Multiplication Operators on Spaces of Vector-Valued Functions

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This talk will examine *operator-valued multiplication operators*.

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They arise from the study of *non-autonomous abstract Cauchy problems*, i.e. problems of the type:

Let E be a Banach space, $x_0 \in E$, $A(t)$ linear operators on E for all $t \in \mathbb{R}_+$. Find $f : \mathbb{R}_+ \to E$ such that

$$\partial_t f(t) = A(t) f(t)$$

$$f(0) = x_0$$

When studying non-autonomous Cauchy problems, it helps studying the associated multiplier operators, for example on the Bochner-Lebesgue spaces:

$$\mathcal{A} : L^p(\mathbb{R}_+, E) \to L^p(\mathbb{R}_+, E)$$
$$f \mapsto (t \mapsto A(t)f(t))$$

Overview:

1. Recapitulation of properties of scalar-valued multiplication operators
2. Exploration of matrix-valued multiplication operators
3. Generalisation to operator-valued multiplication operators

In lecture 2, we defined and explored scalar multiplication operators.

$$L^2(\mathbb{R}, \mathbb{K}) \to L^2(\mathbb{R}, \mathbb{K})$$
$$f \mapsto (x \mapsto V(x)f(x))$$

for a function $V : \mathbb{R} \to \mathbb{K}$. Here, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

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$$\mathrm{dom}(\mathcal{M}_V) = \{f \in \mathcal{E} \,|\, V \cdot f \in \mathcal{E}\}$$

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- A bounded operator is a multiplication operator if and only if it commutes with all bounded multiplication operators.
- $\mathcal{M}_V$ generates a C_0-semigroup if and only if $\mathrm{Re}(V)$ is essentially bounded above.

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Further, we will sometimes need to assume σ*-finiteness*, i.e. $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ for $\Omega_n \in \Sigma$ with $\mu(\Omega_n) < \infty$. This implies semi-finiteness.

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In lecture 6, we define an operator

$$\begin{aligned} L^2(\mathbb{R}, \mathrm{dom}(A)) &\to L^2(\mathbb{R}, H) \\ f &\mapsto (t \mapsto Af(t)) \end{aligned}$$

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What happens when we let A depend on t?

Matrix multipliers

Setup

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  - $\|u\|_{\mathcal{E}} := \left(\int_\Omega \|u(x)\|_{\mathbb{K}^n}^p d\mu(x)\right)^{\frac{1}{p}}$ for $u \in \mathcal{E}$, where $\|\cdot\|_{\mathbb{K}^n}$ defines some norm on $\mathbb{K}^n$.

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where $dom(\mathcal{M}_{q_{ji}}) := \{ f \in L^p(\Omega, \mu) \mid q_{ji} f \in L^p(X, \mu) \}$ *for* $i \in \{1, \dots, n\}$ *is a core for* $\mathcal{M}_Q$*.*

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- One can show $\mathcal{E}_Q \subseteq \mathcal{D} \Rightarrow \mathcal{D}$ is a core for $\mathcal{M}_Q$.

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„$\Rightarrow$“: One indeed makes use of the semi-finiteness.

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Before doing so, we need some preliminaries.

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In the latter case, we have $\mathcal{M}_R = (\mathcal{M}_Q)^{-1}$.

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For $\lambda \in \mathbb{K}, \varepsilon > 0$ *define* $N_{\lambda,\varepsilon} := \{x \in \Omega \mid dist(\lambda, \sigma(Q(x))) < \varepsilon\}$.

Then:

- *Then* $N_{\lambda,\varepsilon}$ *is measurable.*
- *The essential spectrum range is given by*

$$ess\text{-}\sigma(Q(\Omega)) = \{\lambda \in \mathbb{K} \mid \mu(N_{\lambda,\varepsilon}) > 0 \ \forall \varepsilon > 0\}.$$

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Spectral theory

Theorem 2.7 (Spectrum of matrix multiplier operators)

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Notation.

We denote: $\sigma(q(\Omega)) := \bigcup_{x \in \Omega} \sigma(q(x))$ for $q \in [Q]$.

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- Set $M := \|R(\lambda, \mathcal{M}_Q)\| = \|\mathcal{M}_R\| = \|R\|_\infty < \infty$.

