

Theorem of Widder-Arendt

Project E

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- 1 Theorem of Widder-Arendt
- 2 C_0 -Semigroups, Resolvents, Generators
- 3 Integrated semigroups
- 4 Generalisation of Hille-Yosida theorem

Theorem of Widder-Arendt

Theorem (Widder 1934)

Let $r : (0, \infty) \rightarrow \mathbb{R}$ be a function. The following assertions are equivalent:

- ① $r \in C^\infty(0, \infty)$ and

$$\sup_{\lambda > 0, n \in \mathbb{N}_0} \{|\lambda^{n+1} r^{(n)}(\lambda) / n!|\} < \infty.$$

- ② There is $f \in L^\infty(0, \infty)$ s.t.

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt \quad (\lambda > 0)$$

(i.e. r is the Laplace transform of f).

Theorem of Widder-Arendt

Question

Does this statement remain true if we consider functions $r : (0, \infty) \rightarrow E$, where E is a Banach space?

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Definition

Let E be a Banach space. We say, Widder's theorem holds in E , if every $r \in C^\infty((0, \infty), E)$ satisfying $\sup_{\lambda > 0, n \in \mathbb{N}_0} \{|\lambda^{n+1} r^{(n)}(\lambda)/n!|\} < \infty$ can be represented in the form

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt \quad (\lambda > 0)$$

for some $f \in L^\infty([0, \infty), E)$.

Theorem of Widder-Arendt

Theorem (Arendt 1987)

A Banach space E has the Radon-Nikodým property if and only if Widder's theorem holds.

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A Banach space E has the Radon-Nikodým property if and only if Widder's theorem holds.

Definition

A Banach space E has the Radon-Nikodým property if every Lipschitz continuous map $f : [0, 1] \rightarrow E$ is differentiable a.e.

Theorem of Widder-Arendt

Examples for spaces possessing the Radon-Nikodým property:

- \mathbb{R}
- every reflexive space (in particular: every Hilbert space)
- every separable dual space

Theorem of Widder-Arendt

Examples for spaces possessing the Radon-Nikodým property:

- \mathbb{R}
- every reflexive space (in particular: every Hilbert space)
- every separable dual space

Examples for spaces not possessing the Radon-Nikodým property:

$C([0, 1]), L^1([0, 1]), L^\infty([0, 1]), c_0, \dots$

Theorem of Widder-Arendt

Theorem ("integrated version of Widder"; Arendt 1987)

Let E be a Banach space, $r : (0, \infty) \rightarrow E$, $M \geq 0$. The following assertions are equivalent:

- ① $r \in C^\infty((0, \infty), X)$ and

$$\sup_{\lambda > 0, n \in \mathbb{N}_0} \{ \|\lambda^{n+1} r^{(n)}(\lambda) / n!\| \} \leq M.$$

- ② There is $F : [0, \infty) \rightarrow E$ with $F(0) = 0$ and

$$\|F(t+h) - F(t)\| \leq Mh \quad (t, h \geq 0) \text{ s.t.}$$

$$r(\lambda) = \int_0^\infty \lambda e^{-\lambda t} F(t) dt \quad (\lambda > 0).$$

Proof of Widder-Arendt

(1) \Rightarrow (2):

Let $x' \in E'$. The function $\langle r(\cdot), x' \rangle : (0, \infty) \rightarrow \mathbb{R}$ satisfies

$$\sup_{\lambda > 0, n \in \mathbb{N}_0} \{ \|\lambda^{n+1} \langle r^{(n)}(\lambda), x' \rangle / n!\| \} \leq M \|x'\|.$$

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By the classical theorem of Widder

$$\langle r(\lambda), x' \rangle = \int_0^\infty e^{-\lambda t} f(t, x') dt \quad (\lambda > 0)$$

for some $f(\cdot, x') \in L^\infty(0, \infty)$ with $\|f(\cdot, x')\|_\infty \leq M \|x'\|$.

Proof of Widder-Arendt

Define $F(\cdot, x') : [0, \infty) \rightarrow \mathbb{R}$ by $F(t, x') = \int_0^t f(r, x') dr$.

Proof of Widder-Arendt

Define $F(\cdot, x') : [0, \infty) \rightarrow \mathbb{R}$ by $F(t, x') = \int_0^t f(r, x') dr$. Integration by parts yields

$$\begin{aligned}\langle r(\lambda), x' \rangle &= [e^{-\lambda t} F(t, x')]_0^\infty + \int_0^\infty \lambda e^{-\lambda t} F(t, x') dt \\ &= \int_0^\infty \lambda e^{-\lambda t} F(t, x') dt \quad (\lambda > 0, x' \in E').\end{aligned}$$

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Properties:

Proof of Widder-Arendt

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Properties:

$$(i) \quad |F(t+h, x') - F(t, x')| \leq M \|x'\| h \quad (t, h \geq 0, x' \in X'),$$

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- (ii) $F(\cdot, x')$ is continuous for every $x' \in E'$, $F(0) = 0$,

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- (ii) $F(\cdot, x')$ is continuous for every $x' \in E'$, $F(0) = 0$,
- (iii) $F(t, x')$ is linear in x' for every $t \geq 0$ by (ii) and the uniqueness theorem for Laplace transforms.

