26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 07

CHAPTER 5

Regular Dirichlet forms: Approximation, Domain monotonicity and Markov Property

The crucial feature of regular Dirichlet forms is that they are determined by what happens on finite sets. This has various consequences, which are discussed in this section. In particular, we discuss domain monotonicity, the Markov property for semigroups and resolvents and a minimality property of solutions for the Dirichlet Laplacian $L = L^{(D)}$. Recall that the Markov property of the resolvents and semigroups mean that

$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1, \quad \alpha > 0 \qquad \text{and} \qquad 0 \leq e^{-tL} f \leq 1, \quad t \geq 0$$

for $f \in \ell^2(X,m)$, $0 \leq f \leq 1$. While it can easily be derived by approximation it is not a consequence of regularity but rather a general feature of Dirichlet forms. This is elaborated upon in the second part of the section.

5.1. Approximation

We have already seen in the proof of Lemma 4.7 how a regular Dirichlet form is approximated by its restriction to finite set. Here, we are going to elaborate on this approximation and derive various consequences.

Let X be a countable set and m a measure on X with full support. For a graph (b,c) over (X,m) denote the associated energy form by $\mathcal{Q} = \mathcal{Q}_{b,c}$ and the formal Laplacian $\mathcal{L} = \mathcal{L}_{b,c,m}$.

For any finite set $K \subseteq X$, we denote the restriction of m to K by m_K and let $Q_K^{(D)}$ be the form defined on $\ell^2(K,m_K)$ by

$$Q_K^{(D)}(f) := \mathcal{Q}(i_K f)$$

for $f \in \ell^2(K,m_K)$. Here,

$$i_K \colon C(K) \longrightarrow C_c(X)$$

is the canonical embedding, i.e., $i_K f$ is the extension of $f \in C(K)$ to X by setting $i_K f$ to be identically zero outside of K. We call $Q_K^{(D)}$ the *restriction of* $\mathcal{Q}$ to K.

Clearly, $Q_K^{(D)}$ is a closed form on $\ell^2(K,m_K)$ since the domain of $Q_K^{(D)}$ is the entire (finite dimensional) Hilbert space $\ell^2(K,m_K)$. Also, $C \circ (i_K f) = i_K(C \circ f)$ obviously holds for all $f \in C(K)$ and all normal contractions C and this easily gives that $Q_K^{(D)}$ is a Dirichlet form (compare reasoning in the proof of Lemma 4.7 as well).

As $Q_K^{(D)}$ is a Dirichlet form on the finite set K, it must come from a graph (b_K, c_K) over K. It is not hard to determine this graph. Indeed, a short calculation gives

$$Q_K^{(D)}(f) = \mathcal{Q}(i_K f) = \mathcal{Q}_{b_K,c_K}(f) + \sum_{x\in K} d_K(x) f(x)^2 = \mathcal{Q}_{b_K,c_K+d_K}(f),$$

where b_K is the restriction of b to $K\times K$, c_K is the restriction of c to K and

$$d_K(x) := \sum_{y\in X\setminus K} b(x,y)$$

describes the edge deficiency of a vertex $x \in K$ compared to the same vertex in X. Thus, $Q_K^{(D)}$ is the Dirichlet form associated to the graph $(b_K, c_K + d_K)$ over (K, m_K).

We denote the self-adjoint operator associated to $Q_K^{(D)}$ by $L_K^{(D)}$ and call it the *Dirichlet Laplacian with respect to* K. As $\ell^2(K, m_K)$ is finite dimensional this is a bounded operator. Quite remarkably this operator can be computed just from $\mathcal{L}$ independent of K. More specifically, we have the following proposition.

PROPOSITION 5.1 (Computing $L_K^{(D)}$ from $\mathcal{L}$). *Let (b,c) be a graph over (X,m) and $\mathcal{L}$ the associated formal Laplacian. Let $K \subseteq X$ be a finite set and $L_K^{(D)}$ the operator associated to the restriction of $\mathcal{Q}_{b,c}$ to K. Then,*

$$(L_K^{(D)} f)(x) = (\mathcal{L} \circ i_K)(f)(x)$$

holds for any $f \in \ell^2(K, m_K)$ and any $x \in K$.

PROOF. From the definition of $L_K^{(D)}$ and Green's formula for $\mathcal{Q}_{b,c}$ we find by a direct computation

$$\begin{aligned}\sum_{x\in K} \varphi(x) L_K^{(D)} f(x) m(x) &= \langle \varphi, L_K^{(D)} f\rangle_{\ell^2(K,m_K)} = Q_K^{(D)}(\varphi, f) \\ &= \mathcal{Q}_{b,c}(i_K(\varphi), i_K(f)) = \sum_{x\in X} \varphi(x) \mathcal{L} i_K(f)(x) m(x)\end{aligned}$$

for all $\varphi, f \in C(K)$. This then implies the formula for $L_K^{(D)}$. □

Since K is assumed to be finite, we can apply the results from Chapter 1 about finite graphs. This gives that the eigenvalues of $L_K^{(D)}$ are non-negative. Moreover, it gives that the resolvents $(L_K^{(D)} + \alpha)^{-1}$ satisfy the Markov property for all $\alpha > 0$. Specifically, for a function $f \in \ell^2(K, m_K)$ with $0 \le f \le 1$ and $\alpha > 0$, we have

$$0 \le \alpha (L_K^{(D)} + \alpha)^{-1} f \le 1.$$

After these preparations we now turn to the first main result of this section. It says that the resolvent of a restriction to a set becomes larger if the set is increased. This is known as domain monotonicity. A precise version is contained in the next lemma.

LEMMA 5.2 (Domain monotonicity). *Let (b,c) be a graph over (X,m). If K, H are finite subsets of X with $K \subseteq H$ and $\alpha > 0$, then*

$$(L_K^{(D)} + \alpha)^{-1} f \le (L_H^{(D)} + \alpha)^{-1} f$$

on K for all $f \in \ell^2(K, m_K)$ with $f \geq 0$, where f is extended by zero on $H \setminus K$.