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- Set $M := \|R(\lambda, \mathcal{M}_Q)\| = \|\mathcal{M}_R\| = \|R\|_\infty < \infty$.
- Since $\lambda \in \rho(Q(x))$ μ-a.e., we obtain

$$\operatorname{dist}(\lambda, \sigma(Q(x))) \geq \frac{1}{\|(\lambda - Q(x))^{-1}\|} \geq \frac{1}{M} \text{ for } \mu\text{-a.e. } x \in \Omega.$$

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- Hence $\lambda \notin \{\kappa \in \mathbb{K} \mid \mu(N_{\kappa,\varepsilon}) > 0 \ \forall \varepsilon > 0\} \overset{(P2)}{=} \text{ess-}\sigma(Q(\Omega))$.

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- We obtain $|\det(Q(x))| \geq \varepsilon^n > 0 \;\forall x \in \Omega \setminus N_{0,\varepsilon}$
- One can show
$$\|Q(x)^{-1}\| \leq \frac{K}{|\det(Q(x))|}\|Q(x)\|^{n-1} \overset{\mathcal{M}_Q \text{ bounded}}{\leq} \frac{K}{\varepsilon^n}\|\mathcal{M}_Q\|^{n-1},$$
μ-almost everywhere.

Proof. $\sigma(\mathcal{M}_Q) \subseteq \text{ess-}\sigma(Q(\Omega))$

- Define $R : \Omega \to M_n(\mathbb{K})$ by

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- Using (P1), $\mathcal{M}_Q$ is invertible, i.e. $0 \in \rho(\mathcal{M}_Q)$.

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hence $\sigma(\mathcal{M}_Q) \subseteq \overline{\sigma(q_0(\Omega))} = \text{ess-}\sigma(Q(\Omega))$. □

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$\Rightarrow \sigma(\mathcal{M}_Q) = \mathbb{C}$, *i.e.* $\rho(\mathcal{M}_Q) = \emptyset$.

Operator Valued Multiplication Operators

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$$L^\infty(\Omega, \mathcal{L}(E)) := \{ M : \Omega \to \mathcal{L}(E), \forall x \in E : \omega \mapsto M(\omega)x \text{ is measurable}$$

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and will call $M \in L^\infty(\Omega, \mathcal{L}(E))$ a **multiplier**.

Definition 3.1

*We call an operator $\mathcal{M} \in \mathcal{L}(L^p(\Omega, E))$ a **bounded operator valued multiplication operator** if $\mathcal{M} = \mathcal{M}_M$ for some $M \in L^\infty(\Omega, \mathcal{L}(E))$.*

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For every $M \in L^\infty(\Omega, \mathcal{L}(E))$ and $f \in L^p(\Omega, E)$ we have $\mathcal{M}_M(f) = Mf \in L^p(\Omega, E)$.

Furthermore we have $\|\mathcal{M}_M\|_{L^p \to L^p} = \|M\|_\infty$.

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Use that there exists a null set $\mathcal{N}$ such that

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For $\|M\|_\infty \leq \|\mathcal{M}_M\|_{L^p\to L^p}$ use the separability of E.

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Where $\bar{\Omega} \subseteq \Omega$ is called a purely atomic subset, if

$$\bar{\Omega} = \bigcup \{A \subseteq \bar{\Omega} : A \text{ atom}\} \text{ a.e.}$$

Unbounded Operator Valued Multiplication Operators

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For an unbounded operator $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subseteq L^p(\Omega, E) \to L^p(\Omega, E)$ we will write $(\mathcal{A}, \mathrm{dom}(\mathcal{A}))$.

Definition 3.4

We call an unbounded operator $(\mathcal{A}, dom(\mathcal{A}))$ *an* ***unbounded operator valued multiplication operator****, if there exists a family* $(A(\omega), dom(A(\omega)))_{\omega\in\Omega}$ *of linear operators on* E*,*

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*In this case the operators $(A(\omega), dom(A(\omega)))_{\omega\in\Omega}$ are called the **fiber operators** of $\mathcal{A}$.*

Lemma 3.5

Let $(\mathcal{A}, dom(\mathcal{A}))$ *be an unbounded operator valued multiplication operator,* $f \in dom(\mathcal{A})$ *and* $\varphi \in L^\infty(\Omega)$.

