26th Internet Seminar
"Graphs and Discrete Dirichlet Spaces"
Final Workshop

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July 16 to July 22, 2023

Program

The Workshop takes place at the

CVJM Bildungsstätte Bundeshöhe, Bundeshöhe 7, 42285 Wuppertal, Germany.

For virtual participation, the following Zoom meeting will be used throughout the workshop:

`https://tuhh.zoom.us/j/81359919547?pwd=eFlmRm5jVHlrc00yZE9ZeG5sZ0c1Zz09`

Meeting ID: 813 5991 9547
Passcode: ISem26

How to reach the venue:
Airports close to Wuppertal are Düsseldorf, Köln/Bonn, and also Frankfurt/Main. Take a train to Wuppertal main station (Wuppertal HBF), see e.g. `https://www.bahn.com/en`. When you have arrived at Wuppertal main station (HBF), please go to gate 1 and then in direction of waiting area F (please see the blue rectangular signs under the platform roof). On the platform way you will come to the sign: "DB Reisezentrum". Please turn right in order to leave the station, then turn left into the narrow street. This street (without a name) will lead you to "Bahnhofstraße". Please follow "Bahnhofstrasse" the way up, after the traffic lights turn left into "Kleeblatt". The name of the bus station is "Historische Stadthalle/Hauptbahnhof".
Take bus no. 620, in direction of "Ronsdorf/Remscheid-Luettringhausen". Get off the bus at bus stop "Kapellen". Please cross the bridge close to the bus stop in order to cross the street. Continue through ahead and turn left when you come to "Haus Grunewald". After 50 metres you will see stairways on the left. Go down the stairways. After you already passed half the way, then turn right into "Kaethe-Kollwitz-Weg". Continue until the end and then turn left. The "CVJM-Bildungsstätte" will be in front. Bus driving time is about 15 minutes. Operating hours: Monday to Friday every 20 minutes, 4:00 a.m. to 12:00 a.m. On Saturday, Sunday and Holiday every 30 minutes, 06:00 a.m. to 12:00 a.m.
You also may take bus no. CE 62 in direction of Ronsdorf. (This bus does not operate on Sundays!) Get off at bus station "Lichtscheid Wasserturm". Cross the subway/pedestrian underpass in direction of "Hornbach" (DIY store). Then turn into "Obergische Straße". Please turn — five minutes later — into "Böhler Weg" and after 300 metres, turn right, where you will find the "CVJM-Bildungsstätte". Bus driving time is about 10 minutes. Operating hours: Monday to Friday, every 20 minutes, 07:00 a.m. to 06:00 p.m. Saturday, every 30 minutes, 10:00 a.m. to 06:00 p.m.

ISem 26 Phase 3 Workshop July 16, 2023 to July 22, 2023

from	to	Sunday 16th	Monday 17th	Tuesday 18th	Wednesday 19th	Thursday 20th	Friday 21st	Saturday 22nd
08:00	08:15		*Breakfast*	*Breakfast*	*Breakfast*	*Breakfast*	*Breakfast*	*Breakfast*
08:15	08:30							
08:30	08:45							
08:45	09:00							
09:00	09:15		Opening	Project D	Project C	Project I	Plenary Talk Alexander Grigor'yan	
09:15	09:30		Plenary Talk Radoslaw Wojciechowski					
09:30	09:45							
09:45	10:00							
10:00	10:15						*Coffee Break*	
10:15	10:30		*Coffee Break*					
10:30	10:45			*Coffee Break*	*Coffee Break*	*Coffee Break*	Project J	
10:45	11:00		Project M					
11:00	11:15			Plenary Talk Uta Freiberg	Project A	Plenary Talk Noema Nicolussi		
11:15	11:30							
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12:00	12:15							
12:15	12:30		*Lunch Break*	*Lunch Break*		*Lunch Break*	*Lunch Break*	
12:30	12:45				*Lunch Break*			
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14:00	14:15		Project L	Project K	Free Afternoon	Project F	Project B	
14:15	14:30							
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15:30	15:45		*Coffee Break*	*Coffee Break*		*Coffee Break*	*Coffee Break*	
15:45	16:00							
16:00	16:15		Project H	Project N		Project G	Project E	
16:15	16:30							
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17:30	17:45						Closing	
17:45	18:00							
18:00	18:15	*Dinner*	*Dinner*	*Dinner*	*Dinner*	*Conference Dinner: Barbecue*	*Dinner*	
18:15	18:30							
18:30	18:45							
18:45	19:00							
19:00	19:15				Coordinator Meeting			
19:15	19:30							
19:30	19:45							
19:45	20:00							

Plenary Talks

The Neumann semigroup via exhaustion

Radoslaw Wojciechowski (City University of New York, USA)

It is well-known that by taking Dirichlet boundary conditions and exhausting via finite subgraphs, one gets convergence of semigroups and resolvents. We will discuss the corresponding issue for Neumann boundary conditions. In particular, we will give connections to both stochastic completeness and form uniqueness and discuss the Feller property for the Neumann semigroup. This talk is based on joint work with Florentin Münch and Matthias Keller.

