26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 08

CHAPTER 6

Large Time Behaviour

In this chapter we focus on the large time behaviour of the semigroup.

6.1. Connectedness, irreducibility and positivity improving

In this section we show that connectedness of the graph, irreducibility of the form and the property that the semigroup/resolvent are positivity improving are equivalent.

Let X be a countable set and m a measure on X with full support, and let (b, c) be a graph over (X, m). In Theorem 5.13 we showed that the semigroup and resolvent associated to the Dirichlet Laplacian $L := L^{(D)}$ always map positive functions to positive functions. This property is called positivity preserving.

We recall that a subset of X is called connected if any two points in the subset can be connected by a path consisting of vertices in the subset. A maximal connected subset is called a connected component and (b, c) is called connected if it consists of one connected component.

We now introduce the necessary concepts of irreducible forms and positivity improving operators. A quadratic form Q on $\ell^2(X, m)$ with domain $D(Q)$ is called *irreducible* if the only subsets $U \subseteq X$ such that $1_U D(Q) \subseteq D(Q)$ and

$$Q(f) = Q(1_U f) + Q(1_{X \setminus U} f)$$

for all $f \in D(Q)$ are $U = \emptyset$ and $U = X$.

An operator A on $\ell^2(X, m)$ is called *positivity improving* if $Af > 0$ for all non-trivial $f \in D(A)$ with $f \geq 0$.

THEOREM 6.1 (Characterization of connectedness and positivity improving). *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Then, the following statements are equivalent:*

- (i) *(b, c) is connected.*
- (ii) *Q is irreducible.*
- (iii) *$(L + \alpha)^{-1}$ is positivity improving for all $\alpha > 0$.*
- (iv) *e^{-tL} is positivity improving for all $t > 0$.*

PROOF. (i) $\Longrightarrow$ (iv): Let $\varphi \in C_c(X)$ be positive and non-trivial. Let (K_n) be an increasing sequence of connected finite sets such that $X = \bigcup_{n \in \mathbb{N}} K_n$. Denote by $L_{K_n}^{(D)}$ the operators corresponding to the restrictions of Q to $C_c(K_n)$. Then $(L_{K_n}^{(D)} + \alpha)^{-1} \varphi \geq 0$ for all $\alpha > 0$. Therefore,

$$e^{-tL_{K_n}^{(D)}} \varphi = \lim_{n \to \infty} \left(\frac{n}{t} \left(L_{K_n}^{(D)} + \frac{n}{t} \right)^{-1} \right)^n \varphi \geq 0, \quad t > 0$$

so that the semigroup $(e^{-tL^{(D)}_{K_n}})_{t\geq 0}$ is positivity preserving.

Now, let $u(t,x) := e^{-tL^{(D)}_{K_n}}\varphi(x) \geq 0$ and assume that n is large enough so that the support of φ is included in K_n. We want to show that $u(t,x) > 0$ for all $x \in K_n$ and $t > 0$. If there exists $x_0 \in K_n$ and $t_0 > 0$ such that $u(t_0,x_0) = 0$, then (t_0,x_0) is a minimum for u in both variables. Having a minimum at t_0 gives

$$0 = \partial_t u(t_0,x_0) = -L^{(D)}_{K_n} u(t_0,x_0).$$

Now, having a minimum at x_0 yields $u(t_0,y) = 0$ for all $y \sim x_0$. As K_n is connected, this implies $u(t_0,x) = 0$ for all $x \in K_n$. However,

$$e^{t_0 L^{(D)}_{K_n}} u(t_0,x) = \varphi(x),$$

so that $\varphi = 0$ on K_n which gives a contradiction to the assumption on φ. Therefore, $e^{-tL^{(D)}_{K_n}}\varphi(x) > 0$ for all $t > 0$ and $x \in K_n$, so that $e^{-tL^{(D)}_{K_n}}$ is positivity improving for all $t > 0$.

As we assume that $\varphi \geq 0$, by Lemma 5.3, we get $e^{-tL^{(D)}_{K_n}}\varphi \to e^{-tL}\varphi$ as $n \to \infty$ where the convergence is pointwise monotonically increasing. Therefore, we infer that $e^{-tL}\varphi > 0$ for all $t > 0$.