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- (ii) $F(\cdot, x')$ is continuous for every $x' \in E'$, $F(0) = 0$,
- (iii) $F(t, x')$ is linear in x' for every $t \geq 0$ by (ii) and the uniqueness theorem for Laplace transforms.

Then, for every $t \geq 0$ there is $F(t) \in E''$ with $F(t, x') = \langle F(t), x' \rangle$ for every $x' \in E'$.

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To show: $F(t) \in E$ for every $t \geq 0$.

By $q : E'' \rightarrow E''/E$ we denote the quotient mapping. We have

$$0 = q\left(\frac{r(\lambda)}{\lambda}\right) = \int_0^{\infty} e^{-\lambda t} q(F(t)) dt \quad (\lambda > 0).$$

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By $q : E'' \rightarrow E''/E$ we denote the quotient mapping. We have

$$0 = q\left(\frac{r(\lambda)}{\lambda}\right) = \int_0^{\infty} e^{-\lambda t} q(F(t)) dt \quad (\lambda > 0).$$

By the uniqueness theorem for Laplace transforms $q(F(t)) = 0$, i.e.

$F(t) \in E$ for every $t \geq 0$.

Repetition: Theorem of Widder-Arendt

Theorem ("integrated version of Widder"; Arendt 1987)

Let E be a Banach space, $r : (0, \infty) \rightarrow E$, $M \geq 0$. The following assertions are equivalent:

- ① $r \in C^\infty((0, \infty), X)$ and

$$\sup_{\lambda > 0, n \in \mathbb{N}_0} \{ \|\lambda^{n+1} r^{(n)}(\lambda) / n!\| \} \leq M.$$

- ② There is $F : [0, \infty) \rightarrow E$ with $F(0) = 0$ and

$$\|F(t+h) - F(t)\| \leq Mh \quad (t, h \geq 0) \text{ s.t.}$$

$$r(\lambda) = \int_0^\infty \lambda e^{-\lambda t} F(t) dt \quad (\lambda > 0).$$

Proof of Widder-Arendt

(2) \Rightarrow (1): By definition $r \in C^\infty((0, \infty), E)$. Let $\lambda > 0, n \in \mathbb{N}_0$.

$$\begin{aligned}\|r^{(n)}(\lambda)\| &= \left\| \frac{d^n}{d\lambda^n} \int_0^\infty \lambda e^{-\lambda t} F(t) dt \right\| \\ &\leq \int_0^\infty \|F(t)\| \sum_{i=0}^n \binom{n}{i} \frac{d^i \lambda}{d\lambda^i} |(-t)^{n-i} e^{-\lambda t}| dt \\ &\leq M \int_0^\infty t[\lambda t^n e^{-\lambda t} - n t^{n-1} e^{-\lambda t}] dt \\ &= M \int_0^\infty t^n e^{-\lambda t} dt \\ &\vdots \\ &= M \frac{n!}{\lambda^{n+1}}.\end{aligned}$$



C_0 -Semigroups, Resolvents, Generators

Definition

Let E be a Banach space. A function $T : [0, \infty) \rightarrow \mathcal{L}(E)$ is called a C_0 -semigroup if

- 1 $T(0) = I$ (I is the identity operator on E),
- 2 $T(s + t) = T(s)T(t)$ for every $s, t \in [0, \infty)$,
- 3 $\lim_{t \downarrow 0} T(t)x = x$ for every $x \in E$.

C_0 -Semigroups, Resolvents, Generators

Definition

Let E be a Banach space and $T : [0, \infty) \rightarrow \mathcal{L}(E)$ a C_0 -semigroup.

$A : E \supset D(A) \rightarrow E$ is called the generator of T if $\exists \omega \in \mathbb{R}$ s.t.

$(\omega, \infty) \subset \rho(A)$ and $R(\cdot, A) : (\omega, \infty) \rightarrow \mathcal{L}(E)$ defined by

$R(\lambda, A) = (\lambda I - A)^{-1}$ satisfies

$$R(\lambda, A) = \int_0^{\infty} e^{-\lambda t} T(t) dt \quad (\lambda > \omega),$$

i.e. $R(\cdot, A)$ is the Laplace transform of the semigroup T .