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Let $(\mathcal{A}, dom(\mathcal{A}))$ be an unbounded operator valued multiplication operator, $f \in dom(\mathcal{A})$ and $\varphi \in L^\infty(\Omega)$. Then

$$\varphi f \in dom(\mathcal{A}) \text{ and } \mathcal{A}(\varphi f) = \varphi \mathcal{A}(f).$$

Proof.

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$$\omega \mapsto A(\omega)(\varphi(\omega)f(\omega)) = \varphi(\omega)A(\omega)(f(\omega)) \in L^p(\Omega, E).$$
- This shows $\varphi f \in \mathrm{dom}(\mathcal{A})$ and $\mathcal{A}(\varphi f) = \varphi\mathcal{A}(f)$.

□

Lemma 3.6

Let $\mathcal{A}$ *be a multiplication operator (bounded or unbounded) and assume* $\lambda \in \rho(\mathcal{A})$.

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Using Lemma 3.5 one can show $R(\lambda, \mathcal{A})(\varphi f) = \varphi(R(\lambda, \mathcal{A})f)$ for all $f \in L^p(\Omega, E)$ and $\varphi \in L^\infty(\Omega)$.

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Since $\mathrm{dom}(R(\lambda, \mathcal{A})) = L^p(\Omega, E)$ and $R(\lambda, \mathcal{A})$ is bounded,

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Using Lemma 3.5 one can show $R(\lambda, \mathcal{A})(\varphi f) = \varphi(R(\lambda, \mathcal{A})f)$ for all $f \in L^p(\Omega, E)$ and $\varphi \in L^\infty(\Omega)$.

Since $\operatorname{dom}(R(\lambda, \mathcal{A})) = L^p(\Omega, E)$ and $R(\lambda, \mathcal{A})$ is bounded, it follows by Theorem 3.3 that $R(\lambda, \mathcal{A})$ is a bounded multiplication operator. □

Theorem 3.7

Let $\mathcal{A}$ be a densely defined closed operator on $L^p(\Omega, E)$. Assume that there exists an unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subseteq \rho(\mathcal{A})$ such that $\lim_{k\to\infty} \lambda_k R(\lambda_k, \mathcal{A}) = 1$ on $L^p(\Omega, E)$ and that $R(\lambda_k, \mathcal{A})$ is a bounded multiplication operator for every $k \in \mathbb{N}$. Then there exists a family $(A(\omega))_{\omega\in\Omega}$ of densely defined closed operators on E, such that $(\mathcal{A}, dom(\mathcal{A}))$ is a multiplication operator with fiber operators $(A(\omega), dom(A(\omega)))_{\omega\in\Omega}$. Further there exists a null set $\mathcal{N}$ such that for every $\omega \in \Omega\backslash\mathcal{N}$ and for every $k \in \mathbb{N}$ one has $\lambda_k \in \rho(A(\omega))$.

Multiplication Semigroups

I hail a semigroup when I see one and I seem to see them everywhere

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Preliminaries

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Important properties

Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup on $L^p(\Omega; E)$ with generator $(\mathcal{A}, dom(\mathcal{A}))$, such that there exist some constants $W \in \mathbb{R}$ and $M \geq 1$ with $\|T(t)\| \leq Me^{Wt}$ for all $t \geq 0$.

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$$\mathcal{T}(t)f = \lim_{n\to\infty} [n/tR(n/t, \mathcal{A})]^n f, \quad f \in L^p(\Omega; E),$$

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where the convergence is uniform for t in compact intervals of $\mathbb{R}_+$.

Integral representation:

Integral representation: For every $\lambda \in \mathbb{C}$ with $Re(\lambda) > W$, we have $\lambda \in \rho(\mathcal{A})$ and

$$R(\lambda, \mathcal{A})f = \int_0^\infty e^{-\lambda s}\mathcal{T}(s)f ds, \qquad \text{(Integral representation)}$$

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Resolvent limit: We have the following limit

$$\lim_{\lambda \to +\infty} \lambda R(\lambda, \mathcal{A})f = f, \qquad \text{for all } f \in L^p(\Omega; E).$$

(**Resolvent limit**)

Semigroup generated by multiplication operator

Main theorem

Main result

Main theorem: Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup with generator $(\mathcal{A}, dom(\mathcal{A}))$ on $L^p(\Omega; E)$, with $\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\mathrm{Wt}}$.

