26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture 05

3.5. Semigroups and Resolvents

Any positive operator A on a Hilbert space comes with two families of operators. These are its semigroup and its resolvent. Each of these families can be seen as the solution to a certain equation. The two families are equivalent in a certain sense and in particular it is possible to compute one from the other.

We start with a discussion of the semigroup. For $t \geq 0$, we define

$$\Phi^{(t)}\colon [0,\infty) \longrightarrow \mathbb{R}, \quad s \mapsto e^{-ts}.$$

Then, $\Phi^{(t)}$ is a bounded function on $[0,\infty)$ (with bound 1) and $\Phi^{(t)}\Phi^{(r)} = \Phi^{(t+r)}$ holds for all $t, r \geq 0$ as well as $\Phi^{(0)} = 1$. Whenever A is a positive operator we define

$$e^{-tA} := \Phi^{(t)}(A)$$

and call $(e^{-tA})_{t\geq 0}$ the *semigroup* associated to A.

PROPOSITION 3.23 (Basic properties of the semigroup). *Let A be a positive operator on H. Then,*

(a) $e^{0A} = I$ *and for all* $s, t \geq 0$,

$$e^{-(s+t)A} = e^{-sA}e^{-tA}.$$

(b) *For all* $f \in H$,

$$\lim_{t\to 0^+} e^{-tA}f = f.$$

(c) *For all* $t \geq 0$,

$$\|e^{-tA}\| \leq 1.$$

PROOF. (a) This follows immediately from Proposition 3.9 (d).

(b) By Corollary 3.14, for $f \in H$, we have

$$\|e^{-tA}f - f\|^2 = \int_0^\infty (e^{-tx} - 1)^2 \mathrm{d}\mu_f(x) \to 0$$

as $t \to 0^+$ by Lebesgue's dominated convergence theorem. This follows as the integrand is bounded above by 1, converges pointwise to 0 and each spectral measure is finite.

(c) By Corollary 3.14, for $f \in H$ we have

$$\|e^{-tA}f\|^2 = \int_0^\infty e^{-2tx}\mathrm{d}\mu_f(x) \leq \int_0^\infty \mathrm{d}\mu_f = \|f\|^2.$$

This gives the desired conclusion. □

We will now show that the semigroup generates solutions of the parabolic equation involving A. In order to make this precise, we recall that a function $u\colon (0,\infty) \longrightarrow H$ is called *differentiable* if for any $t > 0$ the limit

$$\lim_{h\to 0} \frac{1}{h}(u(t+h) - u(t))$$

exists. In this case, we denote this limit as $\partial_t u(t)$ and call it the *derivative* of u.

THEOREM 3.24 (Solution of the parabolic equation). *Let A be a positive operator on H and let $f \in H$. Then, $u\colon [0,\infty) \longrightarrow H$ given by*

$$u_t := e^{-tA} f$$

is continuous on $[0,\infty)$, differentiable on $(0,\infty)$, satisfies $u_t \in D(A)$ for $t > 0$ and

$$\partial_t u_t = -Au_t$$

for all $t > 0$ as well as $u(t) \to f$ for $t \to 0^+$.

PROOF. We prove the theorem through a series of claims.

Claim. The function u is continuous on $[0,\infty)$.

Proof of the claim. Let $t \geq 0$. Then, for all $h \in \mathbb{R}$ with $t + h \geq 0$, the operator $e^{-(t+h)A}$ is bounded and Corollary 3.14 gives

$$\left\| e^{-(t+h)A} f - e^{-tA} f \right\|^2 = \int_0^\infty \left| e^{-(t+h)x} - e^{-tx} \right|^2 \mathrm{d}\mu_f(x).$$

Now, $[0,\infty) \ni x \mapsto |e^{-(t+h)x} - e^{-tx}|^2$ is bounded by 4 and converges to 0 pointwise as $h \to 0$. Thus, we obtain from Lebesgue's dominated convergence theorem

$$\lim_{h \to 0} \int_0^\infty \left| e^{-(t+h)x} - e^{-tx} \right|^2 \mathrm{d}\mu_f(x) = 0.$$

This proves the continuity of u at t.

Claim. For any $t > 0$, $u_t \in D(A)$.

Proof of the claim. By Proposition 3.13 (a), we have to show that $\int x^2 \mathrm{d}\mu_{u_t}(x) < \infty$ for $t > 0$. Now, by Corollary 3.14, as $f \in H = D(e^{-tA})$ we have $\mu_{u_t} = \mu_{e^{-tA}f} = |e^{-t(\cdot)}|^2 \mu_f$. This easily gives

$$\int x^2 \mathrm{d}\mu_{u_t}(x) = \int_{[0,\infty)} x^2 e^{-2tx} \mathrm{d}\mu_f(x) < \infty,$$

where we used that μ_f is supported on $\sigma(A) \subseteq [0,\infty)$ and $x \mapsto x^2 e^{-2tx}$ is bounded on $[0,\infty)$.

