} \varphi(x)^2 w(x) m(x).$$

Proof. We calculate using Green's formula and the lemma on the ground state transform for operators

$$\begin{aligned}
\mathcal{Q}(u\varphi) &= \sum_{x \in X} (\mathcal{L}\mathcal{T}_u \varphi)(x)(\mathcal{T}_u \varphi)(x) m(x) \\
&= \sum_{x \in X} (\mathcal{T}_{u^{-1}} \mathcal{L} \mathcal{T}_u \varphi)(x) \varphi(x) u(x)^2 m(x) \\
&= \mathcal{Q}_u(\varphi) + \sum_{x \in X} \varphi(x)^2 w(x) u(x)^2 m(x).
\end{aligned}$$

This shows the first part of the statement. The second part of the statement follows by replacing $\varphi$ by $\varphi/u$. □

A main application of the ground state transform concerns the case where there is $u > 0$ with $(\mathcal{L} - \lambda)u \geq 0$. Such a $u$ is sometimes known as *ground state* and this gives the name ground state transform. It is such $u$ that provide a connection between the present section and Section 8.1. We will use this connection to derive a characterization of the infimum of the spectrum in Section 8.3. Here, we note the following corollary.

COROLLARY 8.8. *Let $(b, c)$ be a graph over $(X, m)$. Let $u \in C(X)$ be strictly positive and $\lambda \in \mathbb{R}$ with $(\mathcal{L} - \lambda)u \geq 0$. Then,*

$$\mathcal{Q}(\varphi) - \lambda\|\varphi\|^2 \geq \mathcal{Q}_u(\varphi) \geq 0$$

*for all $\varphi \in C_c(X)$.*

PROOF. Define

$$w\colon X \longrightarrow \mathbb{R}, \quad w(x) := \frac{\mathcal{L}u(x)}{u(x)}.$$

Then,

$$(\mathcal{L} - w)u = 0$$

by definition of $w$. Hence, the preceding theorem gives

$$\mathcal{Q}(\varphi) - \sum_{x\in X} w(x)\varphi(x)^2 m(x) \geq \mathcal{Q}_u(\varphi) \geq 0$$

for all $\varphi \in C_c(X)$.

Moreover, by assumption on $u$ we have

$$w(x) - \lambda = \frac{1}{u(x)}(w(x)u(x) - \lambda u(x)) = \frac{1}{u(x)}((\mathcal{L}u)(x) - \lambda u(x)) \geq 0$$

for all $x \in X$. This implies

$$\begin{aligned}\mathcal{Q}(\varphi) - \lambda\|\varphi\|^2 &= \mathcal{Q}(\varphi) - \sum_{x\in X} w(x)\varphi(x)^2 m(x) + \sum_{x\in X}(w(x) - \lambda)\varphi(x)^2 m(x)\\ &\geq \mathcal{Q}(\varphi) - \sum_{x\in X} w(x)\varphi(x)^2 m(x).\end{aligned}$$

Combining these inequalities we arrive at the desired statement. □

REMARK 8.9. If the graph is connected any $u \geq 0$ with $(\mathcal{L} - \lambda)u = 0$ must satisfy $u > 0$ by the the Harnack inequality provided in Section 8.1.

## 8.3. Agmon–Allegretto–Piepenbrink Theorem

In this section we combine the results of Sections 8.1 and 8.2 in order to characterize the infimum of the spectrum.

THEOREM 8.10 (Agmon–Allegretto–Piepenbrink theorem). *Let $(b, c)$ be a connected graph over $(X, m)$, $L = L^{(D)}$ the Dirichlet Laplacian. For $\lambda \in \mathbb{R}$ the following are equivalent:*

(i) $\lambda \leq \lambda_0(L)$.
(ii) *There exists a positive non-trivial $u \in \mathcal{F}$ with $(\mathcal{L} - \lambda)u \geq 0$.*
(iii) *There exists a strictly positive $u \in \mathcal{F}$ with $(\mathcal{L} - \lambda)u \geq 0$.*

Proof. (ii)$\Longleftrightarrow$(iii): This follows by the local Harnack inequality.

(i)$\Longrightarrow$(ii): Consider first $\lambda < \lambda_0(L)$. Then, for any $o \in X$ the function $u_\lambda := (L-\lambda)^{-1}1_o$ is non-trivial and positive with

$$(\mathcal{L}-\lambda)u_\lambda = (\mathcal{L}-\lambda)(L-\lambda)^{-1}1_o = 1_o \geq 0.$$

This gives (ii) in this case. Note that $u_\lambda$ is strictly positive by the local Harnack inequality.