For a non-trivial positive function $f \in \ell^2(X,m)$, let $f_n := 1_{K_n} f \in C_c(X)$ for $n \in \mathbb{N}$. Then, (f_n) converges monotonically increasingly to f in $\ell^2(X,m)$ as $n \to \infty$. By Corollary 5.6 applied to the functions $f_{n+1} - f_n$, we have that $0 < e^{-tL} f_n \to e^{-tL} f$ where the convergence is pointwise monotonically increasing. Therefore, $e^{-tL} f > 0$ for all $t > 0$.

(iv) $\Longrightarrow$ (iii): This follows directly from the Laplace transform formula in Theorem 3.26, that is,

$$(L+\alpha)^{-1} = \int_0^\infty e^{-\alpha t} e^{-tL} \mathrm{d}t.$$

(iii) $\Longrightarrow$ (ii): If the form Q is not irreducible, then there exists a proper non-trivial subset $U \subseteq X$ such that L decomposes into a direct sum of operators $L_U \oplus L_{X\setminus U}$ on $\ell^2(U,m_U) \oplus \ell^2(X\setminus U, m_{X\setminus U})$. Hence, the resolvent also decomposes into a direct sum $(L_U+\alpha)^{-1} \oplus (L_{X\setminus U}+\alpha)^{-1}$. In this case, taking a non-trivial function $f \in \ell^2(U,m_U)$ with $f \geq 0$ yields a non-trivial function $(f,0) \in \ell^2(U,m_U) \oplus \ell^2(X\setminus U, m_{X\setminus U})$ which is non-negative. However,

$$(L+\alpha)^{-1}(f,0) = ((L_U+\alpha)^{-1} f, (L_{X\setminus U}+\alpha)^{-1} 0) = ((L_U+\alpha)^{-1} f, 0),$$

which is not strictly positive.

(ii) $\Longrightarrow$ (i): For any connected component U, we clearly have that $1_U\varphi \in D(Q)$ and

$$Q(\varphi) = Q(1_U\varphi) + Q(1_{X\setminus U}\varphi)$$

for any $\varphi \in C_c(X)$. We want to show that the same holds for $f \in D(Q)$ so that we may apply irreducibility to conclude connectedness.

Let $f \in D(Q)$ and let (φ_n) in $C_c(X)$ be such that $\|\varphi_n - f\|_Q \to 0$ as $n \to \infty$. Then, $(1_U\varphi_n)$ is a Cauchy sequence in $\|\cdot\|_Q$ since

$$\begin{aligned} Q(1_U\varphi_n - 1_U\varphi_m) &\leq Q(1_U(\varphi_n - \varphi_m)) + Q(1_{X\setminus U}(\varphi_n - \varphi_m)) \\ &= Q(\varphi_n - \varphi_m) \end{aligned}$$

and $\|1_U\varphi_n - 1_U\varphi_m\| \leq \|\varphi_n - \varphi_m\|$ for all $n, m \in \mathbb{N}_0$. Hence, $(1_U\varphi_n)$ converges in $D(Q)$ so that $1_U f \in D(Q)$. Furthermore, as $Q(1_U\varphi_n) \to Q(1_U f)$ and $Q(\varphi_n) \to Q(f)$ as $n \to \infty$, it follows that

$$Q(f) = Q(1_U f) + Q(1_{X\setminus U} f).$$

By irreducibility, we infer that either $U = \emptyset$ or $U = X$. This shows that (b, c) is connected. □

6.2. Toolbox – Variational Characterization for the Bottom of the Spectrum

We prove a characterization of the bottom of the spectrum of a positive operator. To this end we need the following proposition.

PROPOSITION 6.2 (Spectral parts and spectral family). *Let L be a self-adjoint operator on a Hilbert space H and let E be the associated spectral family. Let $\lambda \in \mathbb{R}$.*

(a) $\lambda \in \sigma(L)$ *if and only if* $\lambda \in \operatorname{supp}(E)$, *i.e.,*

$$E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$$

for all $\varepsilon > 0$.