Claim. For any $t > 0$, the function u is differentiable in t and satisfies

$$\partial_t u_t = -Au_t.$$

Proof of the claim. For $h \in \mathbb{R}$ with $|h| \leq t$, we define the function $\psi_h\colon [0,\infty) \longrightarrow \mathbb{R}$ by

$$\psi_h(x) := \frac{e^{-(t+h)x} - e^{-tx}}{h} - xe^{-tx}.$$

Then, (d) and (e) of Proposition 3.9, give

$$\frac{1}{h}(e^{-(t+h)A} f - e^{-tA} f) - Ae^{-tA} f = \psi_h(A) f,$$

where we use $e^{-tA} f \in D(A)$ for $t > 0$, which was established in the preceding claim to write down the expression on the left-hand side. Hence, Proposition 3.13 (a) yields

$$\left\| \frac{1}{h}(e^{-(t+h)A} f - e^{-tA} f) - Ae^{-tA} f \right\|^2 = \int_0^\infty |\psi_h(x)|^2 \mathrm{d}\mu_f(x).$$

Now, ψ_h can easily be seen to converge pointwise to 0 as $h \to 0$ and to be bounded by $x \mapsto 2xe^{-tx}$, which is bounded on $[0, \infty)$. Hence, by Lebesgue's dominated convergence theorem, we see that $\int |\psi_h|^2 \mathrm{d}\mu_f \to 0$ as $h \to 0$ and this gives the desired claim.

Claim. $u_t \to f$ as $t \to 0^+$.

Proof of the claim. This is immediate from the already established continuity of u on $[0, \infty)$ and $u_0 = f$. □

We now gather some basic properties of resolvents.

PROPOSITION 3.25 (Basic properties of resolvents). *Let A be a positive operator on H. Then,*

(a) *For all $\alpha, \beta > 0$,*

$$(A + \alpha)^{-1} - (A + \beta)^{-1} = -(\alpha - \beta)(A + \alpha)^{-1}(A + \beta)^{-1}.$$

(b) *For all $f \in H$,*

$$\lim_{\alpha \to \infty} \alpha(A + \alpha)^{-1} f = f.$$

(c) *For all $\alpha > 0$,*

$$\|\alpha(A + \alpha)^{-1}\| \leq 1.$$

PROOF. (a) This follows directly from the identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

for invertible operators A and B with $D(B) \subseteq D(A)$.

(b) For every $f \in H$ we have by Corollary 3.14

$$\|\alpha(A + \alpha)^{-1} f - f\|^2 = \int_0^\infty \left| \frac{\alpha}{x + \alpha} - 1 \right|^2 \mathrm{d}\mu_f(x) \to 0$$

as $\alpha \to \infty$ by Lebesgue's dominated convergence theorem.

(c) By Corollary 3.14, for every $f \in H$ we have

$$\|\alpha(A + \alpha)^{-1} f\|^2 = \int_0^\infty \left| \frac{\alpha}{x + \alpha} \right|^2 \mathrm{d}\mu_f(x) \leq \int_0^\infty \mathrm{d}\mu_f = \|f\|^2.$$

This completes the proof. □

THEOREM 3.26 (Semigroups and resolvents). *Let A be a positive operator on H. Let $f \in H$ be given.*

(a) *For every $\alpha > 0$,*

$$(A + \alpha)^{-1} f = \int_0^\infty e^{-t\alpha} e^{-tA} f \mathrm{d}t.$$

(Here the integral can be understood as limit of Riemannian sums.)

(*"Laplace transform"*)

(b) *For every $t > 0$,*

$$e^{-tA} f = \lim_{n \to \infty} \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^n f.$$

(*"Exponential formula"*)

PROOF. (a) From the formula

$$(x+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tx} \mathrm{d}t,$$

which holds for all $x \geq 0$ and $\alpha > 0$, we obtain by applying the functional calculus

$$(A+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tA} \mathrm{d}t.$$

This gives the conclusion.

(b) We note that

$$\varphi_n(x) := \left(\frac{n}{t} \left(x + \frac{n}{t} \right)^{-1} \right)^n = \left(1 + \frac{tx}{n} \right)^{-n} \to e^{-tx}$$

as $n \to \infty$ for $x, t \geq 0$. Hence, by Corollary 3.14 and Lebesgue's dominated convergence theorem, we obtain, for every $f \in H$,

$$\left\| e^{-tA} f - \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^n f \right\|^2 = \int_0^\infty \left| e^{-tx} - \varphi_n(x) \right|^2 \mathrm{d}\mu_f(x) \to 0$$

as $n \to \infty$. This completes the proof. □

3.6. Forms

Forms and positive operators can be seen as two faces of the same medal. The advantage of forms is that they have a larger domain of definition and are generally much more easily written down than the underlying operator.

We start with the definition of the basic object.