C_0 -Semigroups, Resolvents, Generators

Consider the abstract Cauchy problem

$$(\text{ACP}_f) \begin{cases} u'(t) = Au(t) + f(t), & t \geq 0 \\ u(0) = x_0 \end{cases}$$

for $A : E \supset D(A) \rightarrow E$ closed, $f \in L^1((0, \infty), E)$, $x_0 \in E$.

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for $A : E \supset D(A) \rightarrow E$ closed, $f \in L^1((0, \infty), E)$, $x_0 \in E$.

If A generates a C_0 -semigroup T then for every $x_0 \in D(A)$ there exists a unique classical solution of (ACP_f) .

C_0 -Semigroups, Resolvents, Generators

Question

When is a given operator the generator of a C_0 -semigroup?

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Theorem (Hille-Yosida)

Let E be a Banach space, $A : E \supset D(A) \rightarrow E$ be a closed linear operator, $M \geq 0$, $\omega \in \mathbb{R}$. The following assertions are equivalent:

- 1 A generates a C_0 -semigroup $T : [0, \infty) \rightarrow \mathcal{L}(E)$.
- 2 $D(A)$ is dense in E , $(\omega, \infty) \subset \rho(A)$ and

$$\|(\lambda - \omega)^{n+1} R(\lambda, A)^{(n)} / n!\| \leq M \quad (\lambda > \omega; n \in \mathbb{N}_0).$$

Purpose

Let $A: D(A) \subset E \rightarrow E$ be an operator, $x \in D(A)$ and $f: \mathbb{R}^+ \rightarrow E$ in an appropriate space.

$$\begin{cases} u'(t) = Au(t) + f(t), & \text{for } t \in \mathbb{R}^+ \\ u(0) = x. \end{cases}$$

A generates a C_0 -semigroup \Rightarrow unique classical solution

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$R(\lambda, A)$ is a Laplace transform \Rightarrow unique classical solution

Question

What about if $\frac{R(\lambda, A)}{\lambda^n}$ is a Laplace transform, for some $n \in \mathbb{N}$?

Let $(T(t))_{t \geq 0}$ a C_0 -semigroup and A its generator. Let $n \in \mathbb{N}$. Then there exist $M, \omega \geq 0$ such that

$$\|T(t)\| \leq M e^{\omega t}, \quad \forall t \geq 0.$$

Defining

$$S^n(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} T(s) ds, \quad \text{for } t \geq 0,$$

we obtain that

$$\frac{R(\lambda, A)}{\lambda^n} = \frac{1}{\lambda^n} \int_0^\infty e^{-\lambda t} T(t) dt = \dots = \int_0^\infty e^{-\lambda t} S^n(t) dt.$$

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$$\Rightarrow \frac{R(\lambda, A)}{\lambda^n} \text{ is a Laplace transform}$$

Integrated semigroups

We are searching for operators $A: D(A) \rightarrow E$ for which there exist $M, \omega \geq 0$ and $S^n: [0, \infty) \rightarrow \mathcal{L}(E)$ strongly continuous such that

- $\|S^n(t)\| \leq M e^{\omega t}, \quad \forall t \geq 0.$
- $(\omega, \infty) \subset \rho(A)$ and

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for some $n \in \mathbb{N}_0$.

Definition

$(S^n(t))_{t \geq 0}$ is called an *n-times integrated semigroup* and A its *generator*.

Theorem (Relations between the semigroup and its generator)

Let $n \in \mathbb{N}$ and $x \in E$. Let $(S^n(t))_{t \geq 0}$ an n -times integrated semigroup and A its generator. Then the following assertions hold:

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- Let $t \geq 0$. If $x \in D(A)$, then

$$\int_0^t S^n(s)Ax \, ds = S^n(t)x - \frac{t^n}{n!}x.$$

In particular, $\frac{d}{dt}(S^n(t)x) = S^n(t)Ax + \frac{t^{n-1}}{(n-1)!}x$.

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In particular, $\frac{d}{dt}(S^n(t)x) = S^n(t)Ax + \frac{t^{n-1}}{(n-1)!}x$.

- Let $t \geq 0$. Then $\int_0^t S^n(s)x \, ds \in D(A)$ and

$$A \int_0^t S^n(s)x \, ds = S^n(t)x - \frac{t^n}{n!}x.$$

In particular, $S^n(0) = 0$ and $S^n(t)x \in \overline{D(A)}$.

Remark

In contrast with C_0 -semigroups, generators of an n -times integrated semigroup (for $n \geq 1$) need not be densely defined.