(b) λ *is an eigenvalue of* L *if and only if* $E(\{\lambda\}) \neq 0$, *in which case the range of* $E(\{\lambda\})$ *is the eigenspace of* λ. *Furthermore,* $f \in H$ *is an eigenfunction corresponding to* λ *if and only if* μ_f *is supported on* $\{\lambda\}$.

PROOF. (a) This has already been shown in Theorem 3.18.

(b) From Proposition 3.13, as $(x - \lambda)^2 = 0$ for $x = \lambda$, we get

$$\|(L-\lambda)f\|^2 = \int (x-\lambda)^2 \mathrm{d}\mu_f(x) = \int_{\mathbb{R}\setminus\{\lambda\}} (x-\lambda)^2 \mathrm{d}\mu_f(x)$$

for any $f \in D(L)$. As $(x-\lambda)^2 > 0$ for all $x \neq \lambda$, we infer that $f \in D(L)$ with $f \neq 0$ is an eigenfunction corresponding to λ if and only if $\mu_f = 1_{\{\lambda\}}\mu_f$, i.e., if and only if $1_{\mathbb{R}\setminus\{\lambda\}}\mu_f = 0$. Now, $1_{\mathbb{R}\setminus\{\lambda\}}\mu_f = 0$ if and only if $\mu_f(\mathbb{R}\setminus\{\lambda\}) = 0$ and Proposition 3.16 gives

$$\|E(\mathbb{R}\setminus\{\lambda\})f\|^2 = \mu_f(\mathbb{R}\setminus\{\lambda\}) = 0.$$

Thus, we infer that $f \in D(L)$ is an eigenfunction corresponding to λ if and only if $E((\mathbb{R}\setminus\{\lambda\}))f = 0$. This, in turn, is equivalent to $f = E(\{\lambda\})f$ as $f = E(\{\lambda\})f + E(\mathbb{R}\setminus\{\lambda\})f$, where the summands are orthogonal. This shows the first statement of (b). The other statement has been shown along the way. □

The equality we show now is also sometimes referred to as the Rayleigh–Ritz formula.

THEOREM 6.3 (Variational characterization of λ_0). *Let Q be a positive closed form and let L be the associated operator on a Hilbert space H. Let $\lambda_0(L)$ denote the bottom of the spectrum of L. Then,*

$$\lambda_0(L) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf \rangle = \inf_{f \in D(Q), \|f\|=1} Q(f).$$

Furthermore, if $f \in D(Q)$ is normalized and satisfies $Q(f) = \lambda_0(L)$, then $f \in D(L)$ and $Lf = \lambda_0(L)f$, i.e., f is a normalized eigenfunction corresponding to the eigenvalue $\lambda_0(L)$.

PROOF. The second equality is clear from the connection between the form and the operator and the fact that $D(L)$ is dense with respect to the form norm $\|\cdot\|_Q$ by Corollary 3.37. Hence, we focus on proving the first equality. In order to do so, we will show two inequalities.

To this end, we recall that by Proposition 3.13 (b) we have

$$\langle f, Lf \rangle = \int x \mathrm{d}\mu_f(x),$$

where the integral is taken over the support of the spectral measure μ_f for $f \in D(L)$ and $\mu_f(\sigma(L)) = \|f\|^2$.

Now, we let $\lambda_0 := \lambda_0(L)$ and let $f \in D(L)$ be normalized. As $\sigma(L) \subseteq [\lambda_0, \infty)$, we obtain

$$\langle f, Lf \rangle = \int_{\lambda_0}^{\infty} x \mathrm{d}\mu_f(x) \geq \lambda_0 \int_{\lambda_0}^{\infty} \mathrm{d}\mu_f(x) = \lambda_0 \|f\|^2 = \lambda_0.$$

This shows $\lambda_0 \leq \inf \langle f, Lf \rangle$ for all normalized $f \in D(L)$.