DEFINITION 3.27 (Symmetric positive form). A *symmetric positive form* Q on H consists of a dense subspace $D(Q) \subseteq H$ called the *domain of* Q together with a map

$$Q \colon D(Q) \times D(Q) \longrightarrow \mathbb{C}$$

satisfying

- $Q(f,g) = \overline{Q(g,f)}$ (*"Symmetry"*)
- $Q(f, \alpha g + \beta h) = \alpha Q(f,g) + \beta Q(f,h)$ (*"Linearity"*)
- $Q(f,f) \geq 0$ (*"Positivity"*)

for all $f, g, h \in D(Q)$ and $\alpha, \beta \in \mathbb{C}$.

We will often refer to positive symmetric forms as just forms.

Whenever such a form Q is given we define for $f \in H$ the value

$$Q'(f) := \begin{cases} Q(f,f) & \text{if } f \in D(Q) \\ \infty & \text{otherwise.} \end{cases}$$

We note that we can recover the form Q from the values $Q'(f)$ for $f \in H$ as the domain of Q is given by

$$D(Q) = \{ f \in H \mid Q'(f) < \infty \}$$

and $Q(f,g)$ can be obtained by using the polarization identity, i.e.,

$$Q(f,g) = \frac{1}{4} \sum_{k=0}^{3} i^k Q(g + i^k f)$$

for $f, g \in D(Q)$. We will often write $Q(f)$ instead of $Q'(f)$.

Any form Q comes with an inner product $\langle \cdot, \cdot \rangle_Q$ on $D(Q)$ given by

$$\langle \cdot, \cdot \rangle_Q : D(Q) \times D(Q) \longrightarrow \mathbb{C}, \quad \langle f, g \rangle_Q := Q(f, g) + \langle f, g \rangle.$$

This inner product induces the norm $\| \cdot \|_Q$ given by

$$\|f\|_Q := \langle f, f \rangle_Q^{1/2}.$$

In the next example we begin to establish the connection between forms and positive operators. In particular, we show how to define a form from a positive operator.

Example 3.28 (Form associated to a positive operator). Let A be a positive operator on H. We define the form Q_A by letting $D(Q_A) := D(\sqrt{A})$ and

$$Q_A(f, g) := \langle \sqrt{A} f, \sqrt{A} g \rangle$$

for all $f, g \in D(\sqrt{A})$. We call Q_A the *form associated to* A.

In particular, if (X, μ) is a measure space, $u \colon X \longrightarrow [0, \infty)$ is measurable and M_u the operator of multiplication by u, then one easily seea that Q_{M_u} has domain

$$D(Q_{M_u}) = D(M_{\sqrt{u}}) = \{ f \in L^2(X, \mu) \mid \int u |f|^2 \mathrm{d}\mu < \infty \}$$

and acts by

$$Q_{M_u}(f, g) = \int u \overline{f} g \mathrm{d}\mu$$

for $f, g \in D(Q_{M_u})$. We note that the integral defining $Q_{M_u}(f, g)$ exists as $u|fg| \leq \frac{1}{2}\big(u|f|^2 + u|g|^2\big)$.

We will show that the converse of the preceding example holds under some additional assumptions. For forms with suitable boundedness properties, this is not hard to see by using the Riesz representation theorem. This is the content of the next proposition.

Proposition 3.29 (Bounded forms and operators). *Let Q be a positive form with $D(Q) = H$ such that there exists a constant $C \geq 0$ with*

$$Q(f, g) \leq C \|f\| \|g\|$$

for all $f, g \in H$. Then, there exists a unique positive operator A with $D(A) = H$, $\|A\| \leq C$ and

$$Q(f, g) = \langle f, Ag \rangle = \langle Af, g \rangle = \langle \sqrt{A} f, \sqrt{A} g \rangle$$

for all $f, g \in H$.

Proof. For a fixed $f \in H$, we consider the map from H to $\mathbb{C}$ given by

$$g \mapsto Q(f, g).$$

This map is linear and bounded by the assumptions on Q. Hence, by the Riesz representation theorem, there exists a unique $f' \in H$ with

$$Q(f, g) = \langle f', g \rangle$$

for all $g \in H$. We define $A \colon H \longrightarrow H$ by

$$Af = f'.$$

It follows that A is linear and

$$Q(f,g) = \langle Af, g\rangle$$

for all $f, g \in H$. In particular, we infer

$$\|Af\| = \sup\{\langle Af, g\rangle \mid \|g\| \leq 1\} \leq C\|f\|$$

and, thus, $\|A\| \leq C$ follows. Moreover, by using the symmetry of Q, we have

$$\langle Af, g\rangle = Q(f,g) = \overline{Q(g,f)} = \overline{\langle Ag, f\rangle} = \langle f, Ag\rangle,$$

so that A is symmetric. As A is bounded, it follows that A is self-adjoint.

Finally, using the positivity of Q, we obtain

$$\langle f, Af\rangle = Q(f,f) \geq 0$$

and, thus, A is positive. The uniqueness of A is clear.

