On Abstract Friedrichs Systems (The Skew-Selfadjoint Case).

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Introduction

Key idea of the solution theory of evolutionary equations:

Introduce an exponential weight function $t \mapsto \exp(-\rho\, t)$, $\rho \in \mathbb{R}$, to generate a weighted L^2-space $H_{\rho,0}(\mathbb{R}, H)$ (inner product $\langle \cdot | \cdot \rangle_{\rho,0,0}$, norm: $|\cdot|_{\rho,0,0}$), H a **real** Hilbert space,

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) \,|\, \psi(t) \rangle_H \exp(-2\rho t)\, dt.$$

Time-differentiation ∂_t is considered as a closed operator in $H_{\rho,0}(\mathbb{R}, H)$ induced by

$$\begin{aligned} \mathring{C}_1(\mathbb{R}, H) \subseteq H_{\rho,0}(\mathbb{R}, H) &\to H_{\rho,0}(\mathbb{R}, H), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

Introduction

Time-differentiation ∂_t is a *normal* operator in $H_{\rho,0}(\mathbb{R},H)$

$$\partial_t = \mathfrak{sym}(\partial_t) + \mathfrak{skew}(\partial_t) = \frac{1}{2}(\partial_t + \partial_t^*) + \frac{1}{2}(\partial_t - \partial_t^*)$$

with $\mathfrak{sym}(\partial_t)$ self-adjoint and , $\mathfrak{skew}(\partial_t)$ skew-selfadjoint and commuting resolvents:

$$\mathfrak{sym}(\partial_t) = \rho.$$

For $\rho \in \mathbb{R}\setminus\{0\}$: continuous invertibility of ∂_t. For $\rho \in]0,\infty[$:

$$\mathfrak{sym}(\partial_t) = \rho > 0.$$

Material laws: $M\left(\partial_0^{-1}\right)$, here for simplicity

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1.$$

Introduction

For solving

$$\overline{\partial_0 M_0 + M_1 + A}\, U = F$$

we want $\overline{\partial_0 M_0 + M_1 + A}$ to be strictly m-accretive, i.e.

$$\overline{\partial_0 M_0 + M_1 + A}, (\partial_0 M_0 + M_1 + A)^* \geq c_0 > 0,$$

since then the associated equation is indeed a continuous linear bijection (normally solvable, i.e. closed range, and trivial null space). This is, however, the case **for sufficiently large ρ, if $M_0 \geq 0$ selfadjoint, $\rho M_0 + \mathfrak{sym}(M_1) \geq c_0 > 0$ (strictly positive definite)** for sufficiently large ρ and A m-accretive, i.e.

$$A, A^* \geq 0,$$

in a real Hilbert space H. In particular, we have (**weak=strong**)

$$(\partial_0 M_0 + M_1 + A)^* = \overline{\partial_0^* M_0 + M_1^* + A^*}.$$

Dynamic abstract Friedrichs system (Friedrichs 1954,1958):

$$\overline{\partial_0 M_0 + M_1 + A} = \overline{E_0 + \mathscr{A}}$$

E_0 symmetric strictly positive definite,

$$\mathscr{A} = \overline{(\partial_0 - \rho)\, M_0 + \mathfrak{skew}\,(M_1) + A}$$

m-accretive in $H_{\rho,0}(\mathbb{R}, H)$. W.l.o.g. $E_0 = 1$, since we have the **congruence**

$$\sqrt{E_0}\left(1 + \sqrt{E_0^{-1}}\mathscr{A}\sqrt{E_0^{-1}}\right)\sqrt{E_0} = E_0 + \mathscr{A},$$

and note that

$$\sqrt{E_0^{-1}}\mathscr{A}\sqrt{E_0^{-1}}$$

remains m-accretive. Such **dynamic abstract Friedrichs systems** are of interest in the following. Béla Szökefalvi-Nagy 1953, [1], (Ian Wood 2016), showed that every m-accretive operator S has a skew-selfadjoint extension $\mathscr{S} \supseteq S$, a so-called **dilation**, such that

$$(1+S)^{-1} = \iota_X^*(1+\mathscr{S})^{-1}\iota_X, X \text{ closed subspace of } H, \iota_X : X \to H, x \mapsto x.$$

Indeed, our core topic focuses on the case, where the operator A (or $\mathscr{A}$) is **skew-selfadjoint**.

Five Tools for Establishing Skew-Selfadjointness
Tool 1: Abstract grad-div Systems.

A standard form of skew-selfadjoint operators is given by

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix},$$

where G is a closed densely defined linear operator (**Tool 0**).

