26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

Matthias Keller, Daniel Lenz, Marcel Schmidt and Christian Seifert

Lecture 04

CHAPTER 3

Toolbox – The Spectral Theorem and Closed Forms

The basic objects of our study are graphs and the associated operators, forms, resolvents and semigroups. In the context of the general infinite graphs that we consider in this course the operators and forms are generally unbounded. To deal with this unboundedness requires some care. The necessary background for this careful dealing is provided by the spectral theorem and its consequences and the theory of closed forms. This is discussed in this chapter.

Throughout this chapter, we let H denote a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We consider complex Hilbert spaces and assume that the inner product is linear in the second argument. Real spaces can be complexified. Thus, all of our results below apply to the real case as well.

We will always assume that the Hilbert space is separable (i.e. contains a countable dense set). This is no restriction for us as the application we have in mind is the Hilbert space $\ell^2(X, m)$ with a countable set X and this Hilbert space is separable. The assumption of separability will allow us to restrict attention to measure spaces which are σ-finite (as separable Hilbert spaces are unitarily equivalent to $L^2(\Omega, \mu)$ with a σ-finite μ).

3.1. Basics on Operator Theory

In this section we present basic theory of unbounded operators. A key point to be taken care of is that such operators will not be defined on the entire Hilbert space but rather on a subspace.

An *operator* on H is a linear map

$$A \colon D(A) \longrightarrow H,$$

where $D(A)$ is a subspace of H which we call the *domain* of A. We say that A is *densely defined* if $D(A)$ is dense in H. We call an operator A *closed* if its graph $\{(f, Af) \in H \times H \mid f \in D(A)\}$ is closed in $H \times H$; put differently, if $f_n \to f$ for (f_n) in $D(A)$ along with $Af_n \to g$ imply $f \in D(A)$ and $Af = g$. We say that an operator A is *bounded* if there exists a constant $C \geq 0$ such that $\|Af\| \leq C\|f\|$ for all $f \in D(A)$. In this case, $\|A\|$, the *norm* of A, is the smallest such constant C.

If A is densely defined and bounded, then A can be uniquely extended to a bounded operator on the entire Hilbert space H and we denote this extension by A as well. We note that a bounded operator defined on the entire space is always closed. We denote the space of bounded operators defined on the entire Hilbert space H by $B(H)$.

For operators A and B on H we define the sum $A+B$ to be the linear map whose domain is

$$D(A+B) := D(A) \cap D(B)$$

and which acts by $(A+B)f := Af + Bf$.

We will also consider operators between different Hilbert spaces H_1 and H_2. In this case, the above definitions hold with the obvious modifications. In particular, an operator A from H_1 to H_2 is a linear map from a subspace $D(A)$ of H_1 into H_2. A most relevant instance is the product AB of operators B from H_1 into H_2 and A from H_2 into H_3. This product is defined on

$$D(AB) := \{f \in D(B) \mid Bf \in D(A)\}$$

and acts by $ABf := A(Bf)$.

Whenever A is an operator on H and $z \in \mathbb{C}$, we write $(A-z)$ for the operator $A - zI$ on $D(A)$, where I denotes the identity operator on H. We define the *resolvent set of* A to be

$$\varrho(A) := \{z \in \mathbb{C} \mid (A-z) \text{ is bijective and } (A-z)^{-1} \text{ is bounded}\}$$

and the *spectrum* of A as

$$\sigma(A) := \mathbb{C} \setminus \varrho(A).$$

We recall the standard fact that $\sigma(A)$ is always a closed set. For $z \in \varrho(A)$, we call the operator $(A-z)^{-1}$ the *resolvent of* A *at* z. For an operator A that is not closed, we have

$$\varrho(A) = \emptyset.$$

Indeed, if for some z the operator $(A-z)$ were bijective and $(A-z)^{-1}$ bounded, then $(A-z)^{-1}$ were bounded on the entire Hilbert space and, therefore, closed. But then $(A-z)$ and, hence, A would also be closed. This shows that the notion of a resolvent set is only relevant for closed operators.

On the other hand, for a closed operator A the definition of the resolvent set can be simplified to

$$\varrho(A) = \{z \in \mathbb{C} \mid (A-z) \text{ is bijective}\}.$$

This follows since if A is closed and $A-z$ is bijective, then $(A-z)^{-1}$ is bounded by the closed graph theorem.