Analysis on self-similar and non self-similar fractals

Uta Freiberg (Technische Universität Chemnitz, Germany)

In this talk, we present the classical definition of the Laplacian on finitely ramified fractals by means of Dirichlet forms as it has been introduced by Kigami in the late 80s. Our model case will be Sierpinski gasket, and we will present the corresponding Weyl asymptotics (obtained by Kigami and Lapidus in 1993). Then we present how these approaches can be applied to fractals where the assumption of strict self similarity is weakend. In particular, so called V-variable fractals (a certain class of statistically self similar fractals) and stretched Sierpinski gaskets are considered. The results presented in the talk have been obtained in collaboration with Ben Hambly (Oxford University), John Hutchinson (ANU Canberra), Patricia Alonso Ruiz (Texas A&M University) and Jun Kigami (Kyoto University).

Laplacians on Infinite Graphs: discrete vs. continuous

Noema Nicolussi (Universität Potsdam, Germany)

There are two different notions of a Laplacian operator associated with infinite graphs: discrete Laplacians and metric graph Laplacians. In the second setting, one turns the graph into a continuous geometric object by replacing edges with intervals and considers Laplacian differential operators on graphs (often called "quantum graphs"). Both objects have a venerable history and their spectral theory relates to diverse branches of mathematics (random walks, combinatorics, geometric group theory, ...). In our talk we explore connections between these two types of operators (spectral, parabolic and geometric properties), and exploit these relations to prove a number of new results in spectral theory for both settings. Based on joint work with Aleksey Kostenko (Ljubljana & Vienna) and Mark Malamud (Moscow).

Path homology and Hodge Laplacian on digraphs

Alexander Grigor'yan (Universität Bielefeld, Germany)

We define a chain complex on a digraph that is based on paths going along the arrows and that defines the path homology of the digraph. This chain complex gives rise to a Hodge Laplacian, and we state some results of about the spectrum of this operator.

Projects

Project A: Positivity preserving and improving C_0-semigroups

Project Coordinator(s): Sahiba Arora and Jochen Glück

Participants: Souhadou Diallo, Leon Berghoff-Flüel, Ines Marzouk and Ines Walha

For Dirichlet Laplacians L on graphs, we have seen in [1, Theorem 6.1] that the corresponding semigroup operators e^{-tL} are positivity improving for each $t > 0$ if and only if the resolvent operators $(L + \alpha)^{-1}$ are positivity improving for all $\alpha > 0$. More generally speaking, the same equivalence holds for each positivity preserving semigroup on a suitable ordered space. In this project, we look at a proof for the case of L^p-spaces.

In addition, we study positivity preserving *irreducible* semigroups. In particular, we will see that a positivity preserving irreducible analytic semigroup is also positivity improving. The main source of references will be [2, 3, 4].

This project is suited for 3 to 4 students.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] A. Bátkai, M. Kramar Fijavž, & A. Rhandi. Positive operator semigroups. From finite to infinite dimensions. *Oper. Theory: Adv. Appl.*, **257** pp. xviii + 364 (2017).

[3] W. Arendt, R. Nagel, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, F. Neubrander, & U. Schlotterbeck. One-parameter semigroups of positive operators. Lecture Notes in Mathematics, vol. 1184, Springer-Verlag Berlin Heidelberg, 1986.

[4] N. Gao & V.G. Troitsky. Irreducible semigroups of positive operators on Banach lattices. *Linear Multilinear Algebra*, **62** pp. 74–95.

Project B: The p-Laplacian on discrete graphs and the torsion function

Project Coordinator(s): Patrizio Bifulco and Delio Mugnolo

Participants: Mohamedahmed Mirghani Hassan Mohamed, Djiby Sow, Gerd Simon Schmidt and Lorenzo Liverani

Given a discrete measure space (X, m) and a graph (b, c) over (X, m), for $p \in (1, \infty)$, we study the *p-Laplacian* $\mathcal{L}^{(p)}_{b,c,m}$: this is an operator formally acting on functions in $C(X)$ by

$$\mathcal{L}^{(p)}_{b,c,m} f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))|f(x) - f(y)|^{p-2} + \frac{c(x)}{m(x)} |f(x)|^{p-2} f(x). \quad (1)$$

For $p = 2$, $\mathcal{L}^{(p)}_{b,c,m}$ is nothing but the formal Laplacian on the graph; but for $p \neq 2$ (1) defines a non-linear operator.