Proof. Set $u_K := i_K(L_K^{(D)} + \alpha)^{-1} f$, $u_H := i_H(L_H^{(D)} + \alpha)^{-1} f$ and $v := u_H - u_K$. On K we have by Proposition 5.1 that

$$\begin{aligned}(\mathcal{L} + \alpha)v &= (\mathcal{L} + \alpha) i_H (L_H^{(D)} + \alpha)^{-1} f - (\mathcal{L} + \alpha) i_K (L_K^{(D)} + \alpha)^{-1} f \\ &= (L_H^{(D)} + \alpha)(L_H^{(D)} + \alpha)^{-1} f - (L_K^{(D)} + \alpha)(L_K^{(D)} + \alpha)^{-1} f \\ &= f - f \\ &= 0.\end{aligned}$$

We use this to show $v \geq 0$: Clearly, $v = 0$ holds on $X \setminus H$. Moreover, $v \geq 0$ holds on $H \setminus K$ as u_K vanishes outside of K and $u_H \geq 0$ holds by the Markov property of the resolvents (of Dirichlet forms on finite dimensional spaces). It remains to show $v \geq 0$ on K. As K is finite, the restriction of v to K must attain a minimum. Consider now $x_0 \in K$ such that $v(x_0)$ is the minimum of v on K. Assume $v(x_0) < 0$. As we have already established $v \geq 0$ outside of K we conclude that the value $v(x_0)$ is the minimal value of v on the whole of X. Using this and the already established vanishing of $(\mathcal{L} + \alpha)v$ on K we then obtain the contradiction

$$\begin{aligned}0 &= (\mathcal{L} + \alpha)v(x_0) \\ &= \frac{1}{m(x_0)} \left(\sum_{y \in X} b(x_0, y) \underbrace{(v(x_0) - v(y))}_{\leq 0} + \underbrace{c(x_0)}_{\geq 0} \underbrace{v(x_0)}_{<0} \right) + \alpha \underbrace{v(x_0)}_{<0} < 0.\end{aligned}$$

This contradiction shows $v(x_0) \geq 0$ and thus $v \geq 0$. □

Our next result will show convergence of the restrictions to finite subsets for both the resolvent and the semigroup. In order to be able to state the result conveniently we will use the following notation.

Notation. Let (b, c) be a graph over (X, m), let $Q = Q_{b,c,m}^{(D)}$ be the associated regular Dirichlet form and $Q_K^{(D)}$ be the restriction of Q to the finite set $K \subseteq X$ with associated Dirichlet Laplacian $L_K^{(D)}$ acting on $\ell^2(K, m_K)$ as defined above. We extend $L_K^{(D)}$ by zero on the orthogonal complement of $\ell^2(K, m_K)$ in $\ell^2(X, m)$. We will extend functions Φ of $L_K^{(D)}$ accordingly, that is, for $f \in \ell^2(X, m)$, we write $\Phi(L_K^{(D)}) f$ for $i_K \Phi(L_K^{(D)})(f|_K)$. This is, in particular, used for the function $\Phi(\lambda) := (\lambda + \alpha)^{-1}$, i.e.,

$$(L_K^{(D)} + \alpha)^{-1} f \quad \text{for} \quad i_K (L_K^{(D)} + \alpha)^{-1}(f|_K),$$

but also applies to $\Phi(\lambda) := (\lambda + \alpha)$ or $\Phi(\lambda) := e^{-t\lambda}$. The extended operators will be denoted by the same symbols as the original ones.

Lemma 5.3 (Convergence of finite approximations). *Let (b, c) be a graph over (X, m) and let Q be the associated regular Dirichlet form with Laplacian L. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n \in \mathbb{N}} K_n$.*

(a) *If $f \in \ell^2(X,m)$ and $\alpha > 0$, then*

$$\lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1} f = (L+\alpha)^{-1} f.$$

(b) *If $f \in \ell^2(X,m)$ and $t \geq 0$, then*

$$\lim_{n\to\infty} e^{-tL_{K_n}^{(D)}} f = e^{-tL} f.$$

Furthermore, if additionally $f \geq 0$, then the sequences in both statements converge not only in $\ell^2(X,m)$ but also pointwise monotonically increasingly, i.e.,

$$(L_{K_n}^{(D)} + \alpha)^{-1} f \nearrow (L+\alpha)^{-1} f \quad \text{and} \quad e^{-tL_{K_n}^{(D)}} f \nearrow e^{-tL} f$$

pointwise as $n \to \infty$.

Remark 5.4. The proof of (b) will actually show

$$\lim_{n\to\infty} \Phi(L_{K_n}^{(D)}) f = \Phi(L) f$$

for any $f \in \ell^2(X,m)$ and any continuous function $\Phi\colon [0,\infty) \longrightarrow \mathbb{R}$ vanishing at infinity (i.e. satisfying $\lim_{s\to\infty} \Phi(s) = 0$).

Proof. (a) In the proof we will use the following characterization of resolvents: Whenever Q is a positive closed form with associated self-adjoint operator L, the function f is an arbitrary element of the underlying Hilbert space and $\alpha > 0$, then $u := (L+\alpha)^{-1} f$ is the unique minimizer of

$$J(v) := Q(v) + \alpha \left\| v - \frac{1}{\alpha} f \right\|^2$$

over $v \in D(Q)$. See Theorem 3.38 for a proof of this characterization.

After decomposing f into positive and negative parts, we can restrict attention to $f \geq 0$. Define

$$u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f, \quad n \in \mathbb{N}.$$

Then, $u_n \geq 0$ by the Markov property in Proposition 5.2 (b).