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In this case we have for all $t \geq 0$ and for $\mu-$a.e., $\omega \in \Omega$, $(T_\omega(t))_{t\geq 0}$ is a C_0-semigroup with generator $(A(\omega), dom(A(\omega)))$ on E, with $\|T_\omega(t)\| \leq M\mathrm{e}^{\mathrm{Wt}}$.

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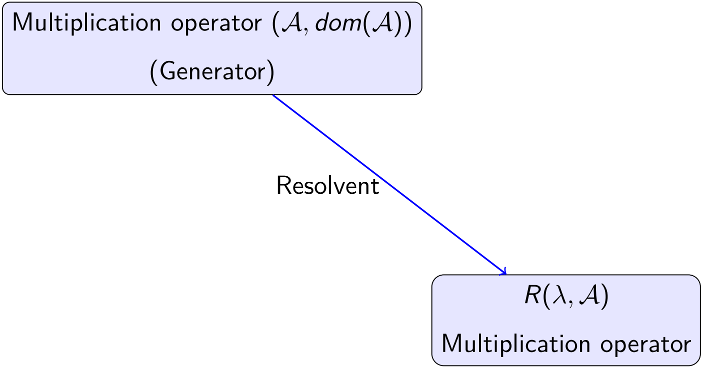

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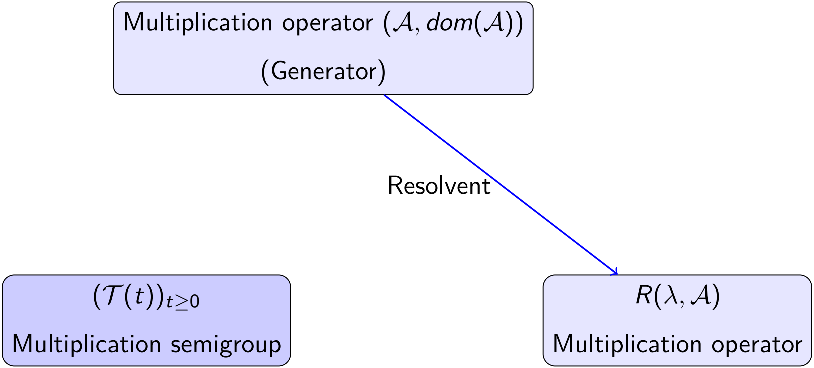

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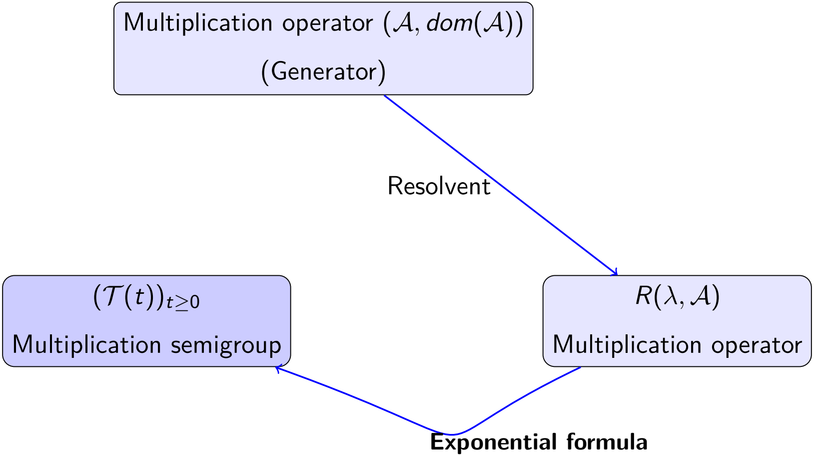