Conversely, since $\lambda_0 \in \sigma(L)$, we have $E([\lambda_0, \lambda_0 + \varepsilon)) \neq 0$ for all $\varepsilon > 0$ by Proposition 3.13 (a). Hence, for every $\varepsilon > 0$ there exists a normalized f with $f = E([\lambda_0, \lambda_0 + \varepsilon))f$. By Proposition 3.17, f has spectral measure supported on $[\lambda_0, \lambda_0 + \varepsilon]$ and $f \in D(L)$. We then find

$$\langle f, Lf \rangle = \int_{\lambda_0}^{\lambda_0 + \varepsilon} x \mathrm{d}\mu_f(x) \leq (\lambda_0 + \varepsilon)\|f\|^2 = \lambda_0 + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this gives $\lambda_0 \geq \inf \langle f, Lf \rangle$, where the infimum is taken over all normalized $f \in D(L)$.

For the furthermore statement, suppose that $f \in D(Q) = D(\sqrt{L})$ is normalized and satisfies $Q(f) = \lambda_0$. By Proposition 3.13 (a), we now get

$$0 = Q(f) - \lambda_0 = \|\sqrt{L}f\|^2 - \lambda_0\|f\|^2 = \int_{\lambda_0}^{\infty} (x - \lambda_0) \mathrm{d}\mu_f(x).$$

As the integrand is non-negative, μ_f is supported on $\{\lambda_0\}$ which, by Proposition 6.2 (b), is equivalent to λ_0 being an eigenvalue with eigenfunction f. □

6.3. Positivity Improving Semigroups and the Ground State

Let X be a countable set and m a measure on X with full support. For the self-adjoint operator L associated to a positive symmetric closed form Q on $\ell^2(X, m)$, the semigroup operators e^{-tL} for $t \geq 0$ are bounded self-adjoint positive operators. By the discreteness of the space X, the semigroup has a kernel, i.e., there exists a map

$$p\colon [0,\infty) \times X \times X \longrightarrow \mathbb{R}$$

such that

$$e^{-tL} f(x) = \sum_{y \in X} p_t(x,y) f(y) m(y)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $t \geq 0$. We call p the *heat kernel* associated to L. An easy calculation gives that

$$p_t(x,y) = \frac{1}{m(x)m(y)} \langle 1_x, e^{-tL} 1_y \rangle$$

for all $x, y \in X$ and $t \geq 0$.

By the variational characterization of the bottom of the spectrum, Theorem 6.3, it follows that

$$\lambda_0(L) = \inf_{f \in D(Q), \|f\|=1} Q(f) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf \rangle.$$

In what follows we denote by $Q := Q^{(D)}_{b,c,m}$ be the Dirichlet form associated to a graph (b,c) over (X,m) with operator $L := L^{(D)}_{b,c,m}$ and the bottom of the spectrum by $\lambda_0 := \inf \sigma(L)$.

The following lemma considers the bottom of the spectrum for connected graphs. Specifically, whenever the bottom of the spectrum is an eigenvalue, then there exists a unique strictly positive normalized eigenfunction.

Lemma 6.4 (Uniqueness of eigenfunctions to λ_0). *Let (b,c) be a connected graph over (X,m). Assume λ_0 is an eigenvalue. Then, there exists a unique strictly positive normalized eigenfunction corresponding to λ_0.*

Proof. Let $u \in D(L)$ be a normalized eigenfunction corresponding to λ_0. We will show that u must be strictly positive or strictly negative. Without loss of generality, we may assume that $u(x) > 0$ for some $x \in X$. Let $u_+ := u \vee 0$ and $u_- := (-u) \vee 0$ so that $u = u_+ - u_-$ and $|u| = u_+ + u_-$. From the variational characterization of the bottom of the spectrum, Theorem 6.3, and the fact that Q is a Dirichlet form we get

$$\lambda_0 \leq Q(|u|) \leq Q(u) = \lambda_0$$

so that $Q(|u|) = Q(u)$. Therefore, $|u|$ is also a normalized eigenfunction corresponding to λ_0 by Theorem 6.3. As both u and $|u|$ are eigenfunctions corresponding to λ_0, we get that

$$u_+ = \frac{u + |u|}{2}$$

is also an eigenfunction corresponding to λ_0. We note that u_+ is non-zero as we assumed that $u(x) > 0$ for some $x \in X$.