It remains to show the formula $\langle f, Ag\rangle = \langle\sqrt{A}f, \sqrt{A}g\rangle$. This, however, is clear as due to positivity of A the operator $\sqrt{A}$ is self-adjoint with $\sqrt{A}\sqrt{A} = A$. □

Any form Q with $D(Q) = H$ which satisfies $Q(f,g) \leq C\|f\|\|g\|$ for all $f, g \in H$ and some constant $C \geq 0$ is called *bounded*. Hence, we see from the proposition above that any positive bounded form gives rise to a unique positive bounded operator. Conversely, if A is a bounded positive operator, then $\sqrt{A}$ is bounded and, thus, Q_A as defined in Example 3.28 is a bounded form. Hence, from the considerations above, we see that there is a one-to-one correspondence between bounded positive operators and bounded positive forms. We will extend this result to a larger class of forms in what follows.

We first show that we can weaken the boundedness assumption on the form to a completeness assumption and still obtain the existence of an operator. This is the content of the next lemma.

LEMMA 3.30 (Associated operator). *Let Q be a positive form on H. If $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space, then there exists a positive operator A with $D(Q) = D(\sqrt{A})$ and*

$$Q(f,g) = \langle\sqrt{A}f, \sqrt{A}g\rangle$$

for all $f, g \in D(Q)$, i.e., $Q = Q_A$ is the form associated to A.

REMARK 3.31. As $\langle\cdot,\cdot\rangle_Q$ is an inner product on $D(Q)$, the assumption that $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space just means that $D(Q)$ is complete with respect to the norm $\|\cdot\|_Q$.

PROOF. We write H_Q to denote the Hilbert space $(D(Q), \langle\cdot,\cdot\rangle_Q)$. Consider

$$\langle\cdot,\cdot\rangle \colon H_Q \times H_Q \longrightarrow \mathbb{C},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on H. Then, as Q is positive,

$$|\langle f, g\rangle| \leq \|f\|\|g\| \leq \|f\|_Q\|g\|_Q,$$

so that $\langle\cdot,\cdot\rangle$ is a bounded form on H_Q. Hence, by Proposition 3.29, there exists a unique positive operator T with $D(T) = H_Q$ and

$$\langle f, g\rangle = \langle f, Tg\rangle_Q = Q(f, Tg) + \langle f, Tg\rangle$$

for all $f, g \in H_Q$.

We will ultimately show that

$$A = T^{-1} - I$$

has the desired properties. Indeed, assuming the definition of A as $T^{-1} - I$ makes sense, letting $g' = Tg$ and noting that $g - g' = Ag'$ we see from the above that

$$\langle f, Ag' \rangle = Q(f, g')$$

for all $f, g \in H_Q$. Using $A = \sqrt{A}\sqrt{A} = (\sqrt{A})^*(\sqrt{A})$ then gives

$$Q(f, g') = \langle \sqrt{A}f, \sqrt{A}g' \rangle$$

for all $f, g \in H_Q$.

To turn this into a rigorous argument, we have to show that T is injective and that $T^{-1} - I$ can be seen as a positive operator on H. One obstacle to overcome is that T and A are only defined on H_Q and we have to extend them to subspaces of H.

After this sketch, we now proceed to give the proof. As

$$\langle f, Tf \rangle_Q = \langle f, f \rangle = \|f\|^2$$

for all $f \in H_Q$, the operator T is positive and bounded on H_Q with $\|T\| \leq 1$. By the spectral theorem, Theorem 3.6, applied to T on H_Q, there exists a σ-finite measure space (X, μ_Q), a measurable function $u\colon X \longrightarrow [0, 1]$ and a unitary map $V\colon H_Q \longrightarrow L^2(X, \mu_Q)$ such that

$$T = V^{-1} M_u V.$$

Here, $0 \leq u \leq 1$ follows from the fact that T is positive and bounded with $\|T\| \leq 1$. Furthermore, as $\langle f, Tf \rangle_Q = \|f\|^2$, the operator T is injective and thus $u > 0$ almost everywhere so that $0 < u \leq 1$ almost everywhere.

We now define $a\colon X \longrightarrow [0, \infty)$ by

$$a := \frac{1}{u} - 1.$$

For all $f, g \in H_Q$, from

$$\langle f, g \rangle = \langle f, Tg \rangle_Q = \int u(\overline{Vf})(Vg)\mathrm{d}\mu_Q$$

we infer

$$\begin{aligned} Q(f, g) = \langle f, g \rangle_Q - \langle f, g \rangle &= \int (\overline{Vf})(Vg)\mathrm{d}\mu_Q - \int u(\overline{Vf})(Vg)\mathrm{d}\mu_Q \\ &= \int (1 - u)(\overline{Vf})(Vg)\mathrm{d}\mu_Q = \int a(\overline{Vf})(Vg)u\mathrm{d}\mu_Q \\ &= \int a(\overline{Vf})(Vg)\mathrm{d}\mu, \end{aligned}$$

where we define the measure $\mu := u\mu_Q$ and use that $1 - u = au$.