Studying this new object, we will extend many results we encountered in the ISem-Lectures [1] in the special case $p = 2$, first on finite and then on infinite graphs: we will explain the details of the introduction of these operators and then sketch the construction of the associated semigroups and perhaps discuss (a nonlinear version of) the Markov property for such semigroups.

Among the main goals of this project is a gentle introduction to the theory of the *p-torsion function*, i.e., of the (positive) solution τ of

$$\mathcal{L}^{(p)}_{b,c,m} \tau = \mathbf{1};$$

for the students of the ISem, this is a new topic already for $p = 2$. It turns out that the p-torsion function play an interesting role in the spectral theory of p-Laplacians: not only do its 1-norm and ∞-norm deliver upper and lower estimates on the lowest strictly positive eigenvalue of $\mathcal{L}^{(p)}_{b,c,m}$, but the p-torsion function also induces pointwise bounds on *all* eigenfunctions [5]. Our methods will be mostly variational, in view of a known characterization of the torsion function that goes back to Pólya.

If time allows we will also compare our results with recent results for the torsional rigidity on metric graphs in [3] and further structures.

This advanced project is suited for 4 to 5 students, ideally with a good understanding of variational methods.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] D. Mugnolo. Parabolic theory of the discrete p-Laplace operator. *Nonlinear Anal., Theory Methods Appl.*, 87:33–60, 2013.

[3] D. Mugnolo and M. Plümer. *On torsional rigidity and ground-state energy of compact quantum graphs*. Calc. of Var. and Part. Diff. Eq. **62**, 27 (2023)

[4] P. Bifulco and D. Mugnolo. *On the torsion of discrete graphs*. (work in progress)

[5] D. Mugnolo. Pointwise eigenvector estimates by landscape functions: some variations on the Filoche–Mayboroda–van den Berg bound. arXiv:2301.06126, 2023.

Project C: Stability of the wave equations on networks

Project Coordinator(s): Boubacar Diao and Salem Nafiri

Participants: Sara Mountassir, Modibo Sissoko, Fatima Ba and Tim Schmatzler

Networks have been studied widely in recent years with motivations from and applications to classical natural sciences (electro-circuits, chemical processes, neural networks, population biology, etc.) as well as to social sciences or even to the WorldWideWeb. Much progress has been made in understanding the structure of these networks, and we refer to M.E.J. Newman [7] for a survey on these developments. However, on p. 224 of [7] he says: "The next logical step after developing models of network structure, $(\cdots)$ is to look at the behavior of models of physical (or biological or social) processes going on on those networks. Progress on this front has been slower than progress on understanding network structure."

The main goal of the present project is to define an appropriate setting and to find the tools to investigate such processes on networks. Here we combine functional analytical and graph theoretical methods in order to study wave equations in networks. We show that these equations can be described by a strongly continuous operator semigroup on a Hilbert space. Using frequency domain analysis we prove that the semigroup behaves asymptotically exponential/polynomial. These results have been already shown for a single elastic string with local Kelvin–Voigt damping, see [5, 6].

The first part of the project will therefore deal with defining the proper functional setting for wave equations on networks, see [1] as well as showing that the resulting system is well-posed, see [3, 4, 2].

Afterwards, and depending on the interests of the participants, we will deal with the asymptotic behavior of the corresponding semigroup defined on the network.

This project is suited for 3 to 4 students.

Bibliography

[1] ISem 26 Lecture Notes, 2023.

[2] Ammari, K., Liu, Z. & Shel, F. *Stability of the wave equations on a tree with local Kelvin–Voigt damping*. Semigroup Forum 100, 364–382 (2020).

[3] Ammari, K., Jellouli, M.: *Stabilization of star-shaped networks of strings*. Differ. Integr. Equ. 17, 1395–1410 (2004)

[4] Ammari, K., Jellouli, M., Khenissi, M.: *Stabilization of generic trees of strings*. J. Dyn. Contin. Syst. 11, 177–193 (2005)

[5] Liu, Z., Rao, B. *Frequency domain characterization of rational decay rate for solution of linear evolution equations*. Z. Angew. Math. Phys. 56, 630–644 (2005).

[6] Liu, K., Liu, Z.: *Exponential decay of energy of vibrating strings with local viscoelasticity*. Z. Angew. Math. Phys. 53, 265–280 (2002).

[7] Newman, M.E.J.: *The structure and function of complex networks*. SIAM Rev. 45, 167–256 (2003).