By domain monotonicity, Proposition 5.2 (c), the sequence $(u_n(x))$ is monotonically increasing for any $x \in X$. Moreover, we have $\|u_n\| \leq \alpha^{-1} \|f\|$ since the operators $(L_{K_n}^{(D)} + \alpha)^{-1}$ are bounded uniformly in norm by $1/\alpha$, as follows from the spectral theorem. This implies that $(u_n(x))$ is also bounded for any $x \in X$. Thus, the sequence (u_n) converges pointwise and in $\ell^2(X,m)$ to a function $u \in \ell^2(X,m)$ by Lebesgue's dominated convergence theorem.

Let $\varphi \in C_c(X)$. Assume without loss of generality that the support of φ is contained in K_1. Then, $Q(\varphi) = Q_{K_n}^{(D)}(\varphi)$ for all n sufficiently large. Since Q is closed and thus lower semi-continuous, convergence of (u_n) to u and the minimizing property of u_n then give

$$\begin{aligned} Q(u)+\alpha\left\|u-\tfrac{1}{\alpha}f\right\|^2 &\le \liminf_{n\to\infty}\left(Q(u_n)+\alpha\left\|u-\tfrac{1}{\alpha}f\right\|^2\right)\\ &= \liminf_{n\to\infty}\left(Q(u_n)+\alpha\left\|u_n-\tfrac{1}{\alpha}f\right\|^2\right)\\ &= \liminf_{n\to\infty}\left(Q^{(D)}_{K_n}(u_n)+\alpha\left\|u_n-\tfrac{1}{\alpha}f\right\|^2\right)\\ &\le \liminf_{n\to\infty}\left(Q^{(D)}_{K_n}(\varphi)+\alpha\left\|\varphi-\tfrac{1}{\alpha}f\right\|^2\right)\\ &= Q(\varphi)+\alpha\left\|\varphi-\tfrac{1}{\alpha}f\right\|^2.\end{aligned}$$

As $\varphi \in C_c(X)$ is arbitrary and Q is regular, this implies

$$Q(u)+\alpha\left\|u-\tfrac{1}{\alpha}f\right\|^2 \le Q(v)+\alpha\left\|v-\tfrac{1}{\alpha}f\right\|^2$$

for any $v \in D(Q)$. Thus, u is a minimizer of

$$Q(v)+\alpha\left\|v-\tfrac{1}{\alpha}f\right\|^2,$$

so that u must then be equal to $(L+\alpha)^{-1}f$ by the characterization of the resolvent stated at the start of the proof.

(b) Let $C_0([0,\infty)) := \{\Phi \in C([0,\infty)) \mid \lim_{s\to\infty}\Phi(s)=0\}$ be the vector space of continuous functions vanishing at infinity. Define for $\alpha > 0$ the function $\varPhi_{(\alpha)} \colon [0,\infty) \longrightarrow \mathbb{R}$ by

$$\varPhi_{(\alpha)}(s) := (s+\alpha)^{-1}.$$

Then, clearly $\varPhi_{(\alpha)} \in C_0([0,\infty))$ for any $\alpha > 0$ and $\varPhi_{(\alpha)}(L) = (L+\alpha)^{-1}$ by the functional calculus, see Definition 3.7.

Let $\mathcal{A}$ be the closure in the supremum norm of the linear span of $\varPhi_{(\alpha)}$ for $\alpha > 0$. Then, by (a) we have

$$\lim_{n\to\infty}\Phi(L^{(D)}_{K_n})f = \Phi(L)f$$

for all $\Phi \in \mathcal{A}$ and $f \in \ell^2(X,m)$. We will show that for every $t \ge 0$, the function $[0,\infty) \longrightarrow \mathbb{R}$ given by $x \mapsto e^{-tx}$ belongs to $\mathcal{A}$, which will complete the proof. The statement for $t = 0$ is clear, so we assume that $t > 0$.

We note that it suffices to show that

$$\mathcal{A} = C_0([0,\infty)).$$

We will do so by proving the following claim and then applying the Stone–Weierstrass theorem.

Claim. The set $\mathcal{A}$ has the following properties:

- $\mathcal{A}$ separates the points of $[0,\infty)$ (i.e., for any $x, y \in [0,\infty)$ with $x \ne y$ there exists a $\Phi \in \mathcal{A}$ with $\Phi(x) \ne \Phi(y)$).
- $\mathcal{A}$ does not vanish identically at any point (i.e., for any $x \in [0,\infty)$ there exists a $\Phi \in \mathcal{A}$ with $\Phi(x) \ne 0$).
- $\mathcal{A}$ is an algebra.

Proof of the claim. The first two points follow directly by considering $\Phi := \varPhi_{(1)}$. As for the last point, by definition, $\mathcal{A}$ is a vector space. Thus, it suffices to show that $\mathcal{A}$ is closed under multiplication. To show this it suffices to show $\Phi_{(\alpha)}\Phi_{(\beta)} \in \mathcal{A}$ for any $\alpha, \beta > 0$. For $\alpha \neq \beta$ this is clear as

$$\Phi_{(\alpha)}\Phi_{(\beta)} = \frac{1}{\alpha - \beta}(\Phi_{(\beta)} - \Phi_{(\alpha)}).$$

For $\alpha = \beta$ we can consider a sequence (β_n) of positive numbers with $\beta_n \to \beta = \alpha$ and $\beta_n \neq \beta$ for all n. Then, by what we have just shown $\Phi_{(\alpha)}\Phi_{(\beta_n)}$ belongs to $\mathcal{A}$ as $\beta_n \neq \alpha$. Thus, $\Phi_{(\alpha)}\Phi_{(\beta)} \in \mathcal{A}$ as $\lim_{n\to\infty} \Phi_{(\alpha)}\Phi_{(\beta_n)} = \Phi_{(\alpha)}\Phi_{(\beta)}$ in the supremum norm. This finishes the proof of the claim.