This is almost the desired formula for Q. It just remains to show that we can use $V\colon H_Q \longrightarrow L^2(X, \mu_Q)$ to define a unitary map

$$U\colon H \longrightarrow L^2(X, \mu)$$

which satisfies $Q(f,g) = \int a(\overline{Uf})(Ug)\mathrm{d}\mu$ for all $f, g \in D(Q)$ and

$$UD(Q) = \{f \in L^2(X,\mu) \mid \int a|f|^2 \mathrm{d}\mu < \infty\}.$$

If so, then we can define $M_{\sqrt{a}}$ on $D(M_{\sqrt{a}}) = UD(Q)$ and

$$\begin{aligned} Q(f,g) &= \int a(\overline{Uf})(Ug)\mathrm{d}\mu = \langle M_{\sqrt{a}}Uf, M_{\sqrt{a}}Ug\rangle_{L^2(X,\mu)} \\ &= \langle U^{-1}M_{\sqrt{a}}Uf, U^{-1}M_{\sqrt{a}}Ug\rangle \end{aligned}$$

for all $f, g \in D(Q)$. We then let

$$\sqrt{A} = U^{-1}M_{\sqrt{a}}U \quad \text{with} \quad D(\sqrt{A}) = U^{-1}D(M_{\sqrt{a}}) = D(Q),$$

which will complete the proof.

To this end, we note that V is isometric as a map from $D(Q) \subseteq H$ to $L^2(X,\mu)$ as

$$\begin{aligned} \langle f, g\rangle = \langle f, Tg\rangle_Q &= \int (\overline{Vf})(Vg)u\mathrm{d}\mu_Q = \int (\overline{Vf})(Vg)\mathrm{d}\mu \\ &= \langle Vf, Vg\rangle_{L^2(X,\mu)}. \end{aligned}$$

Furthermore, as $L^2(X,\mu_Q)$ is dense in $L^2(X,\mu)$, the image of V is dense. As $D(Q)$ is dense in H, we can extend V to an isometric operator $U\colon H \longrightarrow L^2(X,\mu)$ which is onto. As U is also one-to-one, U is unitary.

Moreover, the images of H_Q under U and V are equal. This image, by definition, is $L^2(X,\mu_Q)$ and clearly agrees with

$$\{f \in L^2(X,\mu) \mid \int a|f|^2 \mathrm{d}\mu < \infty\}.$$

Hence, we obtain the asserted formula for $UD(Q)$, which completes the proof. □

We now give the operator constructed above a name.

Definition 3.32 (Associated operator). Let Q be a positive form on H such that $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space. The positive operator A such that

$$D(\sqrt{A}) = D(Q) \quad \text{and} \quad Q(f,g) = \langle\sqrt{A}f, \sqrt{A}g\rangle$$

is called the operator *associated* to Q.

From Lemma 3.30 we see that every form which induces a Hilbert space structure on its domain gives rise to an associated operator. We will now show that all such forms come from positive operators. Along the way, we also characterize the completeness assumption in terms of lower semi-continuity. Recall that for a positive form Q we have defined Q' on the whole Hilbert space by $Q'(f) = Q(f)$ for $f \in D(Q)$ and $Q'(f) = \infty$ for $f \notin D(Q)$.

Theorem 3.33 (Characterization of closed forms). *Let Q be a positive form on H. Then, the following statements are equivalent:*

(i) *There exists a positive operator* A *with* $Q = Q_A$, *i.e.,* $D(Q) = D(\sqrt{A})$ *and*

$$Q(f,g) = \langle \sqrt{A} f, \sqrt{A} g\rangle$$

for all $f, g \in D(Q)$.

(ii) Q' *is lower semi-continuous, i.e.,*

$$Q'(f) \le \liminf_{n\to\infty} Q'(f_n)$$

whenever $f_n \to f$ *as* $n \to \infty$ *in* H.

(iii) $(D(Q), \langle\cdot,\cdot\rangle_Q)$ *is a Hilbert space.*

Proof of Theorem 3.33. (i) $\Longrightarrow$ (ii): We will show that Q is the supremum of continuous functions Q_n, from which (ii) follows easily.

Since $A \ge 0$, the operator $(A+n)^{-1}$ exists and is bounded on H for all $n \in \mathbb{N}$. For $f \in H$, we denote by μ_f the spectral measure associated to f and note by Proposition 3.13 and Lemma 3.19 that $\operatorname{supp}(\mu_f) \subseteq \sigma(A) \subseteq [0,\infty)$. We let $\varphi_n\colon [0,\infty) \longrightarrow \mathbb{R}$ be given by $\varphi_n(x) := nx/(x+n)$ and note that φ_n is bounded for every $n \in \mathbb{N}$. Thus, by the bounded functional calculus, Corollary 3.14, we may define a continuous map $Q_n\colon H \longrightarrow [0,\infty)$ via

$$Q_n(f) := \int_0^\infty \frac{nx}{x+n} \mathrm{d}\mu_f(x) = \langle f, nA(A+n)^{-1} f\rangle.$$

We now claim that

$$Q_n(f) \nearrow \int_0^\infty x \mathrm{d}\mu_f(x) = Q(f)$$

as $n \to \infty$ for every $f \in H$. Here, the convergence follows easily by the monotone convergence theorem as $\varphi_n(x) \nearrow x$ as $n \to \infty$. The equality follows from Proposition 3.13 (a), which gives $f \in D(\sqrt{A}) = D(Q)$ if and only if $\int x \mathrm{d}\mu_f < \infty$, in which case

$$Q(f) = \|\sqrt{A} f\|^2 = \int_0^\infty x \mathrm{d}\mu_f(x).$$

This completes the proof.