Project D: Energy forms on fractals

Project Coordinator(s): Uta Freiberg

Participants: Mahesh Krishna, Youssef Hakiki, Marcel Kleinfeller and Paul Rahlf

There are several approaches in order to define energy forms on a fractal. One of the most popular ones - based on simple calculus - is due to Jun Kigami. However, his monograph (Analysis on fractals, 2001) presents the ideas in a very formal, general language and on a high level of mathematical sophistication. Hence, it might be hardly accessible for beginners. The book of Strichartz [2] is called "a soft introduction" to the subject (by the author himself).
In Chapter 1 of Strichartz' book, energy forms on fractals are obtained as limits of discrete graph energies. To this end, self similar (finitely ramified) fractals are approximated by sequences of finite graphs. On these graphs, we have the notion of graph energies (known from the Internet Seminar, see [1]). Now the challenge is to find the right renormalization to make these sequences of graph energies convergent. Then the limit is an energy form on the fractal! In Chapter 1 of [2], these ideas are developed very detailed with the help of the two model cases *unit interval* and *Sierpinski Gasket*.

This project is suited for 3 to 4 students.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] R. S. Strichartz: *Differential Equations on Fractals. A Tutorial.* Princeton University Press, Princeton, 2006

Project E: Port-Hamiltonian systems on graphs

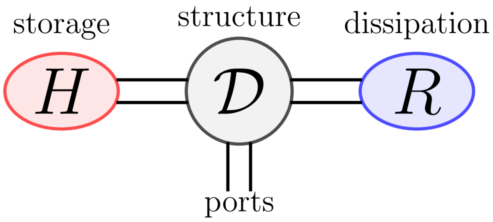
Project Coordinator(s): René Hosfeld and Merlin Schmitz

Participants: Gildas Djanga, Gabriel Diouf, Merten Mlinarzik and Samir Boujijane

Port-Hamiltonian systems (pHS) theory is gaining more and more attention in the last three decades. They are a powerful tool to model and analyze a broad class of energy based physical systems, including multi-body systems, electrical circuits and electromechanical systems. In a simple, finite-dimensional case pHS are of the form

$$\dot{x}(t) = (J - R)\frac{\partial H}{\partial x}(x(t)) + Bu(t),$$
$$y(t) = B^T \frac{\partial H}{\partial x}(x(t)) + Du(t),$$

where the *Hamiltonian* $H\colon \mathbb{R}^n \to \mathbb{R}$ represents the energy of the system, $J = -J^T \in \mathbb{R}^{n\times n}$ the internal interconnection, $0 \leq R \in \mathbb{R}^{n\times n}$ describes dissipation of energy and $0 \leq D \in \mathbb{R}^{m\times m}$ is a feedthrough. The input and collocated output functions u, y, taking values in $\mathbb{R}^m$, are called the *ports* that connect the system to the environment or other pHS via a matrix $B \in \mathbb{R}^{n\times m}$. One characteristic of such a system is its modular structure – it can be seperated into the three building blocks of energy storage, dissipation and the geometrical interconnection which is contained in the so called *Dirac structure*.



Given a (finite) directed graph with a set of boundary vertices, we investigate different Dirac structures respecting the graph's geometry. Following [1], we derive corresponding pHS. The abstract theory is illustrated my means of mass-spring-damper systems. Necessary basics on pHS are given in [2].

Depending on the students background we investigate further analytic properties.

This project is suited for 3 to 4 students.

Bibliography

[1] Arjan van der Schaft and Bernhard Maschke. Port-Hamiltonian systems on graphs. *SIAM Journal on Control and Optimization*. 51:906–937 (2013). https://epubs.siam.org/doi/pdf/10.1137/110840091

[2] Arjan van der Schaft and Dimitri Jeltsema. *Port-Hamiltonian Systems Theory: An Introductory Overview*. Foundations and Trends® in Systems and Control, vol. 1, no. 2-3, pp. 173–378, 2014. https://www.math.rug.nl/~arjan/DownloadVarious/PHbook.pdf

Project F: Index theorems and trace formulas for quantum graphs

Project Coordinator(s): Amru Hussein

Participants: Bahati Chadrack Kilongo, Chtiba Reda, Joachim Hofmann and Maximilian Weinberg

In the ISem lectures discrete graphs and difference operators have been considered. A different graph-like setting are metric graphs where the edges are identified with intervals filling the "space between the vertices". Operators on such metric graphs then consist of one-dimensional ordinary differential operators on the edges (that is a collection of intervals) and coupling boundary conditions on the vertices. For such operators – sometimes called *quantum graphs* – the spectral theory is studied in this project.