Given the claim, the desired statement that $\mathcal{A} = C_0([0,\infty))$ follows directly from the Stone–Weierstrass theorem. This concludes the proof of (b).

In the case of $f \geq 0$, the fact that the sequence (u_n) given by $u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f$ is monotonically increasing pointwise follows from Lemma 5.2 (c). The corresponding statement for $(e^{-tL_{K_n}^{(D)}} f)$ for $t > 0$ follows from the connection between resolvents and semigroups (the case $t = 0$ is clear). That is, from the formula

$$\left(\frac{k}{t}\left(x + \frac{k}{t}\right)^{-1}\right)^{k} = \left(1 + \frac{tx}{k}\right)^{-k} \to e^{-tx}$$

as $k \to \infty$ for any $t > 0$, it follows that

$$e^{-tL_{K_n}^{(D)}} f = \lim_{k\to\infty} \left(\frac{k}{t}\left(L_{K_n}^{(D)} + \frac{k}{t}\right)^{-1}\right)^{k} f$$

for any $f \in \ell^2(X,m)$ and $t > 0$, see Theorem 3.26 for more details. □

Remark 5.5. The convergence given in the previous lemma is a characterization of regularity (Exercise).

Combining the Markov property of the resolvents of restrictions to finite sets proven in Lemma 5.2 (b) along with the convergence statements in Lemma 5.3 gives the Markov properties for the semigroups and resolvents associated to the regular form on the entire graph.

Corollary 5.6 (Markov property of resolvents and semigroups). *Let (b,c) be a graph over (X,m) with associated regular Dirichlet form Q and Laplacian L. Then, for any $f \in \ell^2(X,m)$ with $0 \leq f \leq 1$,*

$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1 \qquad \text{and} \qquad 0 \leq e^{-tL} f \leq 1$$

for all $\alpha > 0$ and $t \geq 0$.

Remark 5.7. It is not necessary for the function to be bounded in order for the positivity preserving property above to hold (Exercise).

Proof. After suitable approximation procedures, it suffices to consider $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$. Consider now an increasing sequence (K_n) of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$. In particular, we may assume that the support of φ is contained in K_n for all $n \in \mathbb{N}$. By Lemma 5.3 we have

$$(L+\alpha)^{-1}\varphi = \lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1}\varphi.$$

By the Markov property for finite sets, Lemma 5.2 (b), we have $0 \leq \alpha(L_{K_n}^{(D)} + \alpha)^{-1}\varphi \leq 1$ for $n \in \mathbb{N}$. Combining these two observation we obtain the desired statement for the resolvents.

We now turn to proving the statement for the semigroups. The case $t = 0$ is clear so we restrict attention to the case $t > 0$. As above, the equality

$$e^{-tL} f = \lim_{k\to\infty} \left(\frac{k}{t}\left(L + \frac{k}{t}\right)^{-1}\right)^k f$$

for any $f \in \ell^2(X, m)$ given in Theorem 3.26 gives the statement from the already shown statement for the resolvents. ☐

Lemma 5.8 (Resolvents as minimal solutions to $(\mathcal{L} + \alpha)u = f$). *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Let $\alpha > 0$ and $f \in \ell^2(X, m)$. Then $u := (L + \alpha)^{-1} f$ belongs to $\mathcal{F}$ and satisfies*

$$(\mathcal{L} + \alpha)u = f.$$

Furthermore, if additionally $f \geq 0$, then u is the smallest $v \in \mathcal{F}$ with $v \geq 0$ and $(\mathcal{L} + \alpha)v \geq f$.

Proof. We first show that u is a solution as stated. For $\alpha > 0$, we note that the resolvent $(L + \alpha)^{-1}$ maps $\ell^2(X, m)$ into $D(L) \subseteq D(Q) \subseteq \mathcal{D} \subseteq \mathcal{F}$, where the last inclusion follows by Proposition 2.9 (b) and the other inclusions follow from the definitions. By Theorem 4.9, the operator L is a restriction of $\mathcal{L}$ so that $u = (L + \alpha)^{-1} f \in \mathcal{F}$ satisfies

$$(\mathcal{L} + \alpha)u = f,$$

as claimed.

We now establish the minimality of u when additionally $f \geq 0$. We first note that $u \geq 0$ whenever $f \geq 0$ as the resolvent is positivity preserving by Corollary 5.6. Now, let $v \geq 0$ be another function with $v \in \mathcal{F}$ and $(\mathcal{L}+\alpha)v \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$ and let $L_{K_n}^{(D)}$ be the Dirichlet Laplacian on $\ell^2(K_n, m_{K_n})$ for $n \in \mathbb{N}$. We recall that $L_{K_n}^{(D)}$ agrees with $\mathcal{L}$ on the set of functions supported in K_n. For $n \in \mathbb{N}$ let $f_n := f 1_{K_n}$,

$$u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f_n$$

and extend u_n by 0 to $X \setminus K_n$. Then, letting $w_n := v - u_n$, for $n \in \mathbb{N}$ the function w_n satisfies

- $(\mathcal{L} + \alpha)w_n = (\mathcal{L} + \alpha)v - (L_{K_n}^{(D)} + \alpha)u_n \geq f - f_n = 0$ on K_n,
- $w_n \wedge 0$ attains a minimum on K_n since K_n is finite,
- $w_n = v \geq 0$ on $X \setminus K_n$.

Hence, we can apply the minimum principle, Theorem 2.10, and find $w_n = v - u_n \geq 0$ on X. Therefore, $v \geq u_n$ on X.