For so-called abstract $\operatorname{grad} - \operatorname{div}$ systems we have

$$G = \begin{pmatrix} G_1 \\ \vdots \\ G_n \end{pmatrix} : D(G) \subseteq H_0 \to H_1 \oplus \cdots \oplus H_n.$$

Thus, the range space is a ***direct sum of real Hilbert spaces*** (in the eponymous case of $\operatorname{grad} - \operatorname{div}$ systems $G_k = \mathring{\partial}_k$ or $G_k = \partial_k$ but in general G_k are just linear operators, which need **not** even be necessarily **closable**).

Application: Acoustics with Damping Boundary Condition

$$\text{Acoustics: } A = \begin{pmatrix} 0 & -\left(\mathring{\operatorname{div}}\right)^* \\ \mathring{\operatorname{div}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{grad} \\ \mathring{\operatorname{div}} & 0 \end{pmatrix}$$

$$\left(\partial_0 \begin{pmatrix} \rho_* & 0 \\ 0 & \kappa^{-1} \end{pmatrix} + A\right) \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

where

$$\kappa = \rho_* c_*^2 \text{ bulk modulus, } \rho_* \text{ mass density, } c_* \text{ speed of sound.}$$

We expand this to

$$\left(\partial_0 \begin{pmatrix} \rho_* & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & \beta_* \end{pmatrix} \end{pmatrix} + \widetilde{A}\right) \begin{pmatrix} v \\ \begin{pmatrix} p \\ \tau \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix}.$$

Application: Acoustics with Damping Boundary Condition

Here

$$\widetilde{A} = \begin{pmatrix} (0) & -\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^* \\ \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

with

$$\delta_{\mathrm{div},\partial\Omega} f = n^\top f \in L^2(\partial\Omega),$$

$$\left(\delta_{\mathrm{div},\partial\Omega} f\right)(\varphi) = \int_{\partial\Omega} \varphi\, n^\top f \,\mathrm{Vol}_{\partial\Omega} = \left\langle \varphi | n^\top f \right\rangle_{L^2(\partial\Omega)}.$$

Third equation:

$$\beta_* \tau + \delta_{\mathrm{div},\partial\Omega} v = h.$$

We note

$$\begin{pmatrix} \mathring{\operatorname{div}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}$$

and so

$$-\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^* \subseteq \left(\operatorname{grad} \;\; 0 \right).$$

Application: Acoustics with Damping Boundary Condition

What does it mean if $\begin{pmatrix} p \\ \tau \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^*\right)$?

From

$$\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^* \subseteq \begin{pmatrix} -\operatorname{grad} & 0 \end{pmatrix}$$

we have

$$\begin{aligned} \langle \operatorname{grad} p | w \rangle &= \left\langle \begin{pmatrix} \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} p \\ \tau \end{pmatrix} \middle| w \right\rangle = \left\langle - \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^* \begin{pmatrix} p \\ \tau \end{pmatrix} \middle| w \right\rangle \\ &= - \left\langle \begin{pmatrix} p \\ \tau \end{pmatrix} \middle| \begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix} w \right\rangle = - \langle p | \operatorname{div} w \rangle - \langle \tau | \delta_{\mathrm{div},\partial\Omega} w \rangle \end{aligned}$$

that is

$$\langle \operatorname{grad} p | w \rangle + \langle p | \operatorname{div} w \rangle = - \int_{\partial\Omega} \tau \left(n^\top w \right) \mathrm{Vol}_{\partial\Omega}.$$

Application: Acoustics with Damping Boundary Condition

On the other hand, we have

$$\langle \operatorname{grad} p | w \rangle + \langle p | \operatorname{div} w \rangle = \int_\Omega (\operatorname{grad} p)\, w \ \mathrm{Vol}_\Omega + \int_\Omega p\,(\operatorname{div} w)\ \mathrm{Vol}_\Omega$$
$$= \int_{\partial\Omega} p \left(n^\top w\right) \mathrm{Vol}_{\partial\Omega}$$

and so by comparison $\tau = -\delta_{\mathrm{grad},\partial\Omega} p$ on $\partial\Omega$. Thus, we found

$$\begin{pmatrix} p \\ \tau \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} \operatorname{div} \\ \delta_{\mathrm{div},\partial\Omega} \end{pmatrix}^*\right) \Longrightarrow \begin{pmatrix} p \\ -\delta_{\mathrm{grad},\partial\Omega} p \end{pmatrix} \in \operatorname{dom}(\operatorname{grad}) \oplus L^2(\partial\Omega).$$