An operator A is called *invertible* if $A\colon D(A) \longrightarrow H$ is bijective. If A and B are invertible operators and $D(B) \subseteq D(A)$, then

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1},$$

as follows by a direct calculation. In particular, if $z_1, z_2 \in \varrho(A)$, then

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}.$$

We refer to these formulae as *resolvent identities*. As a particular consequence, we note that the second formula implies that resolvents commute. Moreover, the second formula implies that the resolvent map

$$\varrho(A) \longrightarrow B(H), \quad z \mapsto (A-z)^{-1},$$

is continuous, i.e., for a sequence (z_n) in $\varrho(A)$ with $z_n \to z \in \varrho(A)$ we have

$$\lim_{n\to\infty} \|(A-z_n)^{-1} - (A-z)^{-1}\| = 0.$$

If A is densely defined, then we define the *adjoint* A^* of A to be the operator with domain

$$D(A^*) := \left\{ f \in H \;\middle|\; \begin{array}{c} \text{there exists a } g \in H \text{ with } \langle f, Ah \rangle = \langle g, h \rangle \\ \text{for all } h \in D(A) \end{array} \right\}$$

acting as

$$A^* f := g,$$

where we note that this is well-defined. Specifically, we have

$$\langle Af, g \rangle = \langle f, A^* g \rangle$$

for all $f \in D(A)$ and $g \in D(A^*)$ and A^* has the maximal domain among all operators with this property. The operator A^* is always closed (as can be easily seen).

We note that $D((A-z)^*) = D(A^*)$ and

$$(A - z)^* = A^* - \overline{z}$$

for all $z \in \mathbb{C}$. Furthermore, for $z \in \varrho(A)$,

$$((A - z)^{-1})^* = (A^* - \overline{z})^{-1}.$$

If A is densely defined, we say that A is *symmetric* if A^* is an extension of A, that is, $D(A) \subseteq D(A^*)$ and $Af = A^* f$ for all $f \in D(A)$. Equivalently, A is symmetric if and only if A is densely defined and

$$\langle Af, g \rangle = \langle f, Ag \rangle$$

for all $f, g \in D(A)$. With these preparations, we now define the class of operators of primary interest.

Definition 3.1 (Self-adjoint operators). We call a densely defined operator A *self-adjoint* if $A = A^*$.

Clearly, a self-adjoint operator is symmetric. Moreover, as the adjoint is always a closed operator, all self-adjoint operators are closed.

Before we develop the general theory further, we present a key example, namely multiplication operators. These operators are self-adjoint if the underlying function is real-valued. The main result on self-adjoint operators (to be discussed in Section 3.2) states a converse to this observation.

Example 3.2 (Multiplication operators). Let (X, μ) be a measure space and let $u \colon X \longrightarrow \mathbb{C}$ be measurable. The operator M_u of *multiplication by u* has domain

$$D(M_u) := \{ f \in L^2(X, \mu) \mid uf \in L^2(X, \mu) \}$$

and acts as

$$M_u f := uf$$

for all $f \in D(M_u)$. As discussed below, the operator M_u is densely defined. Furthermore, it can readily be seen that M_u is closed. The adjoint of M_u is given by $(M_u)^* = M_{\overline{u}}$. In particular, M_u is self-adjoint if u is real-valued. Finally, M_u is bounded if u is bounded.

The proofs of these statements are rather straightforward. We only sketch how to show that the domain of M_u is dense. For $n \in \mathbb{N}$, we define

$$X_n := \{ x \in X \mid |u(x)| \leq n \}.$$

Then, the characteristic functions 1_{X_n} for $n \in \mathbb{N}$ tend pointwise increasingly towards the constant function with value 1. In particular, we have $1_{X_n} f \to f$ for any $f \in L^2(X, \mu)$ by Lebesgue's dominated convergence theorem. On the other hand, by the definition of X_n, the function $1_{X_n} f$ belongs to $D(M_u)$ for any $n \in \mathbb{N}$.

The *essential range* of a measurable function $u\colon X \longrightarrow \mathbb{C}$ over a measure space (X, μ) is defined as

$$\operatorname{ess\,ran} u := \{\lambda \in \mathbb{C} \mid \mu(u^{-1}(B_\varepsilon(\lambda))) > 0 \text{ for all } \varepsilon > 0\},$$

where $B_\varepsilon(\lambda)$ is the closed ball around λ of radius ε.