(ii) $\Longrightarrow$ (iii): Let (f_n) be a Cauchy sequence in $(D(Q), \langle\cdot,\cdot\rangle_Q)$. Then, (f_n) is a Cauchy sequence in H. In particular, there exists an $f \in H$ with $f_n \to f$ with respect to $\|\cdot\|$.

Let $\varepsilon > 0$. As (f_n) is a Cauchy sequence in $(D(Q), \langle\cdot,\cdot\rangle_Q)$, there exists an $N \in \mathbb{N}$ with

$$\|f_n - f_m\|_Q < \varepsilon$$

for all $n, m \ge N$. Consider now $m \ge N$. Then, using (ii), we get

$$Q(f - f_m) \le \liminf_{n\to\infty} Q(f_n - f_m) \le \varepsilon.$$

This implies $f \in D(Q)$ and $Q(f - f_m) \le \varepsilon$ for all $m \ge N$. Therefore, $f_n \to f$ with respect to $\|\cdot\|_Q$.

(iii) $\Longrightarrow$ (i): This is shown in Lemma 3.30. □

We highlight the class of forms appearing in the previous statement by giving a definition.

Definition 3.34 (Closed form). We say that a positive form Q on H is *closed* if Q satisfies one of the equivalent conditions of Theorem 3.33.

For application the following immediate consequence of the theorem is often useful.

PROPOSITION 3.35. *Let Q be a closed form on H. Assume that (f_n) is a sequence in $D(Q)$ converging to $f \in H$. If the sequence $(Q(f_n))$ is bounded, then f belongs to $D(Q)$ and*

$$Q(f) \leq \liminf_n Q(f_n) < \infty$$

holds.

The preceding considerations show that all positive closed forms come from positive operators. We now discuss how to further describe the domain of the operator associated to such a form.

THEOREM 3.36 (Domain and action of the operator). *Let Q be a positive closed form on H. Then, the associated operator A has domain*

$$D(A) = \left\{ f \in D(Q) \;\middle|\; \begin{array}{c} \textit{there exists a } g \in H \textit{ with } Q(h,f) = \langle h, g\rangle \\ \textit{for all } h \in D(Q) \end{array} \right\}$$

and acts on $D(A)$ via

$$Af = g.$$

PROOF. This follows from the definitions of the associated operator and the adjoint of the square root, the fact that $\sqrt{A}$ is self-adjoint, so that $D(\sqrt{A}) = D(\sqrt{A}^*)$, and the fact that $f \in D(A)$ if and only if $\sqrt{A}f \in D(\sqrt{A}^*) = D(\sqrt{A})$, see Lemma 3.22.

More specifically, for $f \in D(\sqrt{A}) = D(Q)$ we have $\sqrt{A}f \in D(\sqrt{A}^*)$ if and only if there exists an element $g \in H$ such that

$$\langle h, \sqrt{A}\sqrt{A}f\rangle = \langle h, g\rangle$$

for all $h \in D(\sqrt{A}) = D(Q)$, which is equivalent to

$$\langle \sqrt{A}h, \sqrt{A}f\rangle = Q(h,f) = \langle h, g\rangle$$

for all $h \in D(Q)$. This completes the proof. □

The following consequence of the previous theorem is a convenient way to think about the operator associated to a closed form. As a further fact, we also show that the operator domain is dense in the form domain with respect to the inner product arising from the form.

COROLLARY 3.37. *Let Q be a positive closed form on H. Then, there exists a unique self-adjoint operator A with*

$$Q(f,g) = \langle f, Ag\rangle$$

for all $f \in D(Q)$ and $g \in D(A)$. The operator A is positive and the form Q satisfies

$$D(Q) = D(\sqrt{A}) \quad \textit{and} \quad Q(f,g) = \langle \sqrt{A}f, \sqrt{A}g\rangle$$

for all $f, g \in D(Q)$. Furthermore, $D(A) \subseteq D(Q)$ is dense with respect to $\|\cdot\|_Q$.

PROOF. We first show uniqueness. Let A be such an operator. Then, by Theorem 3.36 the operator A is a restriction of the operator associated to Q. As both are self-adjoint, they must agree.