The semigroup $(e^{-tL})_{t \geq 0}$ on a connected graph is positivity improving by Theorem 6.1. Therefore, as $u_+ \geq 0$ satisfies $Lu_+ = \lambda_0 u_+$ and is non-zero, by the functional calculus and the positivity improving property we obtain

$$0 < e^{-tL} u_+ = e^{-t\lambda_0} u_+$$

for any $t > 0$. Hence, $u_+ > 0$ so that $u = u_+ > 0$. Therefore, any eigenfunction corresponding to λ_0 which is positive at some vertex is strictly positive.

From the argument above, it follows that any eigenfunction corresponding to λ_0 has a strict sign, i.e., is strictly positive or strictly negative. It is clear that any two functions of strict sign are not orthogonal in $\ell^2(X, m)$. This gives the uniqueness of u. □

If L is a self-adjoint operator arising from a Dirichlet form associated to a connected graph and λ_0 is an eigenvalue, then we have a unique strictly positive eigenfunction which minimizes the energy by the lemma above. In this context, we will refer to this eigenfunction as the ground state and λ_0 as the ground state energy.

We now discuss the case when the ground state energy is zero.

EXAMPLE 6.5 (When $\lambda_0 = 0$ is an eigenvalue). Suppose that (b, c) is a connected graph over (X, m) and L is an operator coming from a Dirichlet form Q associated to (b, c). If $\lambda_0 = 0$ is an eigenvalue for L, then $c = 0$ and $m(X) < \infty$.

Indeed, this follows as if $u > 0$ is a ground state for $\lambda_0 = 0$ given by the lemma above, then

$$0 = \lambda_0 = Q(u) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(u(x) - u(y))^2 + \sum_{x \in X} c(x)u(x)^2.$$

This shows that u is constant and $c = 0$. As $u \in D(L) \subseteq \ell^2(X, m)$, it follows that $m(X) < \infty$. In particular, as u is normalized, we obtain $u = 1/\sqrt{m(X)}$.

In the chapter about recurrence we will see that $\lambda_0 = 0$ is an eigenvalue for $L^{(D)}$ if and only if $c = 0$, $m(X) < \infty$ and the underlying graph is recurrent.

6.4. Theorems of Chavel–Karp and Li

In this section we prove two convergence results. We recall that the heat kernel of an operator L on $\ell^2(X, m)$ is given by

$$p_t(x, y) = \frac{\langle 1_x, e^{-tL} 1_y \rangle}{m(x)m(y)}$$

for $t > 0$ and $x, y \in X$. The following result connects the heat kernel of the Dirichlet Laplacian $L := L^{(D)}$ of a graph (b, c) over (X, m), the bottom of the spectrum $\lambda_0 := \inf \sigma(L)$, and the ground state.

THEOREM 6.6 (Theorem of Chavel–Karp). *Let (b, c) be a connected graph over (X, m). Then, there exists a function $u \colon X \longrightarrow [0, \infty)$ such that*

$$\lim_{t \to \infty} e^{\lambda_0 t} p_t(x, y) = u(x)u(y)$$

for all $x, y \in X$. If λ_0 is not an eigenvalue of L, then $u = 0$. If λ_0 is an eigenvalue of L, then u is the ground state, i.e., the unique normalized positive eigenfunction corresponding to λ_0.

PROOF. The proof is a direct application of the spectral theorem. Let $E := 1_{\{\lambda_0\}}(L)$ be the spectral projection onto the eigenspace of λ_0. By Proposition 6.2, $E = 0$ if λ_0 is not an eigenvalue and, if λ_0 is an eigenvalue, then $E = \langle u, \cdot \rangle u$, where u is the unique positive normalized eigenfunction corresponding to λ_0 given by Lemma 6.4.