Lemma 3.3 (Spectrum of multiplication operators). *Let (X, μ) be a σ-finite measure space. Let $u\colon X \longrightarrow \mathbb{C}$ be measurable and M_u be the operator of multiplication by u. Then, $\sigma(M_u)$ equals the essential range of u.*

Proof. For λ not in the essential range, the operator $M_{1/(u-\lambda)}$ is obviously a bounded inverse for $M_u - \lambda = M_{u-\lambda}$. Conversely, consider λ belonging to the essential range of u. Using the assumption of σ-finiteness, for any $\varepsilon > 0$, we can construct $f \in L^2(X, \mu)$ with $\|f\| = 1$ and $\|(M_u - \lambda)f\| < \varepsilon$. This contradicts the existence of a bounded inverse to $M_u - \lambda$. □

From the definitions it is not hard to derive the following additional properties of multiplication operators. They will be used repeatedly in what follows.

Proposition 3.4 (Further features of multiplication operators). *Let (X, μ) be a σ-finite measure space and let $u\colon X \longrightarrow \mathbb{C}$ be measurable. Then, the following statements hold:*

(a) *The operator M_u is self-adjoint if and only if* $\operatorname{ess\,ran} u \subseteq \mathbb{R}$, *which, in turn, holds if and only if u is real-valued almost everywhere.*
(b) *The operator M_u is bounded if and only if* $\operatorname{ess\,ran} u$ *is bounded, which, in turn, holds if and only if $u \in L^\infty(X, \mu)$. In this case,*

$$\|M_u\| = \|u\|_\infty = \sup\{|\lambda| \mid \lambda \text{ is in the essential range of } u\}.$$

(c) *$M_u = 0$ holds if and only if* $\operatorname{ess\,ran} u = \{0\}$ *if and only if $u = 0$ holds almost everywhere.*

Remark 3.5. In all three statements the if-part does not require the assumption of σ-finiteness of the measure space. This assumption is only used to obtain the only-if statement and the formula for the norm of the operator. Similarly, the assumption is used in obtaining the spectrum as the essential range in the preceding lemma.

3.2. Spectral Theorem and Spectral Calculus

The fundamental result about self-adjoint operators is known as spectral theorem. It states that they are (up to unitary equivalence) operators of multiplication. This can be seen as a (tremendous) generalization of the fact that symmetric matrices can be diagonalized. A consequence of the spectral theorem is the possibility to form functions of a self-adjoint operator. This is known as spectral calculus. In this section we discuss the details.

Without proof we present the spectral theorem.

THEOREM 3.6 (Spectral theorem). *Let A be a self-adjoint operator on the separable Hilbert space H. Then, there exists a measure space (X, μ), which is σ-finite, a measurable function $u\colon X \longrightarrow \mathbb{R}$ and a unitary map $U\colon L^2(X, \mu) \longrightarrow H$ with*

$$A = UM_uU^{-1}.$$

In particular, $\sigma(A)$ is equal to the essential range of u and, hence, contained in $\mathbb{R}$. Moreover, A is bounded if and only if u is essentially bounded.

The spectral theorem allows us to define functions of an operator.

DEFINITION 3.7 (Functional calculus - definition). If $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable, A is self-adjoint and (X, μ), u and U are as in Theorem 3.6, then we define the operator $\varphi(A)$ acting on the domain

$$D(\varphi(A)) := UD(M_{\varphi \circ u})$$

as

$$\varphi(A) := UM_{\varphi \circ u}U^{-1}.$$

REMARK 3.8. The spectral theorem does not state that (X, m), u and U are unique. Indeed, they are not. So, the preceding definition leaves open the possibility that $\varphi(A)$ is not well-defined but rather the definition depends on the choice of (X, m), u and U in the spectral theorem. This is not the case. We do not provide a full proof of independence here but rather sketch the idea: For $z \in \mathbb{C}$ we define $\varphi_z : \mathbb{R} \longrightarrow \mathbb{C}, \varphi_z(t) = \frac{1}{t-z}$. Then, $\varphi_z(A)$ can (by direct computation with multipliciation operators) be seen to satisfy $\varphi_z(A)(A - z) = I_{D(A)}$ and $(A - z)\varphi_z(A) = I$ (where I denotes the identity on H and $I_{D(A)}$ denote the identity on $D(A)$). Hence, $\varphi_z(A) = (A - z)^{-1}$ is independent of the choice of (X, m), u and U and, hence, well-defined. This then applies to all sorts of functions φ, which can be approximated (in a suitable sense) by linear combinations of the φ_z, $z \in \mathbb{C}$. Ultimately, this then yields that $\varphi(A)$ is well-defined for all measurable φ on $\sigma(A)$.