The existence of such an operator as well as the connection to the form follow from Theorem 3.33 and Lemma 3.22. Finally, to show that $D(A)$ is dense in $D(Q)$ with respect to $\|\cdot\|_Q$ we suppose not. Then there exists an $f \in D(Q)$, $f \neq 0$, which is in the orthogonal complement of $D(A)$ with respect to $\langle\cdot,\cdot\rangle_Q$, that is,

$$\langle f, g\rangle_Q = \langle f, g\rangle + Q(f,g) = 0$$

for all $g \in D(A)$. By the connection between the operator and form we then obtain

$$\langle f, Ag\rangle = -\langle f, g\rangle$$

for all $g \in D(A)$. This implies $f \in D(A^*)$ with $A^* f = -f$. As A is self-adjoint, it follows that $f \in D(A)$ with $Af = -f$. As A is positive, -1 can not be an eigenvalue. Hence, we conclude $f = 0$. This contradiction yields the claim. □

A form $\widetilde{Q}$ is called an *extension* of the form Q, written as $Q \subseteq \widetilde{Q}$ if $D(Q) \subseteq D(\widetilde{Q})$ holds and $\widetilde{Q}$ agrees with Q on $D(Q)$.

Sometimes a form Q is not closed but possesses closed extensions. Then, Q is called *closable*. In this case, the form can uniquely be extended to the intersection of the domains of all closed extensions of Q and this extension is a closed form. It is called the *closure* of the form.

3.7. Resolvents as Minimizers

In this section we prove a characterization of the resolvent of an operator. More specifically, given a closed form, we show that the resolvent of the associated operator gives the unique minimizer of an equation involving the form.

THEOREM 3.38 (Characterization of the resolvent as a minimizer). *Let Q be a positive closed form on H with associated operator A. For $f \in H$ and $\alpha > 0$, define $j\colon D(Q) \longrightarrow [0,\infty)$ by*

$$j(v) := Q(v) + \alpha \left\| v - \frac{1}{\alpha} f \right\|^2 .$$

Then, j satisfies the formula

$$j(v) = j\left((A+\alpha)^{-1} f\right) + Q\left((A+\alpha)^{-1} f - v\right) + \alpha \left\|(A+\alpha)^{-1} f - v\right\|^2 .$$

In particular, $(A+\alpha)^{-1} f$ is the unique minimizer of j on $D(Q)$.

PROOF. It suffices to show the formula for j. The statement on the minimizer is then immediate. For ease of notation we set

$$G_\alpha := (A+\alpha)^{-1}$$

and

$$Q_\alpha(u,v) := Q(u,v) + \alpha\langle u, v\rangle, \qquad u, v \in D(Q)$$

for $\alpha > 0$. Given this, the right-hand side of the formula for j can be written as

$$\mathrm{RHS} = j(G_\alpha f) + Q_\alpha(G_\alpha f - v).$$

We will compute the two terms appearing in RHS. In order to do so, we need a little bit of preparation. We obviously have

$$Q_\alpha(G_\alpha f, v) = \langle f, v\rangle$$

for all $f \in H$ and $v \in D(Q)$, which directly yields

$$Q_\alpha(G_\alpha f) = \langle f, G_\alpha f\rangle = \langle G_\alpha f, f\rangle,$$

where we use the self-adjointness of G_α. Furthermore, a direct computation gives

$$j(v) = Q_\alpha(v) - \langle v, f\rangle - \langle f, v\rangle + \frac{1}{\alpha}\|f\|^2.$$

Now, we turn to computing the two terms in RHS: By the last two equalities we obtain for the first term

$$\begin{aligned} j(G_\alpha f) &= Q_\alpha(G_\alpha f) - \langle G_\alpha f, f\rangle - \langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2 \\ &= \langle f, G_\alpha f\rangle - \langle G_\alpha f, f\rangle - \langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2 \\ &= -\langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2. \end{aligned}$$

For the second term, using $Q_\alpha(G_\alpha f, v) = \langle f, v\rangle$ repeatedly we obtain

$$\begin{aligned} Q_\alpha(G_\alpha f - v) &= Q_\alpha(G_\alpha f) - Q_\alpha(G_\alpha f, v) - Q_\alpha(v, G_\alpha f) + Q_\alpha(v) \\ &= \langle f, G_\alpha f\rangle - \langle f, v\rangle - \langle v, f\rangle + Q_\alpha(v). \end{aligned}$$