To deal with the case $\lambda = \lambda_0(L)$ let $(\lambda_n)$ converge to $\lambda_0$ from below. Consider

$$g_n := \frac{u_{\lambda_n}}{u_{\lambda_n}(o)}, \quad n \in \mathbb{N}.$$

(Here we use that the $u_{\lambda_n}$ are strictly positive so that the denominator does not vanish.) Then, the $g_n$ are positive non-trivial supersolutions by what we have just shown. Moreover, they satisfy $g_n(o) = 1$ by assumption. Hence, by the Harnack principle, there exists a subsequence of $(g_n)$ which converges to some positive non-trivial supersolution $g$ for $\lambda_0(L)$.

(iii)$\Longrightarrow$(i): Let $u \in \mathcal{F}$ be a strictly positive solution for $\lambda$. By Corollary 8.8, we have for all $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi) - \lambda\|\varphi\|^2 \geq \mathcal{Q}_u(\varphi/u) \geq 0.$$

Hence, we find

$$\mathcal{Q}(\varphi) \geq \lambda\|\varphi\|^2$$

for all $\varphi \in C_c(X)$. Since $\lambda_0(L) = \inf_{\|\varphi\|=1} Q(\varphi)$ we conclude $\lambda_0(L) \geq \lambda$. □

The theorem begs the question to which extend one can find non-trivial solutions $u$ of

$$(\mathcal{L}-\lambda)u = 0$$

for $\lambda \leq \lambda_0(L)$. For finite graphs the situation is clear. For $\lambda = \lambda_0(L)$ there exists a non-trivial solution (as $\lambda_0(L)$ is an eigenvalue) and to $\lambda < \lambda_0(L)$ there does not exist a solution (as there are no eigenvalues below the infimum of the spectrum). For infinite graphs the situation is more complicated. Still, for locally finite graphs we obtain a rather direct corollary of our considerations. The corollary is based on the following stability feature for pointwise convergent solutions on locally finite graphs.

Lemma 8.11. *Let $(b,c)$ be a locally finite graph over $(X,m)$. Let $(u_n)$ in $\mathcal{F}$ and $(f_n)$ in $C(X)$, $u, f \in C(X)$ with $u_n \to u$ pointwise and $f_n \to f$ pointwise and $\mathcal{L}u_n = f_n$ for all $n \in \mathbb{N}$. Then, $u \in \mathcal{F}$ and $\mathcal{L}u = f$.*

Proof. By Fatou's lemma we have $u \in \mathcal{F}$. For $n \in \mathbb{N}$ and $x \in X$ we have

$$\mathcal{L}u_n(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u_n(x)-u_n(y)) + \frac{c(x)}{m(x)}u_n(x) = f_n(x).$$

By local finiteness we can take the limit as $n \to \infty$ and directly obtain the statement. □

Corollary 8.12. *Let $(b,c)$ be an infinite connected locally finite graph over $(X,m)$ and $L = L^{(D)}$ the Dirichlet Laplacian. Then, for any $\lambda \leq \lambda_0(L)$ there exists a non-trivial solution $u$ of $(\mathcal{L}-\lambda)u = 0$.*

Proof. Fix $o \in X$. Choose a sequence $(o_n)$ in $X$ that leaves any finite set (i.e. that satisfies for each finite set $F \subseteq X$ that $o_n$ does not belong to $F$ for all large enough $n$).

We first deal with the case $\lambda < \lambda_0(L)$. Consider $(L-\lambda)^{-1}1_{o_n}$. This is a non-trivial solution to $(\mathcal{L}-\lambda)u \geq 0$. Hence, by the local Harnack inequality, it is strictly positive. We can therefore define

$$u_n := \frac{1}{(L-\lambda)^{-1}1_{o_n}(o)}(L-\lambda)^{-1}1_{o_n}, \quad n \in \mathbb{N}.$$

Then, each $u_n$ satisfies

$$(\mathcal{L}-\lambda)u_n = \frac{1}{(L-\lambda)^{-1}1_{o_n}(o)}1_{o_n} \geq 0$$

as well as $u_n \geq 0$ and $u_n(o) = 1$. Hence, by the Harnack principle, there exists a pointwise convergent subsequence. Without loss of generality, we assume that $(u_n)$ itself converges pointwise. Call the limit $u$. Then, $u$ satisfies $u \geq 0$ as well as $u(o) = 1$. Moreover, clearly, $(1_{o_n})$ converges pointwise to 0 (as $(o_n)$ leaves every finite set). Now, note that $u_n$ solves

$$\mathcal{L}u_n = \lambda u_n + \frac{1}{(L-\lambda)^{-1}1_{o_n}(o)}1_{o_n},$$

for all $n \in \mathbb{N}$. Thus, Lemma 8.11 gives

$$\mathcal{L}u = \lambda u.$$

As $u$ is positive and non-trivial, we infer $u > 0$ by the local Harnack inequality.