Let μ be the signed spectral measure of L associated to $1_x, 1_y$ for $x, y \in X$. That is, μ is the unique signed measure which is characterized by

$$\langle 1_x, \psi(L) 1_y \rangle = \int_{\lambda_0}^{\infty} \psi(s) \mathrm{d}\mu(s)$$

for all bounded measurable functions on $[\lambda_0, \infty)$, see Proposition 3.15. Assume that λ_0 is an eigenvalue so that $1_{\{\lambda_0\}}(L) = \langle u, \cdot \rangle u$. We then get

$$\begin{aligned} m(x)m(y)|e^{\lambda_0 t} p_t(x,y) - u(x)u(y)| &= |\langle 1_x, (e^{\lambda_0 t} e^{-tL} - 1_{\{\lambda_0\}}(L)) 1_y \rangle| \\ &= \left| \int_{\lambda_0}^{\infty} \left(e^{-t(s-\lambda_0)} - 1_{\{\lambda_0\}}(s) \right) \mathrm{d}\mu(s) \right| \\ &\to 0 \end{aligned}$$

as $t \to \infty$ by Lebesgue's dominated convergence theorem. Note that μ is a finite measure so that the bounding function can be chosen as 1. If λ_0 is not an eigenvalue, then a similar argument gives the conclusion. □

We highlight one immediate corollary of the theorem above which characterizes when there exists a ground state.

COROLLARY 6.7 (Characterization of existence of a ground state). *Let (b, c) be a connected graph over (X, m). Then, λ_0 is an eigenvalue for L if and only if*

$$\lim_{t \to \infty} e^{\lambda_0 t} p_t(x, y) \neq 0$$

for any (all) $x, y \in X$.

We will now state and prove the second of our convergence statements, which gives that the logarithm of the heat kernel converges to the bottom of the spectrum.

THEOREM 6.8 (Theorem of Li). *Let (b, c) be a connected graph over (X, m). Then,*

$$\lim_{t \to \infty} \frac{1}{t} \log p_t(x, y) = -\lambda_0$$

for all $x, y \in X$.

PROOF. Let $e_x := 1_x / \sqrt{m(x)}$ for $x \in X$ and observe that $(e_x)_{x \in X}$ is an orthonormal basis for $\ell^2(X, m)$. Let

$$a_t(x, y) := \langle e_x, e^{-tL} e_y \rangle$$

for $x, y \in X$, $t \geq 0$ and write $a_t(x) := a_t(x, x)$. We will show that the function $t \mapsto \log a_t(x)$ on $[0, \infty)$ is superadditive for all $x \in X$.

Note that, as L is an operator coming from a Dirichlet form, e^{-tL} is positivity improving for $t > 0$ by Theorem 6.1 above and clearly positivity preserving for $t = 0$. Therefore, for all $x \in X$, $s, t \geq 0$, we obtain

$$\begin{aligned} a_{s+t}(x) &= \langle e_x, e^{-(s+t)L} e_x \rangle = \langle e^{-sL} e_x, e^{-tL} e_x \rangle \\ &= \sum_{y \in X} \langle e^{-sL} e_x, e_y \rangle \langle e_y, e^{-tL} e_x \rangle \\ &\geq \langle e^{-sL} e_x, e_x \rangle \langle e_x, e^{-tL} e_x \rangle = a_s(x) a_t(x). \end{aligned}$$

Theorem 6.1 implies $a_t(x) > 0$ for all $t \geq 0$ and $x \in X$, thus, we may take the logarithm of $a_t(x)$ for all $x \in X$ and $t \geq 0$. The estimate above then shows that $t \mapsto \log a_t(x)$ is superadditive, i.e., satisfies

$$\log a_s(x) + \log a_t(x) \leq \log a_{s+t}(x)$$

for $s, t \geq 0$. Furthermore, $a_t(x) \leq 1$ since $a_t(x) = e^{-tL} 1_x(x)$ and semigroups associated to operators coming from Dirichlet forms are contracting by Theorem 5.15. Therefore, $\log a_t(x) \leq 0$. Putting all of this together, we get that the following limit exists for every $x \in X$

$$\lim_{t \to \infty} \frac{1}{t} \log a_t(x) = \sup_{t \in (0,\infty)} \frac{1}{t} \log a_t(x).$$