PROPOSITION 3.9 (Basic properties of $\varphi(A)$). *Let A be a self-adjoint operator on H with spectrum $\sigma(A)$ and let $\varphi, \psi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable on $\sigma(A)$. Then, the following statements hold:*

(a) $\varphi(A)^* = \overline{\varphi}(A)$.

(b) *The operator $\varphi(A)$ is self-adjoint if and only if the essential range of $\varphi|_{\sigma(A)}$ is contained in $\mathbb{R}$.*

(c) *The operator $\varphi(A)$ is bounded if and only if $\varphi|_{\sigma(A)}$ is essentially bounded, in which case*

$$\|\varphi(A)\| = \|\varphi|_{\sigma(A)}\|_\infty.$$

(d) $D(\varphi(A)\psi(A)) \subseteq D((\varphi\psi)(A))$ *and on* $D(\varphi(A)\psi(A))$ *we have*

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

(e) $D(\varphi(A)+\psi(A)) \subseteq D((\varphi+\psi)(A))$ *and on* $D(\varphi(A)+\psi(A))$ *we have*

$$(\varphi + \psi)(A) = \varphi(A) + \psi(A).$$

PROOF. This follows from the definition of $\varphi(A)$ and the corresponding properties of multiplication operators. In particular, (b) and (c) follow from Proposition 3.4. □

The spectral theorem makes it possible to introduce certain measures on the real line. This is done next.

PROPOSITION 3.10 (Spectral measures). *Let A be a self-adjoint operator on H and $f \in H$ be given. Then, the map*

$$\mu_f : \textit{Measurable subsets of } \mathbb{R} \longrightarrow [0,\infty), \quad B \mapsto \langle f, 1_B(A) f\rangle,$$

is a measure on $\mathbb{R}$.

PROOF. As 1_B is a bounded function, the operator $1_B(A)$ is a bounded operator. Moreover, from $1_B = 1_B^2 = \overline{1_B}$ and Proposition 3.9 we find

$$1_B(A) = 1_B(A) 1_B(A) = 1_B(A)^*.$$

This gives

$$\mu_f(B) = \langle f, 1_B(A) f\rangle = \langle f, 1_B(A)1_B(A) f\rangle = \langle 1_B(A) f, 1_B(A) f\rangle \geq 0.$$

This shows that μ_f maps indeed into $[0,\infty)$. It remains to show that μ_f is σ-additive. Let B be a measurable subset of $\mathbb{R}$ and let B_n, $n \in \mathbb{N}$, be pairwise disjoint measurable subsets with $\bigcup_{n\in\mathbb{N}} B_n = B$. Then, we have

$$1_{\bigcup_{n=1}^N B_n}(A) f \to 1_B(A) f, \quad N \to \infty.$$

Indeed, this is clear for $A = M_u$ and then follows for general A by the spectral theorem. From this convergence we obtain

$$\begin{aligned}\sum_{n=1}^N \langle f, 1_{B_n}(A) f\rangle &= \langle f, \sum_{n=1}^N 1_{B_n}(A) f\rangle = \langle f, 1_{\bigcup_{n=1}^N B_n}(A) f\rangle \\ &\to \langle f, 1_B(A) f\rangle = \mu_f(B).\end{aligned}$$

This shows that μ_f is a measure. □

DEFINITION 3.11 (Spectral measure). The measure μ_f appearing in the preceding proposition is called the *spectral measure* of f.

PROPOSITION 3.12 (Computing the spectral measures via spectral theorem). *Let A be a self-adjoint operator on H and let (X,μ), u and U be as in Theorem 3.6. Let $f \in H$ be given and define $\psi = U^{-1} f$. Then, the formula*

$$\int_{\mathbb{R}} \varphi \,\mathrm{d}\mu_f = \int_X (\varphi \circ u) |\psi|^2 \mathrm{d}\mu$$

holds for all measurable $\varphi\colon \mathbb{R} \longrightarrow [0,\infty)$, where both sides may take the value ∞. For $\varphi \in L^1(\mathbb{R}, \mu_f)$ both sides are finite.

PROOF. For $\varphi = 1_B$ with a measurable subset B of $\mathbb{R}$ the formula is immediate from a direct computation, which uses that U is unitary:

$$\begin{aligned}\int_{\mathbb{R}} \varphi \,\mathrm{d}\mu_f &= \langle f, 1_B(A) f\rangle = \langle f, U M_{1_B\circ u} U^{-1} f\rangle = \langle \psi, M_{1_B \circ u}\psi\rangle \\ &= \int_X (1_B \circ u)|\psi|^2 \mathrm{d}\mu.\end{aligned}$$

Now, the general case follows from taking linear combinations and limits. □

Given the preceding computation of the spectral measure, we now give some connections between the spectral calculus and the spectral measures. The arising formulas will be most useful for our subsequent considerations.