We now turn to the case $\lambda = \lambda_0(L)$. Let $(\lambda_n)$ be a sequence converging from below to $\lambda_0(L)$. By what we have shown already, for all $n \in \mathbb{N}$ there exists a $u_n > 0$ with $(\mathcal{L}-\lambda_n)u_n = 0$. Without loss of generality, we can assume $u_n(o) = 1$ for all $n \in \mathbb{N}$. Moreover, invoking the Harnack principle we can assume without loss of generality that $(u_n)$ converges pointwise to a limit. Call the limit $u$. Then, $u$ is positive with $u(o) = 1$ and, by Lemma 8.11, $(\mathcal{L}-\lambda)u = 0$. □

CHAPTER 9

# Recurrence and Transience

Recurrence and its counterpart transience refer to a property of the graph that can be phrased in many different ways. The term itself comes from a stochastic interpretation. In this interpretation the graph gives rise to a Markov process modeling a particle jumping between the vertices of the graph according to certain rules. Recurrence then describes the phenomenon that the particle returns again and again to any given vertex. Transience in turn describes the phenomenon that the particle leaves any vertex for good at one point of time. In the present lecture notes we are not concerned with stochastic processes but rather with an analytic description.

Let $X$ be a countable set and $m$ a measure on $X$ with full support.

## 9.1. The Green Function

Let $(b, c)$ be a graph over $(X, m)$, $Q = Q^{(D)}_{b,c,m}$ the quadratic form and $L = L^{(D)}$ the Dirichlet Laplacian. The *Green function* of the graph is defined by $G\colon X \times X \to [0, \infty]$ with

$$G(x, y) := \int_0^\infty e^{-tL} 1_y(x) \mathrm{d}t.$$

The Green function can be thought of as modelling the (possibly non-existent) inverse $L^{-1}$ of the Laplacian $L$ associated to the graph. Indeed, this point of view is made precise (in two ways) in the first part of the subsequent theorem.

Recall from Section 5.1 that for any finite $K \subseteq X$ the restriction of $Q$ to $C(K)$ gives rise to an operator $L^{(D)}_K$, which is associated to the graph $(b_K, c_K)$ over $(K, m_K)$ with $b_K(x, y) := b(x, y)$ for $x, y \in K$ and $c_K(x) := c(x) + \sum_{y\in X} b(x, y)$ for $x \in K$.

Theorem 9.1 (Basic features of the Green function). *Let $(b, c)$ be a connected graph over $(X, m)$. Then, the following holds.*

(a) *(Approximation property) For all $x, y \in X$*

$$G(x, y) = \lim_{\alpha\to 0+} (L + \alpha)^{-1} 1_y(x).$$

(b) *(Approximation property for infinite $X$) If $X$ is infinite, the operator $L^{(D)}_K$ is invertible for any finite $K \subseteq X$, and for all sequences $(K_n)$ of finite subsets of $X$ with $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$ we have*

$$G(x, y) = \lim_{n\to\infty} (L^{(D)}_{K_n})^{-1} 1_y(x)$$

*for all $x, y \in X$.*

(c) *(Symmetry) For all* $x, y \in X$,

$$G(x,y)m(x) = G(y,x)m(y).$$

(d) *(Dichotomy) The inequality* $G(x,y) > 0$ *holds for all* $x, y \in X$ *and if* $G(x,y) = \infty$ *(respectively* $G(x,y) < \infty$*) for some* $x, y \in X$*, then* $G(x,y) = \infty$ *(respectively* $G(x,y) < \infty$*) for all* $x, y \in X$.

PROOF. (a) By positivity preservation, we see that

$$(L+\alpha)^{-1}1_y(x) = \int_0^\infty e^{-t\alpha}e^{-tL}1_y(x)\mathrm{d}t$$

is monotonically increasing as $\alpha$ is decreasing. Taking the limit $\alpha \to 0+$, we obtain the equality.