One theme of the ISem lecture [1] has been the interplay between geometry and spectral properties. Here, we carry over this theme to the setting of metric graphs discussing *index theorems* and *trace formulas* for quantum graphs. For differential operators on manifolds the so-called index theorems provide a connection between the topology of the manifold and the Fredholm index of certain differential operators, where the Fredholm index describes the difference between dimension of the kernel and the co-range of an operator, that is, it quantifies information on the solvability of the corresponding differential equations. In this project we will work through the corresponding *index theorems for quantum graphs* given in [2] by Fulling, Kuchment, and Wilson. Metric graphs can be seen also as singular one-dimensional manifolds, and therefore – if there is interest – also relations to the famous Atiyah-Singer index theorem for manifolds can be discussed, cf. e.g. [3]. Another instance where spectral quantities and geometry are related are *trace formulas*. Here, we will study the work [4] of Kostrykin, Potthoff, and Schrader where the trace of the heat kernel is related to the Euler characteristic of the graph and closed cycles on the graph. Connections to the celebrated Selberg trace formula for manifolds can be discussed if time permits.

This project is suited for 3 to 4 students.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] Fulling, S. A. and Kuchment, P. and Wilson, J. H., Index theorems for quantum graphs, J. Phys. A, 40(47): 14165–14180, 2007. `https://doi.org/10.1088/1751-8113/40/47/009`

[3] Gilkey, P. B., Invariance theory, the heat equation, and the Atiyah-Singer index theorem, CRC Press, Boca Raton, FL, 1995. `https://pages.uoregon.edu/gilkey/dirPDF/InvarianceTheory1Ed.pdf`

[4] Kostrykin, V. and Potthoff, J. and Schrader, R. , Heat kernels on metric graphs and a trace formula, In *Adventures in mathematical physics* Amer. Math. Soc., Providence, RI, 447: 175–198, 2007. https://doi.org/10.1090/conm/447/08691 Preprint https://arxiv.org/pdf/math-ph/0701009.pdf

Project G: Spectra of periodic quantum graphs

Project Coordinator(s): Joachim Kerner, Ivica Nakić and Matthias Täufer

Participants: Soukaina Mahzoum, Chiara Alessi, Dennis Schmeckpeper and Aymen Bahloul

This project builds on recent results on discrete Laplacians [1] on *periodic graphs* which emerge for instance in physics when modelling two-dimensional materials. The eventual goal is to generalize these results to so-called *quantum graphs* and continuous Schrödinger operators defined on them. We will start with discrete graphs with a two-dimensional symmetry as investigated in [4, 3] and introduce – as a main tool – Floquet theory which allows to reduce periodic operators on infinite graphs to operators with a compact resolvent. In a next step, we will learn about *metric graphs* as well as Schrödinger operators defined on them. Focussing on the Laplacian, we want to understand the correspondence between the spectrum of quantum and the corresponding discrete graphs which holds in certain cases [2]. Finally, if time permits, we shall see whether the results regarding the spectrum of discrete Laplacians as obtained in [3] can be translated to the setting of quantum graphs.

This project is suitable for 3–4 students.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] J. von Below. A characteristic equation associated to an eigenvalue problem on c^2-networks. *Linear Algebra Appl.*, 71:309–325, 1985.

[3] J. Wintermayr, J. Kerner and M. Täufer. Robustness of flat bands on the perturbed Kagome and the perturbed Super-Kagome lattice. arXiv:2301.05076 [math.SP], 2023.

[4] N. Peyerimhoff and M. Täufer. Eigenfunctions and the integrated density of states on Archimedean tilings. arXiv:1807.06406 [math-ph]. *J. Spectr. Theory*, 11(2):461–488, 2021.

Project H: Harnack estimates for the discrete heat and porous medium equation via curvature-dimension conditions

Project Coordinator(s): Sebastian Kräss

Participants: Oumarou Asso, Hind el Baggari and Azam Jahandideh

In order to derive parabolic Harnack estimates for positive solutions u to the heat equation on $(0,\infty) \times \mathbb{R}^n$, the fundamental estimate of Li and Yau

$$-\Delta \log u(t,x) \leq \frac{n}{2t} \tag{2}$$

has turned out to be a very helpful tool. Indeed, by integration over a suitable path in time, one can derive a classical Harnack estimate. In this sense, Li-Yau inequalities are often called differential Harnack estimates. For more details, see [3, Chapter 12]. A discrete version of (2) has been established in [1, Theorem 4.1] in the case of a finite graph. This discrete estimate is based on a so-called curvature dimension inequality, namely the condition $CD(F;0)$ defined in [1, Definition 3.8]. This theory will be the starting point of our project.

We then also have a closer look at the generalization of this theory to the nonlinear case when dealing with positive solutions to the discrete porous medium equation (PME)

$$\partial_t u(t,x) - Lu^m(t,x) = 0 \tag{3}$$

on $(0,\infty) \times X$, where $m > 1$, X is a (possibly infinte) discrete set and L denotes the discrete Laplacian as defined in the lecture notes of this year's ISEM. Note, that for $m = 1$, equation (3) becomes the heat equation.