We formally read off as the last equation of the system

$$\beta_* \tau + \delta_{\mathrm{div},\partial\Omega} v = -\beta_* \delta_{\mathrm{grad},\partial\Omega} p + \delta_{\mathrm{div},\partial\Omega} v = h \text{ on } \partial\Omega,$$

yielding (a dynamic Robin type boundary condition)

$$-\beta_* \delta_{\mathrm{grad},\partial\Omega} p - \delta_{\mathrm{div},\partial\Omega} \partial_0^{-1} \rho_*^{-1} \operatorname{grad} p = h,$$

which reads in more classical terms

$$n^\top \rho_*^{-1} \operatorname{grad} p + \beta_* \partial_0 p = -\partial_0 h \text{ on } \partial\Omega.$$

Tool 2: The Mother-Descendant Mechanism

Theorem

Let $G : D(C) \subseteq H_0 \to H_1$ be a closed densely defined linear operator, H_k, $k = 0, 1$, real Hilbert spaces. If $B_0 : H_0 \to X_0$ is a continuous linear mapping, X_0 real Hilbert space, such that

$$GB_0^* \text{ densely defined.}$$

Then $\overline{\begin{pmatrix} B_0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}}\begin{pmatrix} B_0^* & 0 \\ 0 & 1 \end{pmatrix}$ *is skew-selfadjoint.*

“Mother” and“descendant”: $A \mapsto \overline{W^* A W}$. Can be repeated!

In general, however, we have

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix}\left(\overline{\begin{pmatrix} B_0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}}\begin{pmatrix} B_0^* & 0 \\ 0 & 1 \end{pmatrix}\right)}\begin{pmatrix} 1 & 0 \\ 0 & B_1^* \end{pmatrix}, \overline{\begin{pmatrix} B_0 & 0 \\ 0 & 1 \end{pmatrix}\left(\overline{\begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix}\begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ 0 & B_1^* \end{pmatrix}\right)}\begin{pmatrix} B_0^* & 0 \\ 0 & 1 \end{pmatrix}$$

as two different extensions of $\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}\begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}\begin{pmatrix} B_0^* & 0 \\ 0 & B_1^* \end{pmatrix}$.

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This amounts (in-)formally to considering $B_1 G B_0^*$ but -done properly – leads to the not so subtle difference between

$$\overline{B_1 \left(G B_0^*\right)} \text{ and } \overline{B_1 G} B_0^*.$$

For example consider $G = \partial$ on $L^2(\mathbb{R})$ and $B_0^* = B_1^* = \iota_{L^2(I)}$, I a finite interval, then $\overline{B_1 \left(G B_0^*\right)}$ is ∂ with Dirichlet boundary condition and $\overline{B_1 G} B_0^*$ is ∂ without boundary condition.

"**Mother of All**": Let us consider $G = \overset{\circ}{\operatorname{grad}}$ acting on tensor fields of all ranks with Dirichlet boundary condition, i.e.

$$A := \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{div} \\ \overset{\circ}{\operatorname{grad}} & 0 \end{pmatrix}$$

on $\bigoplus_{k \in \mathbb{N}} \left(L^2(\Omega, \mathbb{T}_k) \oplus L^2(\Omega, \mathbb{T}_{k+1})\right)$.

The linear initial boundary value problems of classical mathematical physics can be produced from this particular "mother" operator A by choosing suitable projections for constructing "descendants".

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Tool 2: The Mother-Descendant Mechanism

To be slightly less ambitious, consider

$$A = \begin{pmatrix} 0 & \operatorname{div}_1 & 0 & 0 \\ \mathring{\operatorname{grad}}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{div}_2 \\ 0 & 0 & \mathring{\operatorname{grad}}_1 & 0 \end{pmatrix}$$ in Cartesian form on elements of

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- tensor rank (or order or degree)
- symmetric/alternating

3-dimensional		
rank 0,1	———	acoustics
rank 1,2	symmetric	classical elasticity
rank 1,2	alternating	electrodynamics

- descent in space dimension (as in method of descent)
- vanishing trace condition (divergence-free; as e.g. for the incompressible Stokes equation)

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- tensor rank (or order or degree)
- symmetric/alternating

3-dimensional		
rank $0,1$	———	acoustics
rank $1,2$	symmetric	classical elasticity
rank $1,2$	alternating	electrodynamics

- descent in space dimension (as in method of descent)
- vanishing trace condition (divergence-free; as e.g. for the incompressible Stokes equation)

Tool 2: The Mother-Descendant Mechanism

To be slightly less ambitious, consider

$$A = \begin{pmatrix} 0 & \operatorname{div}_1 & 0 & 0 \\ \overset{\circ}{\operatorname{grad}}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{div}_2 \\ 0 & 0 & \overset{\circ}{\operatorname{grad}}_1 & 0 \end{pmatrix}$$

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Application: A Connection Between Wave Phenomena.