(b) If $X$ is infinite and $K \subseteq X$ is finite the operator $L_K^{(D)}$ belongs to the graph $(b_K, c_K)$ with $c_K(x) = c(x) + \sum_{y \in X} b(x,y)$ for all $x \in K$. As $(b,c)$ is connected, $c_K$ does not vanish. Hence, $L_K^{(D)}$ is invertible. Furthermore, for $x, y \in X$, $((L_{K_n}^{(D)})^{-1}1_y(x))$ is monotonically increasing by domain monotonicity and therefore we can interchange the limits

$$\lim_{\alpha\to 0}(L+\alpha)^{-1}1_y(x) = \lim_{\alpha\to 0}\lim_{n\to\infty}(L_{K_n}^{(D)}+\alpha)^{-1}1_y(x) = \lim_{n\to\infty}(L_{K_n}^{(D)})^{-1}1_y(x).$$

(c) The symmetry follows directly from the equality

$$e^{-tL}1_x(y)m(y) = \langle e^{-tL}1_x, 1_y\rangle = \langle 1_x, e^{-tL}1_y\rangle = e^{-tL}1_y(x)m(x)$$

for all $x, y \in X$ and $t \geq 0$.

(d) The strict positivity $G > 0$ follows directly from the definition of $G$ and the fact that the semigroup operators $e^{-tL}$ are positivity improving for $t > 0$ by Theorem 6.1 as we assume connectedness.

To show that $G(x,y) = \infty$ for all $x, y \in X$ if $G(x,y) = \infty$ for some $x, y \in X$ let $e_x := 1_x/\sqrt{m(x)}$ for $x \in X$. Let $x, y, x_0 \in X$. We calculate for $t > 1$ that

$$\begin{aligned} e^{-tL}1_y(x) &= e^{-L}e^{-(t-1)L}1_y(x) = \frac{1}{m(x)}\langle e^{-L}e^{-(t-1)L}1_y, 1_x\rangle \\ &= \frac{1}{m(x)}\sum_{z\in X}\langle e^{-(t-1)L}1_y, e_z\rangle\langle e^{-L}1_x, e_z\rangle \\ &\overset{(e^{-tL})\text{ positive}}{\geq} \frac{1}{m(x)}\langle e^{-(t-1)L}1_y, e_{x_0}\rangle\langle e^{-L}1_x, e_{x_0}\rangle \\ &= \frac{m(x_0)}{m(x)}e^{-(t-1)L}1_y(x_0)e^{-L}1_x(x_0). \end{aligned}$$

Since the semigroup is positivity improving on a connected graph by Theorem 6.1, we infer that $C := C_{x,x_0} := e^{-L}1_x(x_0)m(x_0)/m(x) > 0$. Then,

$$\begin{aligned} G(x,y) &= \int_0^\infty e^{-tL}1_y(x)\mathrm{d}t \geq \int_1^\infty e^{-tL}1_y(x)\mathrm{d}t \geq C\int_0^\infty e^{-tL}1_y(x_0)\mathrm{d}t \\ &= CG(x_0,y). \end{aligned}$$

As this holds for all $x_0, x, y \in X$ the we find, by symmetry in (c), that

$$G(x_0,y) \geq C'G(x_0,y_0),$$

where $C' = C_{y,y_0}\frac{m(y)}{m(y_0)}$, for any $y_0 \in X$. As $x, y, x_0, y_0 \in X$ were chosen arbitrarily, the statement follows. □

We have already noted that the Green function models the inverse of $L$. Here is another precise version of this.

THEOREM 9.2 (Characterization of $G(\cdot, o)$). *Let $(b, c)$ be a connected graph over $(X, m)$ and $o \in X$. If $G(x, y) < \infty$ for some $x, y \in X$, then $G(\cdot, o)$ is superharmonic and satisfies*

$$\mathcal{L}G(\cdot, o) = 1_o.$$

*Furthermore, $G(\cdot, o)$ is the smallest $u \in \mathcal{F}$ with $u \geq 0$ such that $\mathcal{L}u \geq 1_o$.*

PROOF. Consider first the case that $X$ is finite. Then, $G(x, y) < \infty$ yields that 0 is not an eigenvalue of $L = \mathcal{L}$ (note that $X$ is finite). Indeed, note that the Dichotomy in Theoem 9.1 yields $G(x, y) < \infty$ for all $x, y \in X$. If 0 is an eigenvalue then 1 is a (strictly positive) eigenfuntion (see Example 6.5), and we observe

$$\infty > \sum_{z \in X} G(x, z) = \int_0^\infty e^{-tL}1(x)\mathrm{d}t = \int_0^\infty 1\mathrm{d}t = \infty,$$

a contradiction. Hence, $\inf \sigma(L) > 0$ and $L$ is invertible. Spectral calculus then gives $G(\cdot, o) = L^{-1}1_o$ and the desired statement follows.