Finally, we show that (u_n) converges to u and thus $v \geq u$, which will complete the proof. Indeed, this can be seen by first fixing $k \in \mathbb{N}$ and considering $(L_{K_n}^{(D)} + \alpha)^{-1} f_k$ for $n \geq k$. Then, Lemma 5.3 (a) gives

$$\lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1} f_k = (L + \alpha)^{-1} f_k.$$

Furthermore, by the spectral theorem

$$\|\alpha(L+\alpha)^{-1}\| \leq 1 \quad \text{and} \quad \|\alpha(L_{K_n}^{(D)}+\alpha)^{-1}\| \leq 1$$

for all $n \in \mathbb{N}$ and all $\alpha > 0$. Therefore, as $f_k \to f$ in $\ell^2(X,m)$ we have

$$\lim_{k\to\infty} (L+\alpha)^{-1} f_k = (L+\alpha)^{-1} f$$

and

$$\|(L_{K_n}^{(D)}+\alpha)^{-1}(f_n - f_k)\| \leq \frac{1}{\alpha}\|f_n - f_k\| \to 0$$

as $k, n \to \infty$. Thus, the triangle inequality implies

$$\begin{aligned}\|u_n - u\| \leq \|(L_{K_n}^{(D)}+\alpha)^{-1}(f_n - f_k)\| + \|(L_{K_n}^{(D)}+\alpha)^{-1} f_k - (L+\alpha)^{-1} f_k\| \\ + \|(L+\alpha)^{-1}(f_k - f)\|,\end{aligned}$$

where we have shown that all three terms go to 0 as $k, n \to \infty$. □

As the resolvent associated to the operator coming from the regular Dirichlet form generates the minimal positive solution of the Poisson equation, so does the semigroup generate the minimal solution of the heat equation. This is discussed next.

We recall that a function

$$u \colon [0,\infty) \times X \longrightarrow \mathbb{R}$$

is called a solution of the heat equation with initial condition f if for all $x \in X$, the mapping $t \mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, $u_t \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t) u_t(x) = 0$$

for all $x \in X$ and $t > 0$ with $u_0 = f$. We call u a supersolution of the heat equation with initial condition f if u satisfies all the assumptions above and instead of equality in the heat equation we have

$$(\mathcal{L} + \partial_t) u_t(x) \geq 0, \quad t > 0, x \in X.$$

We now show that if the initial condition is positive, then the semigroup of the associated Laplacian generates the minimal positive supersolution of the heat equation.

Lemma 5.9 (Semigroup as the minimal solution of the heat equation). *Let (b,c) be a graph over (X,m) with associated regular Dirichlet form Q and Laplacian L. Let $f \in \ell^2(X,m)$. If*

$$u_t(x) := e^{-tL} f(x)$$

for $t \geq 0$ and $x \in X$, then u is a solution of the heat equation with initial condition f.

Furthermore, if additionally $f \geq 0$, then u is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

Proof. As L is a restriction of $\mathcal{L}$ by Theorem 4.9, the fact that u given by $u_t(x) := e^{-tL} f(x)$ for $t \geq 0$ and $x \in X$ is a solution of the heat equation with initial condition f for $f \in \ell^2(X,m)$ is a consequence of the spectral theorem and can be found as Theorem 3.24.

We now show minimality. Let f additionally satisfy $f \geq 0$. Then, by Corollary 5.6 we have $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in X$ as the semigroup is positivity preserving. Thus, u is a positive solution of the heat equation with initial condition f. Now, suppose that w is a positive supersolution of the heat equation with $w_0 \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$ and let $L_{K_n}^{(D)}$ be the Dirichlet Laplacian on $\ell^2(K_n, m_{K_n})$. We recall that $L_{K_n}^{(D)}$ agrees with $\mathcal{L}$ on functions supported in K_n. For $n \in \mathbb{N}$ let $f_n := f 1_{K_n}$ and

$$u_t^{(n)}(x) := e^{-tL_{K_n}^{(D)}} f_n(x)$$

for $x \in K_n$ and $t \geq 0$. We extend $u^{(n)}$ by 0 to $[0,\infty) \times X \setminus K_n$. If $w^{(n)} := w - u^{(n)}$, then for $n \in \mathbb{N}$ the function $w^{(n)}$ satisfies

- $(\mathcal{L} + \partial_t) w^{(n)} \geq 0$ on $(0,T) \times K_n$,
- $w^{(n)} \wedge 0$ attains a minimum on the compact set $[0,T] \times K_n$ since $w^{(n)}$ is continuous,
- $w^{(n)} \geq 0$ on $((0,T] \times (X \setminus K_n)) \cup (\{0\} \times K_n)$.

Hence, we can apply the minimum principle for the heat equation, Theorem 2.13, to obtain $w^{(n)} = w - u^{(n)} \geq 0$ on $[0,T] \times K_n$ for all $n \in \mathbb{N}$. Therefore, $w \geq u^{(n)}$ on $[0,T] \times X$ as $u^{(n)}$ vanishes outside of K_n and w is positive.

We now show that $(u^{(n)})$ converges to u from which it follows that $w \geq u$, thereby completing the proof. Indeed, this can be seen by first fixing $k \in \mathbb{N}$ and considering $e^{-tL_{K_n}^{(D)}} f_k$ for $n \geq k$. Then, Lemma 5.3 (b) gives

$$\lim_{n\to\infty} e^{-tL_{K_n}^{(D)}} f_k = e^{-tL} f_k.$$

Furthermore, by Proposition 3.23 we have

$$\|e^{-tL}\| \leq 1 \quad \text{and} \quad \|e^{-tL_{K_n}^{(D)}}\| \leq 1$$

for all $n \in \mathbb{N}$ and all $t \geq 0$. As $f_k \to f$ in $\ell^2(X,m)$ we have

$$\lim_{k\to\infty} e^{-tL} f_k = e^{-tL} f$$

and

$$\|e^{-tL_{K_n}^{(D)}}(f_n - f_k)\| \leq \|f_n - f_k\| \to 0$$

as $k, n \to \infty$. Thus, the triangle inequality implies

$$\|u_t^{(n)} - u_t\| \leq \|e^{-tL_{K_n}^{(D)}}(f_n - f_k)\| + \|e^{-tL_{K_n}^{(D)}} f_k - e^{-tL} f_k\| + \|e^{-tL}(f_k - f)\|,$$

where we have shown that all three terms go to 0 as $k, n \to \infty$. □

5.2. The Context of General Dirichlet Forms

Domain monotonicity, and minimality of solutions are special features of regular Dirichlet forms. The Markov property of the resolvents and semigroups, however, does not have to do with regularity. It is a general feature of Dirichlet forms. This is discussed in this section.