Now, for $t \geq 1$ and $x, y \in X$, by a similar reasoning as above we obtain

$$\begin{aligned} a_{t-1}(x) a_1(x,y) &= \langle e^{-(t-1)L} e_x, e_x \rangle \langle e_x, e^{-L} e_y \rangle \\ &\leq \sum_{z \in X} \langle e^{-(t-1)L} e_x, e_z \rangle \langle e_z, e^{-L} e_y \rangle \\ &= \langle e^{-(t-1)L} e_x, e^{-L} e_y \rangle = \langle e_x, e^{-tL} e_y \rangle = a_t(x,y). \end{aligned}$$

By the same arguments for $t \geq 0$,

$$a_1(x,y) a_t(x,y) \leq \sum_{z \in X} \langle e^{-L} e_y, e_z \rangle \langle e_z, e^{-tL} e_y \rangle = a_{t+1}(y).$$

Hence, as $a_1(x,y) > 0$, we get

$$a_{t-1}(x) a_1(x,y) \leq a_t(x,y) \leq \frac{1}{a_1(x,y)} a_{t+1}(y).$$

Combining this line of inequalities with the fact that $\lim_{t \to \infty} \frac{1}{t} \log a_t(x)$ exists and $a_t(x,y) = a_t(y,x)$ gives that $\lim_{t \to \infty} \frac{1}{t} \log a_t(x,y)$ exists and is independent of $x, y \in X$.

Let

$$\lim_{t \to \infty} \frac{1}{t} \log a_t(x,y) =: -\lambda.$$

Since

$$a_t(x,y) = \langle e_x, e^{-tL} e_y \rangle = \sqrt{m(x) m(y)}\, p_t(x,y),$$

we conclude that

$$-\lambda = \lim_{t \to \infty} \frac{1}{t} \log a_t(x,y) = \lim_{t \to \infty} \frac{1}{t} \log p_t(x,y).$$

We will now show that $\lambda = \lambda_0$, which will complete the proof. First, we note that

$$\lim_{t \to \infty} \frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) = \lambda_0 - \lambda$$

for all $x, y \in X$. If λ_0 is an eigenvalue for L, it follows from Theorem 6.6 that $\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) = u(x)u(y) > 0$ so that

$$\lim_{t\to\infty} \frac{1}{t} \log\left(e^{\lambda_0 t} p_t(x,y)\right) = 0$$

and, hence, $\lambda = \lambda_0$ in this case.

If λ_0 is not an eigenvalue for L, then Theorem 6.6 yields $e^{\lambda_0 t} p_t(x,y) \to 0$ as $t \to \infty$. Therefore, $\log\left(e^{\lambda_0 t} p_t(x,y)\right) < 0$ for all t large enough and since $\frac{1}{t}\log\left(e^{\lambda_0 t} p_t(x,y)\right) \to \lambda_0 - \lambda$ as $t \to \infty$, it follows that $\lambda_0 \leq \lambda$.

We will now show that $\lambda_0 \geq \lambda$. Let $\varepsilon > 0$. From Proposition 6.2 we get

$$1_{[\lambda_0, \lambda_0+\varepsilon]}(L) \neq 0$$

since $\lambda_0 \in \sigma(L)$. As the set of functions $\{1_x \mid x \in X\}$ is total in $\ell^2(X,m)$, it follows that there exists an $x \in X$ such that

$$1_{[\lambda_0, \lambda_0+\varepsilon]}(L) 1_x \neq 0.$$

Let μ_x be the spectral measure of L associated to 1_x. Proposition 3.13 gives

$$\frac{p_t(x,x)}{m(x)^2} = \langle 1_x, e^{-tL} 1_x \rangle = \int_{\lambda_0}^{\lambda_0+\varepsilon} e^{-ts} \mathrm{d}\mu_x(s) \geq e^{-t(\lambda_0+\varepsilon)} \mu_x([\lambda_0, \lambda_0+\varepsilon])$$

as the spectral measure μ_x is supported on $[\lambda_0, \lambda_0+\varepsilon]$ by Proposition 3.17. Therefore,

$$-\lambda = \lim_{t\to\infty} \frac{1}{t} \log p_t(x,x) \geq -(\lambda_0 + \varepsilon),$$

that is, $\lambda \leq \lambda_0 + \varepsilon$. As $\varepsilon > 0$ was arbitrary, it follows that $\lambda \leq \lambda_0$, which concludes the proof. □

From the theorem above we immediately obtain the following corollary which states that the existence of a positive eigenfunction implies that the eigenvalue is the bottom of the spectrum.