Consider now the case that $X$ is infinite. We first show that $\mathcal{L}G(\cdot, o) = 1_o$ holds. This implies in particular that $G(\cdot, o)$ is superharmonic.

Let $(K_n)$ be an arbitrary sequence of increasing finite subsets of $X$ such that $\bigcup_{n \in \mathbb{N}} K_n = X$ and $o \in K_n$ for all $n \in \mathbb{N}$. Set

$$g_n := (L_{K_n}^{(D)})^{-1}1_o, \quad n \in \mathbb{N}.$$

By domain monotonicity we see that $(g_n)$ is monotonically increasing in $n$ and, by the previous theorem, it converges to $G(\cdot, o)$. By monotone convergence we get

$$1_o = \mathcal{L}g_n \to \mathcal{L}G(\cdot, o), \qquad n \to \infty.$$

This is the desired equality.

We now show that $G(\cdot, o)$ is the smallest function $u \in \mathcal{F}$ such that $u \geq 0$ and $\mathcal{L}u \geq 1_o$. So, let $u \in \mathcal{F}$ satisfy $u \geq 0$ and $\mathcal{L}u \geq 1_o$. Then, for $n \in \mathbb{N}$, $v_n := u - g_n$ is superharmonic on $K_n$, satisfies $v_n \geq 0$ outside of $K_n$ and $v_n \wedge 0$ assumes its minimum on the finite set $K_n$. Hence, by the minimum principle, Theorem 2.10, we infer $v_n \geq 0$ and, therefore, $u \geq g_n$. Since $(g_n)$ converges to $G(\cdot, o)$, it follows that $u \geq G(\cdot, o)$. □

## 9.2. Null Sequences and $\mathcal{D}_0(X)$

In this section we use the space $\mathcal{D}_0(X)$ defined earlier. Let $(b, c)$ be a connected graph over $X$. Note that in this section we do not need the measure $m$. For $o \in X$, let the semi-scalar product $\langle \cdot, \cdot \rangle_o$ be given by

$$\langle f, g \rangle_o := \mathcal{Q}(f, g) + f(o)g(o)$$

for $f, g \in \mathcal{D}$ and let $\|\cdot\|_o$ be the corresponding semi-norm. Then, $\langle\cdot,\cdot\rangle_o$ is a scalar-product and $\|\cdot\|_o$ is a norm on $\mathcal{D}$ whenever the graph is connected. Moreover, recall that

$$\mathcal{D}_0 := \mathcal{D}_0(X) = \overline{C_c(X)}^{\|\cdot\|_o}.$$

In Lemma 2.5 it was shown that for any $x \in X$ the norms $\|\cdot\|_o$ and $\|\cdot\|_x$ are equivalent and $(\mathcal{D}, \|\cdot\|_o)$ is a Hilbert space. Also, it was shown there that a sequence $(f_n)$ in $\mathcal{D}$ converges to $f$ w.r.t. $\|\cdot\|_o$, if and only if $f_n \to f$ pointwise and

$$\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f).$$

We now turn to the question whether the constant function 1 can be approximated by functions in $C_c(X)$ with respect to $\|\cdot\|_o$ for one (all) $o \in X$. In this context we provide the following definition.

Definition 9.3 (Null sequence). Let $(b, c)$ be a connected graph over $X$. A sequence $(e_n)$ in $C_c(X)$ with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ is called a *null sequence* if $e_n \to 1$ pointwise and $\mathcal{Q}(e_n) \to 0$ as $n \to \infty$.

Corollary 9.4 (Characterization of null sequences). *Let $(b, c)$ be a connected graph over $X$.*

(a) *If $(b, c)$ admits a null sequence then $c = 0$ holds.*
(b) *Assume $c = 0$. Then, a sequence $(e_n)$ in $C_c(X)$ is a null sequence if and only if $(1 - e_n)$ converges to 0 with respect to $\|\cdot\|_o$ for one (all) $o \in X$.*
(c) *The graph $(b, 0)$ admits a null sequence if and only if $1 \in \mathcal{D}_0$.*

Proof. (a) Let $(e_n)$ be a null sequence. Then, $e_n(x) \to 1$ for all $x \in X$. By

$$\sum_{x\in X} c(x)e_n(x)^2 \leq \mathcal{Q}(e_n) \to 0$$

the claim on $c$ is immediate.