We first define the concept of a Dirichlet form. Let (X, μ) be a σ-finite measure space. Let H be the Hilbert space of square integrable real-valued functions on X, i.e., $H = L^2(X, \mu)$. We let Q be a positive closed form with domain $D(Q) \subseteq H$. In particular, Q is a positive symmetric form and $D(Q)$ is complete with respect to the form norm $\|\cdot\|_Q$ given by $\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$ for all $f \in D(Q)$, where $\|\cdot\|$ denotes the norm arising from the inner product on H. We recall that Q is extended on the diagonal to all of H via $Q(f) = \infty$ for $f \in H \setminus D(Q)$.

Recall that a map $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a normal contraction if $C(0) = 0$ and $|C(s) - C(t)| \leq |s - t|$ for all $s, t \in \mathbb{R}$.

Definition 5.10 (Dirichlet form). A positive closed form Q with domain $D(Q)$ in $H = L^2(X, \mu)$ is called a *Dirichlet form* if $C \circ f \in D(Q)$ and

$$Q(C \circ f) \leq Q(f)$$

for all $f \in D(Q)$ and all normal contractions C.

This condition has a number of surprising consequences which we will discuss. We note that while the definition requires compatibility with all normal contractions, it actually suffices to check the condition for the normal contraction $C_{[0,1]}$ given by

$$C_{[0,1]}(s) := 0 \vee (s \wedge 1), \quad s \in \mathbb{R},$$

that is, cutting below by 0 and above by 1. This follows directly from the proof of Theorem 5.15 given below.

We recall that whenever Q is a positive closed form, the associated operator L is positive, that is, L is self-adjoint and $\sigma(L) \subseteq [0, \infty)$, see Theorem 3.33 for more details and Lemma 3.30 for the construction of L. As $\sigma(L) \subseteq [0, \infty)$, it follows that we can use the functional calculus to define both the resolvent $(L + \alpha)^{-1}$ for $\alpha > 0$ and the semigroup e^{-tL} for $t \geq 0$, which are bounded operators on H, see Propositions 3.13.

We now give some consequences for both semigroups and resolvents when the associated operator comes from a Dirichlet form. We say that an operator A with domain $D(A) \subseteq L^2(X, \mu)$ is *positivity preserving* if $Af \geq 0$ whenever $f \in D(A)$ satisfies $f \geq 0$. We say that A is *Markov* if $0 \leq Af \leq 1$ holds for all $f \in D(A)$ with $0 \leq f \leq 1$. It is not hard to see that any Markov operator is positivity preserving.

We start with a lemma which will be applied to the semigroup and resolvent in what follows.

Lemma 5.11. *Let A be a bounded self-adjoint positivity preserving operator on $H = L^2(X, \mu)$. Then, the quadratic form Q_{I-A} defined by*

$$Q_{I-A}(f, g) := \langle (I - A)f, g \rangle$$

satisfies

$$Q_{I-A}(|f|) \leq Q_{I-A}(f)$$

for all $f \in L^2(X, \mu)$. If, furthermore, A is Markov, then Q_{I-A} is a Dirichlet form and for $f, g \in L^2(X, \mu) \cap L^\infty(X, \mu)$ we have

$$Q_{I-A}(fg) \leq 2\|g\|_\infty^2 Q_{I-A}(f) + 2\|f\|_\infty^2 Q_{I-A}(g).$$

PROOF. We show the first statement for simple functions. The statement for functions in $L^2(X,\mu)$ then follows by approximation. Let $f := \sum_{k=1}^n f_k 1_{U_k}$ for $f_1,\ldots,f_n \in \mathbb{R}$ and $U_1,\ldots,U_n \subseteq X$ which are measurable disjoint sets of finite measure. Then, by a direct calculation we find the following explicit formula for $Q_{I-A}(f)$

$$Q_{I-A}(f) = \frac{1}{2}\sum_{k,l=1}^n b_{k,l}(f_k - f_l)^2 + \sum_{k=1}^n c_k f_k^2,$$

where $b_{k,l} := \langle 1_{U_k}, A1_{U_l}\rangle$ and $c_k := \mu(U_k) - \sum_{l=1}^n b_{k,l}$.

If A is positivity preserving, then $b_{k,l} \geq 0$ and the explicit formula for $Q_{I-A}(f)$ above easily gives $Q_{I-A}(|f|) \leq Q_{I-A}(f)$.

If A is Markov, then for $U = \bigcup_{l=1}^n U_l$ we have $0 \leq A1_U \leq 1$ and thus

$$\sum_{l=1}^n b_{k,l} = \langle 1_{U_k}, A1_U\rangle \leq \mu(U_k).$$

Therefore, $c_k \geq 0$ by definition. Then, the explicit formula for $Q_{I-A}(f)$ above easily gives $Q_{I-A}(C \circ f) \leq Q_{I-A}(f)$ for any normal contraction C. As Q_{I-A} is clearly symmetric positive and closed, this shows that Q_{I-A} is a Dirichlet form.