Our **main references** for the project are [1] and **[2]**. In the case of the PME, we will prove estimates of the form

$$-L\left(\frac{m}{m-1}u^{m-1}\right)(t,x) \leq \frac{c}{t}, \tag{4}$$

which are called Aronson-Bénilan estimates in the literature. Note, that (4) becomes a discrete version of the Li-Yau inequality (2) for $m \to 1$ (as the left hand side in (4) converges to $-L\log u$ when $m \to 1$).

Our procedure will be the following: We start by discussing the linear case $m = 1$ considered in [1]. We will see how a certain curvature-dimension inequality, namely $CD(F;0)$, is sufficient to prove discrete Li-Yau estimates on finite graphs. We then study the generalization of these results to the nonlinear case $m > 1$ by formulating an adapted curvature-dimension condition that is sufficient to prove estimates of the form (4). Several examples, the most important being the one of a general complete graph, will be treated for the PME case. We also have a closer look at the lattice $\mathbb{Z}$ where the proof of an Aronson-Bénilan estimate turns out to be much more difficult. In both, the linear and the nonlinear case, we will use our differential estimates to derive Harnack type estimates by simple integration arguments.

This project is suited for 3 to 4 students.

Bibliography

[1] D. Dier, M. Kassmann, R. Zacher: *Discrete versions of the Li-Yau gradient estimate.* Preprint 2017. https://arxiv.org/abs/1701.04807.

[2] S. Kräss, R. Zacher: *Aronson-Bénilan and Harnack estimates for the discrete porous medium equation.* Preprint 2023. https://arxiv.org/abs/2301.07683.

[3] P. Li: *Geometric Analysis.* Cambridge Studies in Advanced Mathematics **134**. Cambridge University Press, Cambridge, 2012.

Project I: Laplacians on infinite graphs: from continuous to discrete and back

Project Coordinator(s): Noema Nicolussi

Participants: Tchato Ngahane Mikael, Salma Lahbabi, Paco Ezquerra and Katharin Klioba

There are two main notions of Laplacian operators associated with infinite graphs: in the first setting, graphs are treated as discrete objects and discrete Laplacians are defined by difference expressions. In the second setting, one turns the graph into a continuous geometric object by replacing edges with intervals and glueing them together at vertices. This geometric object is called a metric graph and allows to consider Laplacian differential operators on graphs (often called quantum graphs).

In this project, we will study the relationship between these two settings. A systematic connection has been established recently in [3, 4]. To each Laplacian on a metric graph, one can associate a discrete Laplacian which has similar spectral and parabolic properties. Moreover, all discrete Laplacians on locally finite graphs arise in this way. This leads to a systematic way of connecting the settings and several applications.

We first introduce metric graphs and their Laplacians [1, 3]. Following [4], we then study connections between Laplacians on discrete and metric graphs on different levels. Here the participants can choose topics which are interesting to them, e.g.,

- Intrinsic metrics and quasi-isometric spaces
- Spectral estimates, isoperimetric constants and Cheeger's inequality
- Stochastic completeness and recurrence/transience
- Ultracontractivity estimates for the heat semigroup

For finding out whether this project is interesting to them, the participants can also consult [5, Section 5], which contains a short overview of these thematics.

This project is suited for 3 to 4 students.

Bibliography

[1] G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Amer. Math. Soc., Providence, RI, 2013.

[2] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[3] A. Kostenko and N. Nicolussi, *Laplacians on infinite graphs: discrete vs. continuous*, Proceedings of the 8ECM, to appear, arXiv:2110.03566 (2021).

[4] A. Kostenko and N. Nicolussi, *Laplacians on infinite graphs*, Mem. Eur. Math. Soc., to appear; preprint available at `www.mat.univie.ac.at/~kostenko/list/GraphLaplInf.pdf`.

[5] N. Nicolussi, *A note on Spectral Analysis of Quantum graphs*, arXiv:2209.02968 (2022).

Project J: Surjectivity of formal Graph Laplacians and magnetic Schrödinger operators

Project Coordinator(s): Marcel Schmidt

Participants: Ilham Oueld Dris, Soufiane Boumasmoud, Tobias Habacker and Micky Barthmann

Let b be a graph on an at most countable set X as in the lectures. In this project we study surjectivity of the formal Laplacian (and related formal magnetic Schrödinger operators)

$$\mathcal{L}\colon \mathcal{F} \to C(X), \quad \mathcal{L}f(x) = \sum_{y \in X} b(x,y)(f(x) - f(y))$$

with

$$\mathcal{F} = \{f \in C(X) \mid \sum_{y \in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\}.$$

If X is finite, then $\mathcal{L}$ is not injective because $\mathcal{L}1 = 0$ and hence it cannot be surjective. On infinite connected graphs one can show with the help of local Harnack inequalities that any $f \in \mathcal{F}$ with $\mathcal{L}f = 0$ must have infinite support, making the restriction $\mathcal{L}|_{C_c(X)}$ injective.