PROPOSITION 3.13 (Functional calculus and spectral measures). *Let A be a self-adjoint operator on H, let $f \in H$ and let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable.*

(a) *$f \in D(\varphi(A))$ if and only if $\varphi \in L^2(\mathbb{R}, \mu_f) = L^2(\sigma(A), \mu_f)$, in which case*

$$\|\varphi(A)f\|^2 = \int |\varphi|^2 \mathrm{d}\mu_f.$$

In particular, $f \in D(A)$ if and only if $\int x^2 \mathrm{d}\mu_f(x) < \infty$.

(b) *If $f \in D(\varphi(A))$, then*

$$\langle f, \varphi(A)f\rangle = \int \varphi \mathrm{d}\mu_f \qquad \text{and} \qquad |\varphi|^2 \mu_f = \mu_{\varphi(A)f}.$$

PROOF. Let (X, μ), u and U be as in Theorem 3.6, i.e.,

$$A = UM_uU^{-1},$$

where $U\colon L^2(X, \mu) \longrightarrow H$ is unitary. Then, by definition,

$$\varphi(A) = UM_{\varphi \circ u}U^{-1}$$

holds for all measurable $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$. We set

$$\psi := U^{-1}f.$$

(a) We first show the characterization of the domain of $\varphi(A)$. By Proposition 3.12, we have $\varphi \in L^2(\mathbb{R}, \mu_f)$ if and only if

$$\int |\varphi \circ u|^2 |\psi|^2 \mathrm{d}\mu < \infty$$

which, by the definition of the domain of a multiplication operator, is equivalent to

$$\psi \in D(M_{\varphi \circ u}).$$

As $\psi = U^{-1}f$, this holds if and only if $f \in D(\varphi(A))$ from the definition of the domain of $\varphi(A)$.

Now, if φ belongs to $L^2(\mathbb{R}, \mu_f)$, then, as U is unitary, Proposition 3.12 gives

$$\|\varphi(A)f\|^2 = \|M_{\varphi \circ u}\psi\|^2 = \int |\varphi \circ u|^2 |\psi|^2 \mathrm{d}\mu = \int |\varphi|^2 \mathrm{d}\mu_f,$$

which proves the formula given in (a). The last statement of (a) is immediate by taking $\varphi = \mathrm{id}$.

(b) As U is unitary and $U^{-1}\varphi(A) = M_{\varphi \circ u}U^{-1}$, we obtain

$$\langle f, \varphi(A)f\rangle = \langle U^{-1}f, U^{-1}\varphi(A)f\rangle = \langle \psi, M_{\varphi \circ u}\psi\rangle = \int (\varphi \circ u)|\psi|^2 \mathrm{d}\mu.$$

Since we assume $f \in D(\varphi(A))$, part (a) gives $\varphi \in L^2(\mathbb{R}, \mu_f)$. As μ_f is finite, $\varphi \in L^2(\mathbb{R}, \mu_f)$ implies $\varphi \in L^1(\mathbb{R}, \mu_f)$ and Proposition 3.12 yields

$$\int (\varphi \circ u)|\psi|^2 \mathrm{d}\mu = \int \varphi \mathrm{d}\mu_f.$$

Putting these equations together gives the first formula claimed in (b).

We now show

$$|\varphi|^2 \mu_f = \mu_{\varphi(A)f}.$$

It suffices to show

$$\int \chi|\varphi|^2 \mathrm{d}\mu_f = \int \chi \mathrm{d}\mu_{\varphi(A)f}$$

for all bounded measurable functions $\chi\colon \mathbb{R} \longrightarrow \mathbb{C}$. As χ is bounded and μ_f is finite for all $f \in H$, by part (a) the operator $\chi(A)$ is defined on the entire Hilbert space H. Hence, from the already established first formula of (b), the fact that U is unitary and the definitions of $\varphi(A)$ and $\chi(A)$, we find

$$\begin{aligned}\int \chi \mathrm{d}\mu_{\varphi(A)f} &= \langle \varphi(A)f, \chi(A)\varphi(A)f\rangle \\ &= \langle U^{-1}\varphi(A)f, U^{-1}\chi(A)UU^{-1}\varphi(A)f\rangle \\ &= \langle M_{\varphi\circ u}\psi, M_{\chi\circ u}M_{\varphi\circ u}\psi\rangle = \int (\chi\circ u)|\varphi\circ u|^2|\psi|^2 \mathrm{d}\mu \\ &= \int \chi|\varphi|^2\mathrm{d}\mu_f,\end{aligned}$$

where we used Proposition 3.12 in the last equality. This is the desired statement. □