For the last statement, we let $g := \sum_{k=1}^n g_k 1_{U_k}$, where we alter the sets $U_1,\ldots,U_n$ appearing in the definition of f if necessary. Then, using Young's inequality we get

$$(f_k g_k - f_l g_l)^2 = (g_k(f_k - f_l)) + f_l(g_k - g_l))^2 \leq 2g_k^2(f_k - f_l)^2 + 2f_l^2(g_k - g_l)^2,$$

which, along with the estimates

$$\sum_{k=1}^n c_k f_k^2 g_k^2 \leq \|f\|_\infty^2 \sum_{k=1}^n c_k g_k^2 \text{ and } \sum_{k=1}^n c_k f_k^2 g_k^2 \leq \|g\|_\infty^2 \sum_{k=1}^n c_k g_k^2,$$

yields

$$\begin{aligned} Q_{I-A}(fg) &= \frac{1}{2}\sum_{k,l=1}^n b_{k,l}(f_k g_k - f_l g_l)^2 + \sum_{k=1}^n c_k f_k^2 g_k^2 \\ &\leq 2\|g\|_\infty^2 Q(f) + 2\|f\|_\infty^2 Q(g). \end{aligned}$$

This concludes the proof. □

Define the quadratic forms $Q^{(t)}\colon H \longrightarrow \mathbb{R}$ associated to the semigroup by

$$Q^{(t)}(f) := \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

for $t > 0$. As the semigroup consists of bounded self-adjoint operators, see Propositions 3.23, we have $D(Q^{(t)}) = H$ as well as

$$Q^{(t)}(f,g) = \frac{1}{t}\langle (I - e^{-tL})f, g\rangle$$

for all $f, g \in H$ and $t > 0$ by polarization. Moreover, for $\alpha > 0$ we define the quadratic form $Q_{(\alpha)}\colon H \longrightarrow \mathbb{R}$ associated to the resolvent by

$$Q_{(\alpha)}(f) := \alpha\langle (I - \alpha(L+\alpha)^{-1})f, f\rangle.$$

We now show that the value of a closed form on the diagonal is the limit of the value of the quadratic forms associated to the resolvents and the semigroup.

LEMMA 5.12. *Let Q be a positive closed form on H and Q' the map on H with $Q'(f) := Q(f,f)$ for $f \in D(Q)$ and $Q'(f) := \infty$ otherwise. Then, for all $f \in H$,*

$$Q'(f) = \lim_{\alpha \to \infty} Q_{(\alpha)}(f) = \lim_{t \to 0^+} Q^{(t)}(f).$$

In particular, the limits are finite if and only if $f \in D(Q)$.

PROOF. The statement follows directly from the connection between the operator and form, properties of the functional calculus given in Proposition 3.13, and the monotone convergence theorem as

$$\alpha\left(1 - \alpha(s+\alpha)^{-1}\right) \nearrow s \qquad \text{and} \qquad \frac{1}{t}(1 - e^{-tx}) \nearrow x$$

as $\alpha \to \infty$ and $t \to 0^+$, respectively, for all $s \geq 0$ and $x \geq 0$, respectively. □

We now state and prove the Beurling–Deny criteria for positive closed forms. The first criterion shows that a form being compatible with the absolute value is equivalent to the fact that the heat semigroup and the resolvent are positivity preserving.

THEOREM 5.13 (First Beurling–Deny criterion). *Let Q be a positive closed form on $H = L^2(X,\mu)$ and let L be the associated positive operator. Then, the following statements are equivalent:*

(i) *$Q(|f|) \leq Q(f)$ for all $f \in H$.*
(ii) *$\alpha(L+\alpha)^{-1}$ is positivity preserving for every $\alpha > 0$.*
(iii) *e^{-tL} is positivity preserving for every $t \geq 0$.*

PROOF. (i) $\Longrightarrow$ (ii): Let $f \geq 0$ be given. By the characterization of the resolvent as a minimizer we know that $h := (L+\alpha)^{-1}f$ is the unique minimizer of ψ given by

$$\psi(v) := Q(v) + \alpha\|v - \frac{f}{\alpha}\|^2.$$

Now, by assumption (i) we have $Q(|h|) \leq Q(h)$. Moreover, for $f \geq 0$ clearly

$$\||h| - \frac{f}{\alpha}\|^2 \leq \|h - \frac{f}{\alpha}\|^2$$

holds. Thus, $|h|$ is a minimizer of ψ as well. This shows $h = |h| \geq 0$.

(ii) $\Longrightarrow$ (iii): This follows directly from Theorem 3.26 (b).

(iii) $\Longrightarrow$ (i): By Lemma 5.11 we have

$$\frac{1}{t}\langle (I - e^{-tL})|f|, |f|\rangle \leq \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

for all $t > 0$ and $f \in H$. Letting $Q^{(t)}(f) := \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$, Corollary 5.12 gives $\lim_{t \to 0^+} Q^{(t)}(f) = Q(f)$ for all $f \in H$. Thus, we conclude

$$Q(|f|) \leq Q(f).$$

This finishes the proof. □

REMARK 5.14. One can check that (i) in Theorem 5.13 is equivalent to:

(i.a) $Q(f_+) \leq Q(f)$ for all $f \in H$,

where $f_+ := f \vee 0$ denotes the positive part of f.

Indeed, as $f_+ = (f + |f|)/2$, it is clear that (i) implies (i.a). On the other hand, (i.a) implies $Q(f_-) \leq Q(f)$ and, by considering $f_s := f_+ - sf_-$ for $s > 0$, so that $(f_s)_+ = f_+$ and using bilinearity, $Q(f_+, f_-) \leq 0$, where $f_- := (-f) \vee 0$ is the negative part of f. Now, using the bilinearity of the form once more implies (i).

The second Beurling–Deny criterion deals with Dirichlet forms. In particular, being a Dirichlet form turns out to be equivalent to the Markov property for both the heat semigroup and the resolvent.

THEOREM 5.15 (Second Beurling–Deny criterion). *Let Q be a positive closed form on $H = L^2(X, \mu)$ and let L be the associated positive operator. Then, the following statements are equivalent:*

(i) Q is a Dirichlet form.
(i') $Q(C_{[0,1]} \circ f) \leq Q(f)$ holds for all $f \in D(Q)$.
(ii) $\alpha(L + \alpha)^{-1}$ is Markov for every $\alpha > 0$.
(iii) e^{-tL} is Markov for every $t \geq 0$.