In Nowacki's non-symmetric elasticity we are dealing with a skew-selfadjoint spatial operator of the form

$$\begin{pmatrix} 0 & \operatorname{div}_2 \\ \mathring{\operatorname{grad}}_1 & 0 \end{pmatrix} \text{ on } L^2\left(\Omega, \mathbb{R}^3\right) \oplus L^2\left(\Omega, \mathbb{R}^{3\times 3}\right).$$

Here, focusing on the Cartesian case and letting

$$\widetilde{\mathrm{Op}}\, W := \left(\mathrm{Op}\, W^\top\right)^\top,$$

where Op denotes a matrix PDE operator, we have in suggestive matrix notation with $\nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$

$$\operatorname{grad}_1 v := \widetilde{\nabla} v = \left(\nabla\, v^\top\right)^\top \text{ (Jacobian of } v\text{)},$$

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Application: A Connection Between Wave Phenomena.

As possible descendants we obtain with $\iota^*_{\mathfrak{sym}} T = \frac{1}{2}\left(T + T^\top\right)$

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & -\iota^*_{\mathfrak{sym}} \end{pmatrix} \begin{pmatrix} 0 & \operatorname{div}_2 \\ \mathring{\operatorname{grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -\iota_{\mathfrak{sym}} \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{Div} \\ -\mathring{\operatorname{Grad}} & 0 \end{pmatrix},$$

i.e. classical symmetric elasticity, and with $\iota^*_{\mathfrak{skew}} T = \frac{1}{2}\left(T - T^\top\right)$,

$$I_0 : \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix},$$

$$2\,\overline{\begin{pmatrix} 1 & 0 \\ 0 & I_0^*\iota^*_{\mathfrak{skew}} \end{pmatrix} \begin{pmatrix} 0 & \operatorname{div}_2 \\ \mathring{\operatorname{grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathfrak{skew}} I_0 \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix},$$

where $\operatorname{curl} = \nabla\times = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}$, i.e. classical electrodynamics.

Also acoustics can be recovered as a descendant from non-symmetric elasticity via

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{trace} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{div}_2 \\ \mathring{\mathrm{grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{trace}^* \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{grad}_0 \\ \mathring{\mathrm{div}}_1 & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathring{\mathrm{div}}_1 \\ \mathrm{grad}_0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here trace denotes the standard matrix trace and its adjoint evaluates simply to

$$\mathrm{trace}^* p = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} = p\, \mathbb{1}_{3\times 3}.$$

It may be of interest to note that also

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{trace} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{Div}_2 \\ -\mathring{\mathrm{Grad}}_1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{trace}^* \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{grad}_0 \\ \mathring{\mathrm{div}}_1 & 0 \end{pmatrix},$$

which establishes acoustics as a part of continuum mechanics.

Tool 3: Coupling of Different Physical Phenomena

Block-diagonal operator matrix:

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}$$

skew-selfadjoint in $H = \bigoplus_{k=0,\dots,n} H_k$, if diagonal block entries $A_k : D(A_k) \subseteq H_k \to H_k$, $k = 0, \dots, n$, are skew-selfadjoint.

Proper coupling: M contains off-diagonal block entries

$$M_k := \begin{pmatrix} M_{k,00} & \cdots & M_{k,0n} \\ \vdots & \ddots & \vdots \\ M_{k,n0} & \cdots & M_{k,nn} \end{pmatrix}, \; k = 0, 1.$$

Not coupling elasticity and acoustics:

$$\partial_0 \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \nu_1 & 0 & 0 \\ 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A$$

with

$$A = \begin{pmatrix} 0 & \operatorname{grad}_0 & 0 & 0 \\ \mathring{\operatorname{div}}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{Div} \\ 0 & 0 & -\mathring{\operatorname{Grad}} & 0 \end{pmatrix}.$$

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By projecting onto $\ker\left(\partial_3^\#\right) = L^2(\Omega_0) \otimes \mathbb{R} = L^2(\Omega_0)$ assuming $\Omega := \Omega_0 \times \mathbb{T} \subseteq \mathbb{R}^2 \times \mathbb{T}$, $\mathbb{T} = [-1/2, 1/2]$, with identification of the interval ends (**mother-descendant mechanism**), we can reduce the associated evolutionary problem by one spatial dimension to a 2-dimensional situation.