COROLLARY 3.14 (Bounded functional calculus). *Let A be a self-adjoint operator on H and $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable and bounded on $\sigma(A)$. Then, $D(\varphi(A)) = H$ and, for every $f \in H$,*

$$\|\varphi(A)f\|^2 = \int |\varphi|^2 \mathrm{d}\mu_f, \qquad \langle f, \varphi(A)f\rangle = \int \varphi \mathrm{d}\mu_f$$

and

$$|\varphi|^2\mu_f = \mu_{\varphi(A)f}.$$

PROOF. As φ is bounded on $\sigma(A)$ and μ_f is a finite measure supported on $\sigma(A)$, we have $D(\varphi(A)) = H$ from (a) of Proposition 3.13, which also gives the first equality. The remaining equalities then follow from (b) of Proposition 3.13. □

PROPOSITION 3.15. *Let A be a self-adjoint operator on H and let $f, g \in H$. Then, there exists a unique finite signed regular Borel measure μ on $\mathbb{R}$ with*

$$\langle f, (A-z)^{-1}g\rangle = \int \frac{1}{x-z}\mathrm{d}\mu(x)$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$. If $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable and $f, g \in D(\varphi(A))$, then

$$\langle f, \varphi(A)g\rangle = \int \varphi \mathrm{d}\mu.$$

In particular, this holds for all $f, g \in H$ when φ is bounded and measurable.

PROOF. The existence of such a signed measure is given by the existence of μ_f and μ_g and polarization. The remaining statements follow by Proposition 3.13 (b) and polarization as well. □

3.3. Spectral Projections

Of particular relevance for applications of the functional calculus are characteristic functions of measurable sets. They are discussed next. Recall that $B(H)$ denotes the space of bounded operators on H. Given a self-adjoint operator A we define

$$E\colon \text{measurable subsets of } \mathbb{R} \longrightarrow B(H)$$

via

$$E(B) := 1_B(A).$$

We call the operator $E(B)$ the *spectral projection* associated to B. Note that we have already encountered $E(B)$ in the proof of Proposition 3.10. In particular, we have used $1_B = 1_B 1_B = \overline{1_B}$ to conclude that $E(B)$ satisfies

$$E(B) = E(B)E(B) = E(B)^*$$

and hence is an orthogonal projection. Similarly, we infer from $1_{B_1}1_{B_2} = 1_{B_1 \cap B_2}$ that

$$E(B_1)E(B_2) = E(B_1 \cap B_2) = E(B_2)E(B_1)$$

whenever B_1, B_2 are measurable subsets of $\mathbb{R}$. Moreover, we obviously have $E(\emptyset) = 0$ as $1_\emptyset = 0$. These considerations give, in particular,

$$E(B_1)E(B_2) = E(\emptyset) = 0$$

whenever $B_1 \cap B_2 = \emptyset$. Moreover, as $1_{\cup_{n\in\mathbb{N}} B_n}$ is the monotone pointwise limit of $\left(\sum_{n=1}^N 1_{B_n}\right)_{N\in\mathbb{N}}$ whenever the sets B_n are mutually disjoint, we infer $E(\bigcup_{n\in\mathbb{N}} B_n) = \oplus_{n\in\mathbb{N}} E(B_n)$. Furthermore, $E(\mathbb{R}) = I$ is the identity operator.

To summarize, we note that E satisfies the following properties:

- $E(B)$ is an orthogonal projection for each measurable $B \subseteq \mathbb{R}$.
- $E(\bigcup_{n\in\mathbb{N}} B_n) = \oplus_{n\in\mathbb{N}} E(B_n)$ for mutually disjoint measurable sets.
- $E(\emptyset) = 0$.

In this sense, the map E resembles a measure. We refer to E as the *projection-valued measure associated to A* or the *spectral family*.

The map E is intimately linked to the spectral measures. This is discussed in the subsequent two propositions. The first proposition is a direct consequence of Proposition 3.10.