PROOF. (i) $\Longrightarrow$ (i'): This is clear.

(i')$\Longrightarrow$(ii): Let $0 \leq f \leq 1$ be given and consider $h := (L + \alpha)^{-1} f$. By the characterization of the resolvent as a minimizer we infer that h is the unique minimizer of the functional

$$\psi : D(Q) \longrightarrow [0, \infty), \quad \psi(f) := Q(f) + \alpha \| f - \frac{g}{\alpha} \|^2.$$

Now, consider the map that cuts off at 0 and α, i.e. consider

$$C_{[0,\alpha]} : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \alpha C_{[0,1]}(\frac{1}{\alpha} t).$$

Then, $C_{[0,\alpha]}$ is a normal contraction with

$$Q'(C_{[0,\alpha]} \circ f) = \alpha^2 Q'(C_{[0,1]}(\frac{1}{\alpha} f)) \leq \alpha^2 Q'(\frac{1}{\alpha} f) = Q'(f)$$

for all $f \in L^2(X, m)$ due to (i'). In particular,

$$Q(C_{[0,\alpha]} \circ h) \leq Q(h) < \infty$$

holds. Also, $\frac{g}{\alpha}$ is invariant under $C_{[0,\alpha]}$ due to $0 \leq g \leq 1$ and this implies

$$\|C_{[0,\alpha]} \circ h - \frac{g}{\alpha}\|^2 = \|C_{[0,\alpha]}(h) - C_{[0,\alpha]}(\frac{g}{\alpha})\|^2 \leq \|h - \frac{g}{\alpha}\|^2.$$

Thus, $C_{[0,\alpha]} \circ h$ is (another) minimizer of ψ. Uniqueness of the minimizer implies

$$C_{[0,\alpha]} \circ h = h$$

and this shows (ii).

(ii) $\Longrightarrow$ (iii): This follows directly from Theorem 3.26 (b).

(iii) $\Longrightarrow$ (i): As e^{-tL} is Markov for every $t \geq 0$, the form $Q^{(t)}$ defined by

$$Q^{(t)}(f) := \frac{1}{t} \langle (I - e^{-tL}) f, f \rangle$$

is a Dirichlet form for $t > 0$, by Lemma 5.11. Since $Q(f) = \lim_{t\to 0^+} Q^{(t)}(f)$ by Corollary 5.12, the statement follows. □

Remark 5.16. It is remarkable that compatibility with all normal contractions is equivalent to compatibility with the normal contraction $C_{[0,1]}$ with $C_{[0,1]}(s) := s \vee (s \wedge 1)$. In fact, compatibility with all normal contractions is also equivalent to compatibility with the normal contraction C_1 with $C_1(s) := s \wedge 1$. Indeed, whenever Q satisfies (i) it will also satisfy the following condition:

(i") $Q(C_1 \circ f) \le Q(f)$ for all $f \in L^2(X, \mu)$.

Conversely, using $s \vee (-\varepsilon) = -\varepsilon(-\varepsilon^{-1}s \wedge 1)$ for $s \in \mathbb{R}$ and $\varepsilon > 0$, one easily sees that (i") also implies $Q(0 \vee f) \le Q(f)$ for all $f \in L^2(X, \mu)$. Another application of (i") then gives

$$Q(C_{[0,1]} \circ f) = Q(0 \vee (f \wedge 1)) \le Q(f \wedge 1) \le Q(f).$$

Hence, (i") implies (i').

Remark 5.17. By monotone convergence we see that (iii) in Theorem 5.15 is equivalent to

(iii.a) $0 \le e^{-tL} f \le 1$ for all $f \in L^\infty(X, \mu)$ with $0 \le f \le 1$.

By duality and the Riesz–Thorin interpolation theorem, one sees that (iii.a) is equivalent to

(iii.b) $0 \le e^{-tL} f \le 1$ for all $f \in L^p(X, \mu)$ with $0 \le f \le 1$ and $1 \le p \le \infty$.

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Sheet 7

Approximation and Dirichlet forms

Exercise 1 ($Q^{(D)} \neq Q^{(N)}$) 4 points

Let (b,c) be a graph over (X,m) such that $m(X) = 1$ and $\lambda_0 = \inf \sigma(L^{(D)}) > 0$. Show that $Q^{(D)} \neq Q^{(N)}$.

Exercise 2 ($Q^{(D)} = Q^{(N)}$) 4 points

Let (b,c) be a graph over (X,m) such that $\mathrm{Deg}_0 : X \longrightarrow [0,\infty)$

$$\mathrm{Deg}_0(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)$$

is bounded. Show that $Q^{(D)} = Q^{(N)}$.

Exercise 3 (Regularity and resolvent convergence) 4 points

Let (X,m) be a discrete measure space. Let Q be a Dirichlet form on (X,m) such that $C_c(X) \subseteq D(Q)$ and let L be the self-adjoint operator associated to Q. For an increasing sequence of finite sets $K_n \subseteq X$ such that $X = \bigcup_n K_n$, let L_{K_n} be the operators corresponding to the restriction of Q to $C_c(K_n)$. Assume

$$\lim_{n\to\infty} (L_{K_n} + \alpha)^{-1}\varphi = (L+\alpha)^{-1}\varphi$$

for all $\alpha > 0$ and $\varphi \in C_c(X)$. Show that Q is regular.

Exercise 4 (Bounded functions in domain are an algebra) 4 points

Let (X,μ) be a σ-finite measure space and Q be a Dirichlet form on $L^2(X,\mu)$ with domain of Q. Show that $D(Q) \cap L^\infty(X,\mu)$ is an algebra.