(b) Given that $c = 0$ holds, we directly see

$$\mathcal{Q}(e_n) = \mathcal{Q}(1 - e_n).$$

From this (b) follows.

(c) This is just a reformulation of (b). □

By definition, null sequences give a way to approximate the constant function $1 \in \mathcal{D}$ by functions with finite support. In fact, they offer the possibility to approximate any function in $\mathcal{D}$ by functions with finite support. Details are given in the subsequent lemma.

Lemma 9.5. *Let $(b, c)$ be a connected graph over $X$ and $o \in X$. If $(e_n)$ is a null sequence, then for any $f \in \mathcal{D}$ we have $e_n f \to f$ with respect to $\|\cdot\|_o$ as $n \to \infty$.*

Proof. We first deal with bounded functions $f$ in $\mathcal{D}$. So, let $f \in \mathcal{D} \cap \ell^\infty(X)$ be given. Now, $e_n f \in C_c(X)$ for all $n \in \mathbb{N}$. Moreover, simple algebraic manipulations give

$$
\begin{aligned}
&(f(x)(1-e_n)(x) - f(y)(1-e_n)(y))^2 \\
&= (f(x)(1-e_n)(x) - f(y)(1-e_n)(x) + f(y)(1-e_n)(x) - f(y)(1-e_n)(y))^2 \\
&= ((1-e_n)(x)(f(x)-f(y)) + f(y)(e_n(x)-e_n(y)))^2 \\
&\leq 2(1-e_n(x))^2(f(x)-f(y))^2 + 2f(y)^2(e_n(x)-e_n(y))^2
\end{aligned}
$$

for all $x, y \in X$. This yields

$$
\begin{aligned}
\mathcal{Q}(f - e_n f) &= \mathcal{Q}(f(1-e_n)) \\
&= \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x)(1-e_n)(x) - f(y)(1-e_n)(y))^2 \\
&\leq \sum_{x \in X} (1-e_n(x))^2 \sum_{y \in X} b(x,y)(f(x)-f(y))^2 \\
&\qquad + \sum_{y \in X} f(y)^2 \sum_{x \in X} b(x,y)(e_n(x)-e_n(y))^2 \\
&\leq \sum_{x \in X} (1-e_n(x))^2 \sum_{y \in X} b(x,y)(f(x)-f(y))^2 + 2\|f\|_\infty^2 \mathcal{Q}(e_n) \\
&\to 0.
\end{aligned}
$$

Here, we used in the last step that $(e_n)$ is a null sequence to treat the second term and the Lebesgue's dominated convergence theorem to treat the first term. Note that Lebesgue's theorem is applicable since $f$ belongs to $\mathcal{D}$. Altogether, we have shown that $(e_n f)$ converges to $f$ with respect to $\|\cdot\|_o$ for bounded $f$ in $\mathcal{D}$.

Now, consider an arbitrary function $f \in \mathcal{D}$. Then for any $k \in \mathbb{N}$, the function $f_k := (-k) \vee f \wedge k$ is bounded and belongs to $\mathcal{D}$ (as they arise from $f$ by taking a normal contraction). Hence, it suffices to show $f_k \to f$ w.r.t. $\|\cdot\|_o$ as $k \to \infty$. Clearly $f_k \to f$ pointwise as $k \to \infty$. Moreover, as $\mathcal{Q}$ is compatible with normal contractions we find

$$\limsup_{k \to \infty} \mathcal{Q}(f_k) \leq \mathcal{Q}(f).$$

Together with the characterization of convergence with respect to $\|\cdot\|_o$ discussed at the beginning of the section, we infer the desired statement. □

We get the following immediate consequence.

THEOREM 9.6. *Let $b$ be a connected graph over $X$. The following statements are equivalent:*

(i) $\mathcal{D}_0 = \mathcal{D}$.
(ii) $1 \in \mathcal{D}_0$.
(iii) *There exists a null sequence.*
(iv) *For any $w \geq 0$ such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x \in X} w(x)\varphi(x)^2, \qquad \varphi \in C_c(X)$$

*we have $w = 0$.*

Proof. The equivalence between (ii) and (iii) is shown in Lemma 9.4.