Proposition 3.16 (Spectral measure via projection-valued measures). *Let A be a self-adjoint operator on H with associated projection-valued measure E. Then, for any $f \in H$, we have*

$$\mu_f(B) = \langle f, E(B)f\rangle = \|E(B)f\|^2$$

for any measurable set $B \subseteq \mathbb{R}$.

Proposition 3.17. *Let A be a self-adjoint operator on H with associated projection-valued measure E. Let $B \subseteq \mathbb{R}$ be measurable. Then, for all $f \in H$,*

$$\mu_{E(B)f} = 1_B \mu_f.$$

In particular, for any $g \in E(B)H$, we have $\mu_g = 1_B\mu_g$,

$$\operatorname{supp}(\mu_g) \subseteq \overline{B},$$

where $\overline{B}$ denotes the closure of B, and $g \in D(\varphi(A))$ if and only if $\varphi \in L^2(\sigma(A), \mu_g)$ for any measurable $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$.

Proof. The first statement follows from (b) of Proposition 3.13. Now, for $g \in E(B)H$, we have $g = E(B)g$ as $E(B)$ is an orthogonal projection and we find $\mu_g = 1_B\mu_g$. Clearly, $1_B\mu_g$ is supported on $\overline{B}$. Now, if $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable, then the statement on the domain of $\varphi(A)$ follows from Proposition 3.13 (a). This finishes the proof. □

It is possible to characterize the spectrum of A via E. To do so we define the *support of E* as

$$\operatorname{supp}(E) := \{\lambda \in \mathbb{R} \mid E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0 \text{ for all } \varepsilon > 0\}.$$

With this definition, we can show that the spectrum of A is equal to the support of E.

Theorem 3.18. *Let A be a self-adjoint operator and E the associated projection-valued measure. Then,*

$$\sigma(A) = \operatorname{supp}(E).$$

Proof. As the spectrum is preserved by unitary equivalence, we may assume that $A = M_u$, where (X, μ) is a σ-finite measure space and $u\colon X \longrightarrow \mathbb{R}$ is measurable by Theorem 3.6. In particular, the spectrum of A is given by the essential range of u. Hence, it remains to show that $\operatorname{supp}(E)$ equals the essential range of u. Now, for a measurable set $B \subseteq \mathbb{R}$, we have

$$E(B) = M_{1_B \circ u}$$

and, hence, $E(B)$ is not trivial if and only if

$$0 \neq 1_B \circ u = 1_{u^{-1}(B)}$$

if and only if $\mu(u^{-1}(B)) > 0$. This easily shows that $\operatorname{supp}(E)$ is equal to the essential range of u. □

3.4. Positive Operators

We will now restrict our attention further to those operators whose spectrum is contained in the non-negative real numbers.

Lemma 3.19. *Let A be a self-adjoint operator on H with domain $D(A)$. Then, the following statements are equivalent:*

(i) $\sigma(A) \subseteq [0, \infty)$.
(ii) *A is unitarily equivalent to multiplication by an almost everywhere non-negative function.*
(iii) $\langle f, Af\rangle \geq 0$ *for all* $f \in D(A)$.
(iv) *There exists a self-adjoint operator S with $A = S^2$.*

Proof. According to the spectral theorem, Theorem 3.6, we can assume without loss of generality that A is the operator M_u of multiplication by a measurable function $u\colon X \longrightarrow \mathbb{R}$, where (X, μ) is a σ-finite measure space and $D(M_u) = \{f \in L^2(X, \mu) \mid uf \in L^2(X, \mu)\}$. The spectrum of A is then the essential range of u by Lemma 3.3. Now, the essential range is contained in $[0, \infty)$ if and only if $u \geq 0$ almost everywhere and this in turn holds if and only if $\int u|f|^2 \mathrm{d}\mu = \langle f, M_u f\rangle \geq 0$ for all $f \in D(M_u)$. This shows the equivalence between (i), (ii) and (iii).

Now, if $u \geq 0$ almost everywhere, then $M_u = M_v^2$ with $v = \sqrt{u}$ and, thus, (ii) implies (iv). Finally, (iv) implies (iii) via

$$\langle f, Af\rangle = \langle f, S^2 f\rangle = \langle Sf, Sf\rangle = \|Sf\|^2 \geq 0.$$

This finishes the proof. □

We highlight the class of operators appearing in the previous statement by giving a definition.

Definition 3.20 (Positive operator). We say that a self-adjoint operator A is *positive* if A satisfies one of the equivalent conditions of Lemma 3.19. We write $A \geq 0$ in this case.