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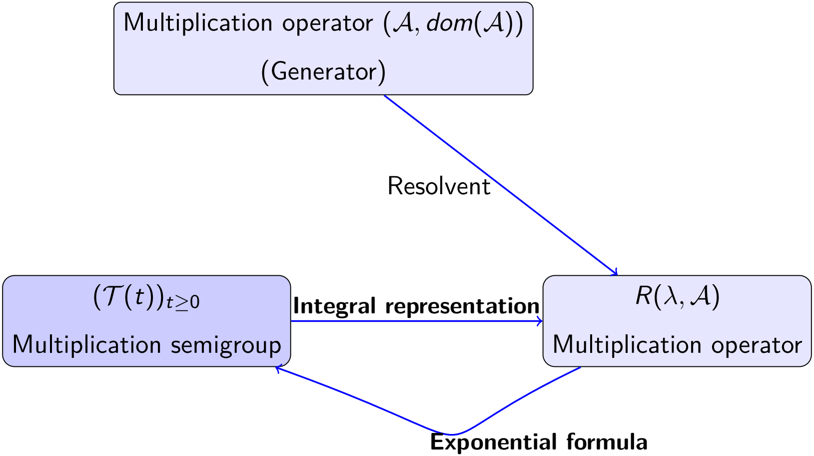

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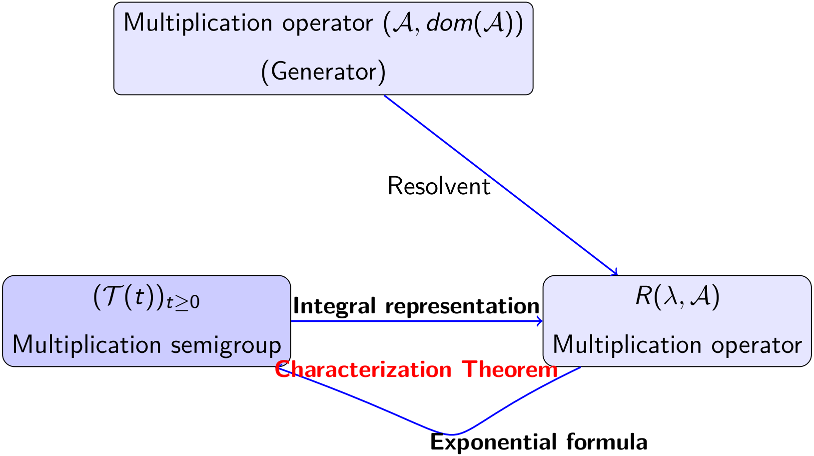

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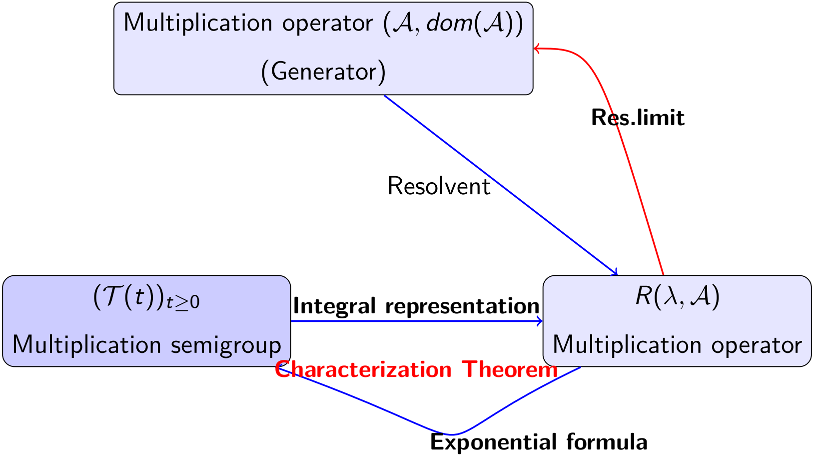

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$$(\mathcal{T}(t)f_{n,m})(\omega) = T_\omega(t)(\chi_{\Omega_n} \cdot x_m)(\omega) = \chi_{\Omega_n}(\omega)T_\omega(t)x_m$$

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Hence for all $\omega \in \Omega \setminus \mathcal{N}$ there exists $n \in \mathbb{N}$ such that $\omega \in \Omega_n$ ($\mathcal{N} = \bigcup_{n,\, m\in\mathbb{N}} \mathcal{N}_{n,m}$)

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$$T_\omega(t)x_m = (\mathcal{T}(t)f_{n,m})(\omega)$$

$\hookrightarrow$

$$T_\omega(0)x_m = x_m$$

$$T_\omega(t+s)x_m = T_\omega(t)T_\omega(s)(\omega)x_m$$

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for all $m \in \mathbb{N}$.