It was observed in [3] that on locally finite graphs injectivity of $\mathcal{L}|_{C_c(X)}$ leads to surjectivity of $\mathcal{L}$. This is based on an old surjectivity critertion of Eidelheit [2], which relies on some abstract surjectivity results for spaces of type (B_0) due to Mazur and Orlicz (Fréchet spaces in modern terminology, the name had not yet been coined in 1936). A self-contained account on the surjectivity of $\mathcal{L}$ on locally finite graphs without much reference to abstract Fréchet space theory can be found in [4]. There it is also shown that on infinite connected graphs which are not locally finite surjectivity of $\mathcal{L}$ need not hold anymore. A different approach to surjectivity of $\mathcal{L}$ with a Mittag-Leffler type argument is given in [1]. Indeed, it preceeded and inspired the previously mentioned [3].

The first aim of this project is to study and present the two different apporoaches to surjectivity of $\mathcal{L}$ in [3, 4] and [1]. Eidelheit's surjectivity criterion should lead to surjectivity results on some not locally finite graphs, a case which apart from a counterexample has not been treated in the literature. The second aim of this project is to explore and in the best case characterize those not locally finite graphs whose formal Laplacian is surjective with the help of Eidelheit's results from [2].

This project is suited for 3 to 4 students. Since one of the references is in German, at least one German speaker is required.

Bibliography

[1] Tullio Ceccherini-Silberstein, Michel Coornaert, and Józef Dodziuk. The surjectivity of the combinatorial Laplacian on infinite graphs. *Enseign. Math. (2)*, 58(1-2):125–130, 2012.

[2] M. Eidelheit. Zur Theorie der Systeme linearer Gleichungen. *Studia Math.*, 6:139–148, 1936.

[3] T. Kalmes. A short remark on the surjectivity of the combinatorial Laplacian on infinite graphs. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 110(2):695–698, 2016.

[4] J. Koberstein, M.Schmidt A note on the surjectivity of operators on vector bundles over discrete spaces. *Arch. Math. (Basel)*, 114 (2020), no. 3, 313–329.

Project K: General Dirichlet Form Theory

Project Coordinator(s): Sascha Trostorff and Marcus Waurick

Participants: Lafsioui Brahim, El-Houcine Ouali, Jan Hausmann and Robert Wilke

In this year's ISem [1] we have encountered Dirichlet Forms in Lecture 7 and have applied the Beurling–Deny Theorems to the concrete form arising from a graph. It is the purpose of this project to provide more examples for Dirichlet forms and to study the associated Markov semigroups.
One main goal of the project is to prove the following representation result due to Beurling and Deny:

Theorem ([2, Theorem 3.2.1])**.** *Let $\mathcal{E}$ be a regular Dirichlet form on $L_2(X, m)$. Then there exist uniquely determined Radon measures σ and k on $X \times X \setminus \Delta$ and X respectively, such that*

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \int_{X \times X \setminus \Delta} (u(x) - u(y))(v(x) - v(y))\,\mathrm{d}\sigma(x, y) + \int_X u(x)v(x)\,\mathrm{d}k(x)$$

for each $u, v \in C_c(X) \cap \mathrm{dom}(\mathcal{E})$. Here, Δ is the diagonal of $X \times X$ and $\mathcal{E}^{(c)}$ is a strongly local symmetric form.

Moreover, if time permits, we study the concept of recurrence and transience for general Dirichlet forms and their associated Markov semigroups and compare them with the results presented in the ISem.

This project is suited for 3 to 4 students.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, 2nd Edition, De Gruyter, Berlin/New York, 2011

Project L: A probabilistic perpective on recurrence and transience

Project Coordinator(s): Hendrik Vogt

Participants: Thomas Pomberger, Tatang Demano, Pablo Lummerzheim and Zouhair El Hadiq

In the ISem lectures, the concepts of recurrence and transience for a graph over X were defined analytically via the spaces $\mathcal{D}(X)$ and $\mathcal{D}_0(X)$. The aim of this project is to understand the probabilistic meaning of these notions: recurrence means that the Markov process associated with the graph returns infinitely often to any given vertex, whereas transience means that eventually the process will never return again.