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The resulting evolutionary equation looks the same, but now it has to be interpreted (by dropping zero rows and columns in A and adapting the material law) in $L^2(\Omega_0,\mathbb{R}^2)\oplus L^2(\Omega_0,\mathbb{R})\oplus L^2(\Omega_0,\mathbb{R}^2)\oplus L^2_2(\Omega_0,\mathfrak{sym}[\mathbb{R}^{2\times 2}])$ with $\Omega_0\subseteq\mathbb{R}^2$.

This is the Reissner-Mindlin plate system commonly used in engineering models.

Remark (Kirchhoff-Love plate): Letting $\kappa=0$ and $\nu_2=0$ (in consequence destroying well-posedness for associated initial boundary value problems) and eliminating the first and third unknown and equation we get (in-)formally

$$\partial_0\begin{pmatrix}\nu_1 & 0\\ 0 & C^{-1}\end{pmatrix}+\begin{pmatrix}d & 0\\ 0 & 0\end{pmatrix}+A$$

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Finally eliminating the stress yields for isotropic homogeneous media

$$\partial_0^2\nu_1\eta+d\partial_0\eta+(2\kappa+\lambda)\Delta^2\eta=\partial_0 f.$$

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Tool 4: Weak = Strong

Let A_k, $k = 1, 2$, be closed, densely defined linear operators from H to K, $\mathrm{dom}\,(A_1 + A_2) = \mathrm{dom}\,(A_1) \cap \mathrm{dom}\,(A_2)$ dense in H. Then we would want

$$(A_1 + A_2)^* = \overline{A_1^* + A_2^*},$$

which is the functional analytical formulation for "weak=strong" (here applied to summation).

Transmutator as technical tool:

$$[L, C, R] := LC - CR$$

assumed to be defined on $\mathrm{dom}\,(C)$ (usually L, R continous).

The commutator

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is a special case.

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- $L_\varepsilon^* \underset{\varepsilon \to 0+}{\overset{s}{\to}} 1$, $R_\varepsilon^* \underset{\varepsilon \to 0+}{\overset{s}{\to}} 1$ *and* $[L_\varepsilon, A_1 + A_2, R_\varepsilon]^* \underset{\varepsilon \to 0+}{\overset{s}{\to}} 0$.

Then (weak = strong)

$$(A_1 + A_2)^* = \overline{A_1^* + A_2^*}.$$

Corollary

Let A_1, A_2 be skew-selfadjoint, then under the assumptions of the previous theorem we have

$$\overline{A_1 + A_2} \text{ skew-selfadjoint.}$$

Tool 4: Weak = Strong

Let A_k, $k = 1, 2$, be closed densely defined operators from H to K, $\operatorname{dom}(A_1 + A_2) = \operatorname{dom}(A_1) \cap \operatorname{dom}(A_2)$ dense in H.

Theorem

Let $(L_\varepsilon)_{\varepsilon \in]0,1[}$, $(R_\varepsilon)_{\varepsilon \in]0,1[}$ be bounded families of continuous linear mappings in K and H, respectively, and $[L_\varepsilon, A_1 + A_2, R_\varepsilon]$ defined on $\operatorname{dom}(A_1) \cap \operatorname{dom}(A_2)$ such that $\overline{[L_\varepsilon, A_1 + A_2, R_\varepsilon]} \in \mathscr{L}(H, K)$. Moreover,

- $L_\varepsilon^* \left[\operatorname{dom}\left((A_1 + A_2)^*\right)\right] \subseteq \operatorname{dom}(A_1^* + A_2^*)$,
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Application: (acoustics in moving media, joint work with Sascha Trostorff) assuming that $\mathfrak{sym}\left(\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)$ is continuous

$$\partial_0\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\begin{pmatrix}0 & \operatorname{grad}\\ \mathring{\operatorname{div}} & 0\end{pmatrix}=$$
$$=\partial_0\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\mathfrak{sym}\left(\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)+A_1+A_2$$

with

$A_1=\overline{\mathfrak{skew}\left(\partial_3\alpha\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)}$, (skew-selfadjoint for suitable α,Ω)

$A_2=\begin{pmatrix}0 & \operatorname{grad}\\ \mathring{\operatorname{div}} & 0\end{pmatrix}$,

$R_\varepsilon=L_\varepsilon=(1+\varepsilon\partial_3)^{-1}$.

With plenty of differential geometry this transfers to open subsets of Riemannian manifold.

The End

Thank You for Your Attention!

Béla Sz.-Nagy.
Sur les contractions de l'espace de Hilbert.
Acta Sci. Math., 15:87–92, 1953.