(iv) $\Longrightarrow$ (ii): For $x \in X$, let

$$\mathrm{cap}(x) := \inf_{\varphi \in C_c(X), \varphi(x)=1} \mathcal{Q}(\varphi).$$

Then,

$$\mathcal{Q}(\varphi) \geq \mathrm{cap}(x)\varphi(x)^2$$

for any $\varphi \in C_c(X)$. In particular, (iv) implies

$$\mathrm{cap}(x) = 0$$

for all $x \in X$. Thus, for $x \in X$, there exists $(\varphi_n)$ in $C_c(X)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n(x) = 1$ for all $n \in \mathbb{N}$ such that $\mathcal{Q}(\varphi_n) \to 0$. Hence, $(\varphi_n)$ is a Cauchy sequence with respect to $\|\cdot\|_x$. By completeness, we infer that there exists $f \in \mathcal{D}$ such that $(\varphi_n)$ converges to $f$ with respect to $\|\cdot\|_x$. Then, $f = 1$ must hold as otherwise $\mathcal{Q}(f) > 0$ must hold (by connectedness of the graph) and, by Fatou's lemma, we had the contradiction

$$0 < \mathcal{Q}(f) \leq \liminf_{n\to\infty} \mathcal{Q}(\varphi_n) = 0.$$

This shows the desired implication (iv) $\Longrightarrow$ (ii).

(iii) $\Longrightarrow$ (iv): This is clear.

(ii) $\Longrightarrow$ (i): We have already discussed that $1 \in \mathcal{D}_0$ is equivalent to existence of a null sequence. Hence, (i) follows from (ii) by Lemma 9.5.

(i) $\Longrightarrow$ (ii): From $\mathcal{D} = \mathcal{D}_0$ and $1 \in \mathcal{D}$ we infer $1 \in \mathcal{D}_0$. □

### 9.3. Characterization of Recurrence and Transience

In Section 9.2 we have been concerned with $\mathcal{D}_0$ and it being equal to $\mathcal{D}$. Here, we consider and characterize the non-equality.

We call a connected graph $b$ over $X$ *transient* if $\mathcal{D} \neq \mathcal{D}_0$, and *recurrent* otherwise.

Theorem 9.7 (Characterization of transience). *Let $b$ be a connected graph over $X$. The following statements are equivalent:*

(i) *$\mathcal{D}_0 \neq \mathcal{D}$, i.e. $b$ is transient.*

(ii) *There exists a non-trivial $w \geq 0$ such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x\in X} w(x)\varphi^2(x), \qquad \varphi \in C_c(X).$$

(iii) *There exists a positive non-constant superharmonic function.*

(iv) *The set $X$ is infinite and there exist infinitely many linearly independent positive superharmonic functions.*

(v) *$G(x, y) < \infty$ for some (all) $x, y \in X$.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is already given in Theorem 9.6.

(ii) $\Longrightarrow$ (v): By the dichotomy given in Theorem 9.1 it suffices to find one $x \in X$ with $G(x, x) < \infty$. Let $x \in X$ with $w(x) > 0$ be given. Now, note that (ii) implies that $X$ is infinite (as otherwise $\varphi = 1$ would give a contradiction to (ii)). Let $(K_n)$ be an increasing sequence of finite subsets of $X$ with $\bigcup_{n\in\mathbb{N}} K_n = X$, and let $g_n := (L^{(D)}_{K_n})^{-1} 1_x$ for $n \in \mathbb{N}$. Then, $g_n \to G(\cdot, x)$ pointwise by Theorem 9.1. Moreover, a direct computation invoking the Green formula and (ii) gives

$$g_n(x)m(x) = \langle 1_x, g_n \rangle = \langle \mathcal{L} g_n, g_n \rangle = \mathcal{Q}(g_n) \geq \sum_{y \in X} w(y) g_n(y)^2 \geq w(x) g_n(x)^2.$$

Hence,

$$g_n(x) \leq \frac{m(x)}{w(x)}, \quad n \in \mathbb{N},$$

which implies $G(x, x) < \infty$ after taking the limit $n \to \infty$.