Corollary 6.9 (Positive eigenfunctions are multiples of ground states). *Let (b,c) be a connected graph over (X,m). If there exists a non-trivial $u \geq 0$ with $u \in D(L)$ such that*

$$Lu = \lambda u$$

i.e., u is a positive eigenfunction corresponding to λ, then $\lambda = \lambda_0$. Furthermore, $u > 0$ and u is the ground state.

Proof. As λ is an eigenvalue, $\lambda \in \sigma(L)$ so that $\lambda_0 \leq \lambda$ by definition. Now, if $u \in D(L)$, $u \geq 0$ is non-trivial and satisfies $Lu = \lambda u$, then the functional calculus gives

$$e^{-tL} u = e^{-t\lambda} u$$

for $t \geq 0$. Therefore, for an arbitrary $x \in X$, using the positivity of u we get

$$p_t(x,x)u(x)m(x) \leq \sum_{y\in X} p_t(x,y)u(y)m(y) = e^{-tL}u(x) = e^{-t\lambda}u(x).$$

Applying Theorem 6.8 and choosing $x \in X$ such that $u(x) \neq 0$, we get

$$-\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log\left(p_t(x,x)u(x)m(x)\right) \leq \lim_{t\to\infty} \frac{1}{t} \log\left(e^{-t\lambda}u(x)\right) = -\lambda$$

so that $\lambda_0 \geq \lambda$. Therefore, $\lambda_0 = \lambda$.

The strict positivity of u follows from the proof of Lemma 6.4, which shows that $u = u_+ > 0$ and also gives the uniqueness of u. □

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Sheet 8

Large time behavior

Exercise 1 (Theorem of Chavel–Karp for resolvents) 4 points

Let Q be a Dirichlet form associated to (b,c) with operator L and $\lambda_0 = \inf \sigma(L)$. Let

$$g\colon (0,\infty) \times X \times X \longrightarrow \mathbb{R}$$

be such that

$$(L+\alpha)^{-1} f(x) = \sum_{y \in X} g_\alpha(x,y) f(y) m(y)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $\alpha > 0$. Show that $g_\alpha > 0$ and that there exists a $u\colon X \longrightarrow [0,\infty)$ such that

$$\lim_{\alpha \to 0^+} \alpha g_\alpha(x,y) = u(x)u(y)$$

for all $x,y \in X$. Show furthermore that $\lambda_0 = 0$ is an eigenvalue of L if and only if $u \neq 0$, in which case u is the ground state, i.e., the unique normalized positive eigenfunction for $\lambda_0 = 0$.

Exercise 2 (Is 0 an eigenvalue?) 4 points

Let (b,c) be a graph over (X,m) and let Q be an associated form with operator L. Show that $1 \in D(Q)$ and $Q(1) = 0$ if and only if 0 is an eigenvalue for L.

Exercise 3 (Vanishing of the heat kernel) 4 points

Let (b,c) be a connected graph over (X,m) with $m(X) = \infty$. Let Q be a Dirichlet form associated to (b,c) with operator L. Let p be the heat kernel associated to L. Show that

$$p_t(x,y) \to 0$$

as $t \to \infty$ for all $x,y \in X$.

Exercise 4 (Fractional Laplacian) 4 points

Let Δ be the Laplacian with standard weights on $\ell^2(\mathbb{Z})$. Show that for $s \in (0,1)$ the operator Δ^s is the operator associated to a non-locally finite graph b over $\mathbb{Z}$. Show specifically that $b(x,y) > 0$ if and only if $x \neq y$ and $c = 0$.

Hints:

- Exercise 3 on Sheet 5.
- Show $\sum_{y \in X} e^{-t\Delta} 1_y(x) = 1$ for all $x \in X$, $t \geq 0$. ("$e^{-t\Delta} 1 = 1$")