This implies that that $(T_\omega(t))_{t\geq 0}$ is a C_0 -semigroup for all $\omega \in \Omega \setminus \mathcal{N}$. (With generators $(B(\omega), dom(B(\omega)))_{\omega \in \Omega \setminus \mathcal{N}}$)

$$B(\omega) = A(\omega) \text{ ?!}$$

Proof of the main result

In a similar way there exists a $\mu-$nullset $\tilde{\mathcal{N}} \supseteq \mathcal{N}$ such that for all $\omega \in \Omega \setminus \tilde{\mathcal{N}}$ there exists $n \in \mathbb{N}$ such that $\omega \in \Omega_n$

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$$\begin{aligned} R(\lambda, A(\omega))x_m &= (R(\lambda, \mathcal{A})f_{n,m})(\omega) \\ &= \int_0^\infty \mathrm{e}^{-\lambda t} (\mathcal{T}(t)(f_{n,m})(\omega) dt \\ &= \int_0^\infty \mathrm{e}^{-\lambda t} T_\omega(t) x_m dt, \\ &= R(\lambda, B(\omega))x_m \end{aligned}$$

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for all $m \in \mathbb{N}$.

Therefore $A(\omega)$ is the generator of $(T_\omega(t))_{t\geq 0}$.

Finally we have for almost all $\omega \in \Omega$,

$$\|T_\omega(t)\| \leq \left\|T_{(\cdot)}(t)\right\|_\infty = \|\mathcal{T}(t)\| \leq M\mathrm{e}^{\mathrm{W}t}.$$

Definition:

C_0-semigroups satisfying the assertion a) from the above Theorem are called **multiplication semigroups**.

The next theorem gives a sufficient condition on the fiber operators so that $\mathcal{A}$ is a generator on $L^p(\Omega; E)$.

Theorem:

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Let $(\mathcal{A}, dom(\mathcal{A}))$ be a multiplication operator, assume that

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Moreover $\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\mathrm{w}t}$ for all $t \geq 0$.

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1. $\mathcal{T}(0) = Id$
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3. Strong continuity follows from the strong continuity of $(T_\cdot(t))_{t\geq 0}$ and the estimation (2)

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$$dom(\tilde{\mathcal{A}}) = \left\{ f \in L^p(\Omega, E) : \lim_{t \to 0} \frac{\mathcal{T}(t)f - f}{t} \text{ exists in } L^p(\Omega, E) \right\}$$

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$$\tilde{\mathcal{A}}f = \lim_{t \to 0} \frac{\mathcal{T}(t)f - f}{t}$$

Hence by the previous theorem $\tilde{\mathcal{A}}$ is a multiplication operator with fiber operators $(A(\omega), dom(A(\omega)))_{\omega \in \Omega \setminus \mathcal{N}_0}$. Therefore

$$\tilde{\mathcal{A}} = \mathcal{A}.$$

Illustrating Example

Let $\Omega = \mathbb{R}$, $V \subset \mathbb{R}^n$ a bounded domain with smooth boundary and consider the following operators

$$A(\omega) = \Delta + a(\omega), \quad dom(A(\omega)) = H_0^1(V) \cap H^2(V)$$

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$\hookrightarrow$ $T_\omega(t) = e^{a(\omega)t} e^{t\Delta}$.

Then the multiplication operator $\mathcal{A}$ associated to the fiber operators $(A(\omega))_{\omega \in \Omega}$ is a generator.

Matrix multipliers :

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(Engel's book and references therein.)

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The following two assertions are equivalent:

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1. *The matrix multiplier* $\mathcal{M}_Q$ *generates a strongly continuous semigroup on* $\mathrm{L}^p(\Omega;\mathbb{K})^n$,
2. *There exist* $K \geq 1$ *and* $W \in \mathbb{R}$ *such that* $\{z \in \mathbb{C} : \mathrm{Re}(z) > W\} \subseteq \rho(\mathcal{M}_Q)$ *and the resolvent of* $\mathcal{M}_Q$ *satisfies*

$$\|R(\lambda, \mathcal{M}_Q)\| \leq \frac{K}{\mathrm{Re}(\lambda) - W},$$

for all $\mathrm{Re}(z) > W$.

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Thank You!