(v) $\Longrightarrow$ (iv): First we note that $X$ must be infinite if $G(x, y) < \infty$ holds. (Otherwise, by $c = 0$ we had that 1 were an eigenfunction to the eigenvalue 0 of $L$ and this would imply $G(x, y) = \infty$ by the very definition of $G$. Compare reasoning within the proof of Theorem 9.2.) As discussed in Theorem 9.2 the functions $G(\cdot, x)$ are positive and superharmonic with $\mathcal{L} G(\cdot, x) = 1_x$ for all $x \in X$. By $c = 0$ they can not be constant. We show that they are linearly independent. Assume

$$\sum_{x \in X} \lambda_x G(\cdot, x) = 0$$

for some $\lambda_x \in \mathbb{R}$ with $\lambda_x = 0$ for all but finitely many $x \in X$. Then,

$$\sum_{x \in C} \lambda_x 1_x = \sum_{x \in X} \lambda_x \mathcal{L} G(\cdot, x) = L(\sum_{x \in X} \lambda_x G(\cdot, x)) = L0 = 0.$$

This gives $\lambda_x = 0$ for all $x \in X$.

(iv) $\Longrightarrow$ (iii): This is clear.

(iii) $\Longrightarrow$ (ii): Let $u$ be a non-constant postive superharmonic function. By the Harnack inequality we have $u > 0$. Then, $v := u^{1/2}$ satisfies

$$\begin{aligned}
\mathcal{L} v(x) &= \frac{1}{m(x)} \sum_{y \in X} b(x, y)(u(x)^{1/2} - u(y)^{1/2}) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(u(x)^{1/2} - u(y)^{1/2}) \\
&= \frac{1}{m(x) u(x)^{1/2}} \sum_{y \in X} b(x, y)(\frac{1}{2} u(x) - \frac{1}{2} u(y) + \frac{1}{2} u(x) - u(x)^{1/2} u(y)^{1/2} + \frac{1}{2} u(y)) \\
&= \frac{1}{2m(x) u(x)^{1/2}} \mathcal{L} u(x) + \frac{1}{2m(x) u(x)^{1/2}} \sum_{x \in X} b(x, y)(u(x)^{1/2} - u(y)^{1/2})^2 \\
&> 0.
\end{aligned}$$

Hence, by the ground state transform, Theorem 8.7, we get with

$$w(x) := \frac{1}{v(x)} \mathcal{L} v(x) m(x), \quad x \in X,$$

that

$$\mathcal{Q}(\varphi) = Q_v(\varphi / v) + \sum_{x \in X} \frac{\varphi(x)^2}{v(x)} \mathcal{L} v(x) m(x) \geq \sum_{x \in X} w(x) \varphi(x)^2.$$

This finishes the proof. □

*Matthias Keller*
*Daniel Lenz*
*Marcel Schmidt*
*Christian Seifert*

# Sheet 10

## Superharmonic functions and Recurrence

**Exercise 1 (Superharmonic functions)** 4 points

Let $b$ be a graph over $X$. Show the following statements:

(a) The pointwise infimum of a set of superharmonic functions is superharmonic whenever the infimum is a finite function.

(b) The sum of two superharmonic functions is superharmonic.

(c) The limit of any monotonically increasing sequence of superharmonic functions is superharmonic whenever the limit is pointwise finite.

(d) The composition $\varphi \circ u$ of a monotonically increasing concave function $\varphi : [0,\infty) \longrightarrow [0,\infty]$ with a positive superharmonic function $u$ is a superharmonic function which is non-harmonic whenever $\varphi$ is strictly concave.

**Exercise 2 (Positive eigenfunctions)** 4 points

Let $(b,c)$ be a connected graph over a measure space $(X,m)$. Let $L$ denote the Laplacian associated to $(b,c)$ over $(X,m)$. If there is an eigenvalue $\lambda$ of $L$ with a positive eigenfunction, then $\lambda = \lambda_0(L)$. Show that this statement is false if the graph is not connected.

**Exercise 3 (Finite measure and bounded degree implies recurrent)** 4 points

Let $b$ be a graph over $(X,m)$ such that $m(X) < \infty$ and Deg is bounded. Show that the graph is recurrent.

**Exercise 4 (Khasminskii criterion)** 4 points

Let $b$ be a connected graph over $X$. Show that $b$ is recurrent if and only if there exists a function $f \in \mathcal{D}$ such that $f(x) \to \infty$ as $x \to \infty$ where $X \cup \{\infty\}$ is the one point compactification of $X$.

**Bonus Exercise 1 (Deg $\geq w$)** 1 point

Let $(b,c)$ be a graph over $(X,m)$ and $w : X \longrightarrow \mathbb{R}$ such that there is $u \in \mathcal{F}$, $u \geq 0$ such that $(\mathcal{L} - w)u \geq 0$. Show that

$$\mathrm{Deg} \geq w.$$