There are actually two ways to associate a stochastic process with a graph: a time discrete version that corresponds to a random walk on the graph, and a time continuous version that is related to the semigroup $(e^{-tL})_{t\geqslant 0}$ and kernels p_t from the ISem lectures. The basic idea is that $p_t(x, y)$ is related to the probability that a particle starting at the vertex x at time 0 will be at the vertex y at time t. Background material on discrete- and continous-time Markov chains can be found in [2].

Starting point of the project is the book [1] on which the ISem lectures are based. In Sections 0.10 and 2.5 one can find the definition of the process associated to a graph. Then Section 6.6 deals with the probabilistic view on recurrence; in particular, in Theorem 6.35 the equivalence of the analytic and probabilistic descriptions of recurrence is proved. An interesting 'side quest' can be found in Section 7.9, where a probabilistic interpretation of stocastic completeness is given. The precise selection of topics will be decided among the participants of the project.

This project is suited for 3 to 4 students.

Bibliography

[1] M. Keller, D. Lenz, R. Wojciechowski, *Graphs and Discrete Dirichlet Spaces.* Grundlehren der mathematischen Wissenschaften 358, Springer, Cham, 2021.

`https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-GraphTh/Keller/KellerLenzWojciechowski_GraphsAndDiscreteDirichletSpaces_personal.pdf`

[2] J. R. Norris, *Markov Chains.* Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 1997.

Project M: Uniform transience of spherically symmetric graphs

Project Coordinator(s): Radosław K. Wojciechowski

Participants: Till Matthes, Uwe Blechschmidt, Marta Imke and Hakil Haxhiu

We will investigate the notion of uniform transience as discussed in [1] for weakly spherically symmetric graphs. The overall goal is to obtain a characterization of uniform transience for this case to complement the characterizations of transience and stochastic completeness for weakly spherically symmetric graphs as presented in Chapter 9 of [2] and Lecture 13 of [3].

This project is suited for 3 to 4 students.

Bibliography

[1] Matthias Keller, Daniel Lenz, Marcel Schmidt and Radosław K. Wojciechowski, *Note on uniformly transient graphs*, Rev. Mat. Iberoam. **33** (2017), no. 3, 831–860.

[2] Matthias Keller, Daniel Lenz, Marcel Schmidt and Radosław K. Wojciechowski, *Graphs and discrete Dirichlet spaces,* Grundlehren der mathematischen Wissenschaften, vol. 358, Springer, 2021, 682 pp.

[3] ISem 26, Lecture Notes, 2023. https://urldefense.com/v3/__https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf__;!!GekbXoL5ynDpFgM!RmrAN3cGqPzO-GQdRdD3rjDUJ83n7ukvkZnrC_7kAjlrQR2WEJWeMX_kIYcwmi2FESHQk9xoWnCOSOMLvEUIIjME$

Project N: Infinite trees over non-Archimedean ordered fields

Project Coordinator(s): Anna Muranova

Participants: Max Schmidt, Ragon Ebker and Brian Voss

An ordered field is a field together with a total ordering ($\succ$) of its elements that is compatible with the field operations (see [4, p. 449]). An ordered field $\mathbb{K}$ is *non-Archimedean* if there exist an *infinitesimal*, i.e. $\epsilon \in \mathbb{K}$ such that

$$\epsilon \prec \frac{1}{n} = \frac{1}{\underbrace{1 + \cdots + 1}_{n \text{ times}}}$$

for any $n \in \mathbb{N} \subset \mathbb{K}$.

One very important example of non-Arcimedean field is the Levi-Civita field, which is described in e.g. [2].

Instead of usual real weights we consider weights from non-Archimedean field $\mathbb{K}$ on infinite locally finite trees. Since the monotone convergence theorem does not hold in non-Archimedean ordered fields, the property of graph to be either recurrent or transient fails, which is pointed out in [5].

The goal of this project is to investigate behaviour of infinite locally finite trees using the energy of the solutions of the Dirichlet problems on finite approximations (see [3], [5]), starting with spherically symmetric trees, described in [1].

This project is suited for 3 to 4 students.

Bibliography

[1] ISem 26, Lecture Notes, 2023. `https://www.mat.tuhh.de/veranstaltungen/isem26/_media/lecturenotes.pdf`

[2] Khodr Shamseddine, Martin Berz. *Analysis on the Levi-Civita Field: A Brief Overview*, Contemporary Mathematics, 508 pp 215-237 (2010)

[3] A. Grigor'yan. *Introduction to Analysis on Graphs*, volume 71 of *AMS University Lecture Series*. Providence, Rhode Island, 2018.

[4] Serge Lang. Algebra. Springer, 2002.

[5] Anna Muranova. *Effective impedance over ordered fields*. Journal of mathematical physics 62, 033502(2021)