Remark 3.21. Note that positivity for self-adjoint operators means spectral positivity. If the Hilbert space has an additional lattice structure, which is the case in our typical situation of $\ell^2(X, m)$, this notion should not be confused with the property of being positivity preserving. An easy and instructive example is given by $A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $Af := \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} f$. Then A is clearly self-adjoint and $\sigma(A) = \{2, 4\}$ implies that A is positive. However, for $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have that f is positive, but $Af = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is not, so A is not positivity preserving. Here, note that $\mathbb{R}^2 = \ell^2(\{1, 2\})$.

Lemma 3.22 (Square root). *Let A be a positive operator on H. Then, the following statements hold:*

(a) *$\sqrt{A}$ is self-adjoint and positive.*
(b) *$(\sqrt{A})^2 = A$, i.e., $f \in D(A)$ if and only if $f \in D(\sqrt{A})$ and $\sqrt{A}f \in D(\sqrt{A})$ and in this case $Af = (\sqrt{A})^2 f$. In particular, $D(A) \subseteq D(\sqrt{A})$.*

Proof. By Lemma 3.19 we may assume that A is unitarily equivalent to multiplication by a function u which is positive almost everywhere on a σ-finite measure space (X, μ). It follows by the definition of the spectral calculus that $\sqrt{A}$ is then unitarily equivalent to multiplication by $\sqrt{u}$. As $\sqrt{u}$ is real-valued almost everywhere, the self-adjointness of $\sqrt{A}$ follows from Proposition 3.4. Positivity of $\sqrt{A}$ then follows by Lemma 3.19. This proves (a). Property (b) follows by a short argument involving the definition of the domain of a multiplication operator, see Example 3.2, and the definition of powers of unbounded operators. □

Matthias Keller
Daniel Lenz
Marcel Schmidt
Christian Seifert

Sheet 4

Operators on Hilbert spaces

Exercise 1 (Resolvents are continuous) 4 points

Show that the resolvent map of an operator A on a Hilbert space H

$$\varrho(A) \longrightarrow B(H), \quad z \mapsto (A - z)^{-1},$$

is continuous.

Exercise 2 (Multiplication operators I) 4 points

Let (X,μ) be a measure space and let $u\colon X \longrightarrow \mathbb{C}$ be measurable. The operator M_u of *multiplication by* u has domain

$$D(M_u) = \{f \in L^2(X,\mu) \mid uf \in L^2(X,\mu)\}$$

and acts as

$$M_u f = uf$$

for all $f \in D(M_u)$. Show the following statements:

(a) The operator M_u is densely defined.

(b) The operator M_u is closed.

(c) The adjoint of M_u is given by $(M_u)^* = M_{\overline{u}}$. In particular, M_u is self-adjoint if u is real-valued.

(d) The operator M_u is bounded if $u \in L^\infty(X,\mu)$.

Exercise 3 (Multiplication operators II) 4 points

Let (X,μ) be a σ-finite measure space and M_u the multiplication operator for a measurable function $u\colon X \longrightarrow \mathbb{C}$.

(a) The operator M_u is self-adjoint if and only if the essential range of u is contained in $\mathbb{R}$, which, in turn, holds if and only if u is real-valued almost everywhere.

(b) The operator M_u is bounded if and only if the essential range of u is bounded, which, in turn, holds if and only if $u \in L^\infty(X,\mu)$. In this case,

$$\|M_u\| = \|u\|_\infty = \sup\{|\lambda| \mid \lambda \text{ is in the essential range of } u\}.$$

(c) $M_u = 0$ holds if and only if the essential range of u is $\{0\}$ which, in turn, holds if and only if $u = 0$ holds almost everywhere.

Exercise 4 (Closure convergence) 4 points

Let (L_n) be a sequence of self-adjoint operators on a Hilbert space and let L be a self-adjoint operator. Assume that for a family $(\Phi_\alpha)_{\alpha \in I}$ of measurable bounded functions from $\mathbb{R}$ to $\mathbb{R}$ and some index set I we have

$$\lim_{n\to\infty} \Phi_\alpha(L_n)f = \Phi_\alpha(L)f$$

for all f in the Hilbert space and for all $\alpha \in I$. Let $\mathcal{A}$ be the closure of $\{\Phi_\alpha \mid \alpha \in I\}$ with respect to the supremum norm. Show that

$$\lim_{n\to\infty} \Phi(L_n)f = \Phi(L)f$$

for all $\Phi \in \mathcal{A}$ and f in the Hilbert space.