Putting the two terms together we can now compute

$$\begin{aligned} \mathrm{RHS} &= j(G_\alpha f) + Q_\alpha(G_\alpha f - v) \\ &= Q_\alpha(v) - \langle f, v\rangle - \langle v, f\rangle + \frac{1}{\alpha}\|f\|^2 = j(v), \end{aligned}$$

which finishes the proof. □

Remark 3.39 (Geometric interpretation). It is possible to interpret the previous result in terms of Hilbert space geometry on a suitably chosen Hilbert space. First, $v = G_\alpha f$ is equivalent to $(A + \alpha)v = f$, which in turn is equivalent to the fact that $v \in D(Q)$ with $Q_\alpha(v, w) = \langle f, w\rangle$ for all $w \in D(Q)$. We can write this as

$$Q(v, w) + \alpha\langle v - \frac{1}{\alpha} f, w\rangle = 0$$

for $w \in D(Q)$. Rewriting this with the (semi)-inner product

$$\langle (a, b), (c, d)\rangle_* := Q(a, c) + \alpha\langle b, d\rangle$$

on $D(Q) \times D(Q)$ we infer that $v = G_\alpha f$ if and only if $(v, v - \alpha^{-1} f)$ is perpendicular to the diagonal, i.e.,

$$(v, v - \frac{1}{\alpha} f) = (v, v) - (0, \frac{1}{\alpha} f) \perp U,$$

where U is the subspace

$$U := \{(w, w) \mid w \in D(Q)\}.$$

So, if $x = -(0, \alpha^{-1} f)$, then we want to find an element $\widetilde{v} \in U$ such that $x + \widetilde{v}$ is perpendicular to U.

By standard theory this problem has a unique solution, which is given by the minimizer of $\|\cdot\|_*$ on $x+U$ whenever $\langle\cdot,\cdot\rangle_*$ is an inner product inducing a Hilbert space structure. Now, in general, $\langle\cdot,\cdot\rangle_*$ is not an inner product and completeness may fail on $D(Q) \times D(Q)$. So, the basic theory does not apply directly. However, it is not necessary for $\langle\cdot,\cdot\rangle_*$ to be an inner product giving a Hilbert space structure on the entire space, it suffices that $\langle\cdot,\cdot\rangle_*$ is an inner product on U making U into a Hilbert space. This is indeed the case in our situation and we infer that $v = G_\alpha f$ holds if and only if v minimizes $\|\cdot\|_*$ on $x+U$. As, $j(\cdot) = \|\cdot\|_*^2$ on $x+U$ we obtain the statement of the theorem.

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Sheet 5

Forms and Operators

Exercise 1 (Forms of multiplication operators) 4 points

Let (X,μ) be a measure space and $u\colon X \longrightarrow [0,\infty)$ is measurable. Let

$$D(Q) = \{f \in L^2(X,\mu) \mid \int u|f|^2 \mathrm{d}\mu < \infty\}$$

and

$$Q(f,g) = \int u\overline{f}g \mathrm{d}\mu$$

for $f, g \in D(Q)$.

(a) Show that Q is a closed form.

(b) Show that Q is bounded if $u \in L^\infty(X,\mu)$.

(c) The associated operator to Q is the operator of multiplication by u denoted by M_u. Moreover,

$$D(Q_{M_u}) = D(M_{\sqrt{u}}).$$

(d) Compute $Q' : L^2(X,\mu) \longrightarrow [0,\infty]$ which is defined as $Q'(f) = Q(f,f)$ for $f \in D(Q)$ and ∞ otherwise.

Exercise 2 (Closable forms) 4 points

Let $Q \geq 0$ be a form on a Hilbert space H which allows for a closed extension $Q^\#$.

(a) Let (f_n) in $D(Q)$ and $f \in H$. Show that the following statements are equivalent:

(i) $f_n \to f$ with respect to $\|\cdot\|_{Q^\#}$.

(ii) (f_n) is a Cauchy sequence with respect to $\|\cdot\|_Q$ and $f_n \to f$ in H.

(b) Show that

$$D(\overline{Q}) = \{f \in D(Q^\#) \mid \text{ there is } (f_n) \text{ in } D(Q) \text{ such that } f_n \to f \text{ with respect to } \|\cdot\|_{Q^\#}\}$$

and

$$\overline{Q}(f,g) = Q^\#(f,g)$$

is a closed form.

(c) Show that $\overline{Q}$ is included in every closed extension of Q.

Exercise 3 (Fractions of Operators) 4 points

Assume that T is a positive self-adjoint operator and $\varphi : [0,\infty) \longrightarrow [0,\infty)$, $t \longmapsto t^s$ for $0 < s < 1$. Let $T^s = \varphi(T)$. Show that $f \in D(T^s)$ for $f \in D(T)$ and

$$\|T^s f\| \leq \|Tf\|^s \|f\|^{1-s}.$$

Show furthermore that

$$T^s f = -\frac{s}{|\Gamma(1-s)|} \int_0^\infty (e^{-tT} - I) f \frac{\mathrm{d}t}{t^{1+s}},$$

for $f \in D(T^s)$ where Γ is Euler's Gamma Function.

Exercise 4 (Cosine formula for the resolvent) 4 points

Let $T \geq 0$ be a positive self-adjoint operator in a Hilbert space H, and let $\lambda > 0$. Then for all $f \in H$

$$(\lambda + T)^{-1} f = \int_0^\infty e^{-s} \cos\left(\lambda^{-1/2} s T^{1/2}\right) f \mathrm{d}s.$$

Hint: This exercise is completely irrelevant but curious.