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Remark

It follows that

$$S^n(\cdot)x \text{ is right-sided differentiable in } t \geq 0 \Leftrightarrow S^n(t)x \in D(A)$$

In particular,

$$x \in D(A) \Rightarrow S^n(t)x \in D(A) \Rightarrow \frac{d}{dt}S^n(t)x \in \overline{D(A)}$$

Equation for integrated semigroups

$$(T(t))_{t \geq 0} \text{ semigroup} \quad \Rightarrow \quad T(t+s) = T(t)T(s), \quad \forall s, t \geq 0$$

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Question

Can we associate an n -times integrated semigroup with an equation?

$$(S^n(t))_{t \geq 0} \text{ } n\text{-times integrated semigroup} \quad \Rightarrow \quad ?$$

Theorem

Let $S: [0, \infty) \rightarrow \mathcal{L}(E)$ be a strongly continuous function and suppose that there exist $M, \omega \geq 0$ s.t. $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$. Let $n \in \mathbb{N}$ and define

$$\frac{R(\lambda)}{\lambda^n} := \int_0^\infty e^{-\lambda t} S(t) dt, \quad \forall \lambda > \omega.$$

Then the following assertions are equivalent:

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Then the following assertions are equivalent:

1. There exists an operator $A: D(A) \rightarrow E$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) = R(\lambda)$.

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1. There exists an operator $A: D(A) \rightarrow E$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) = R(\lambda)$.

2. $S(t)x = 0$, for all $t \geq 0$ implies $x = 0$ and

$$S(t)S(s) = \frac{1}{(n-1)!} \left[\int_t^{s+t} (s+t-r)^{n-1} S(r) dr - \int_0^s (s+t-r)^{n-1} S(r) dr \right],$$

for all $s, t \geq 0$.

In particular, $S(t)S(s) = S(s)S(t)$, for all $s, t \geq 0$.

Order: $n \rightarrow n + 1$

Proposition

If the operator A generates an n -times integrated semigroup, then A generates an $(n + 1)$ -times integrated semigroup.

$$S^{n+1}(t) := \int_0^t S^n(s) ds$$

is the $(n + 1)$ -times integrated semigroup generated by A .

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Problem

If A generates an $(n + 1)$ -times integrated semigroup, do we need additional conditions to make A generating an n -times integrated semigroup?

By intuition

If $(S^{n+1}(t))_{t \geq 0}$ is an $(n + 1)$ -times integrated semigroup, we would like to define:

$$S^n(t)x := \frac{d}{dt} S^{n+1}(t)x, \quad \forall t \geq 0.$$

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$$S^n(t)x := \frac{d}{dt} S^{n+1}(t)x, \quad \forall t \geq 0.$$

- If $x \in D(A)$, then $\frac{d}{dt} S^{n+1}(t)x$ exists and belongs to $\overline{D(A)}$.

$$\Rightarrow S^n(t): D(A) \rightarrow \overline{D(A)}, \quad \forall t \geq 0.$$

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If $(S^{n+1}(t))_{t \geq 0}$ is an $(n + 1)$ -times integrated semigroup, we would like to define:

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- If $x \in D(A)$, then $\frac{d}{dt} S^{n+1}(t)x$ exists and belongs to $\overline{D(A)}$.

$$\Rightarrow S^n(t): D(A) \rightarrow \overline{D(A)}, \quad \forall t \geq 0.$$

- If $x \in \overline{D(A)}$, $\frac{d}{dt} S^{n+1}(t)x$ might not make sense.

\Rightarrow We need an additional hypothesis

Consequence of "Integrated Widder"

Theorem

Let $n \in \mathbb{N}_0$ and $\omega \geq 0$. Let A a linear operator. The following assertions are equivalent:

Consequence of "Integrated Widder"

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Let $n \in \mathbb{N}_0$ and $\omega \geq 0$. Let A a linear operator. The following assertions are equivalent:

1. A generates an $(n + 1)$ -times integrated semigroup $(S^{n+1}(t))_{t \geq 0}$ satisfying

$$\limsup_{h \rightarrow 0} \frac{1}{h} \left\| S^{n+1}(t+h) - S^{n+1}(t) \right\| \leq M e^{\omega t}, \quad \forall t \geq 0. \quad (\text{Lip})$$

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$$\limsup_{h \rightarrow 0} \frac{1}{h} \left\| S^{n+1}(t+h) - S^{n+1}(t) \right\| \leq M e^{\omega t}, \quad \forall t \geq 0. \quad (\text{Lip})$$

2. $(\omega, \infty) \subset \rho(A)$ and

$$\left\| \frac{(\lambda - \omega)^{k+1}}{k!} \left[\frac{R(\lambda, A)}{\lambda^n} \right]^{(k)} \right\| \leq M, \quad \forall \lambda > \omega, \forall k \in \mathbb{N}_0. \quad (\text{Hi-Yo})$$

Order: $n + 1 \rightarrow n$

Let the equivalent conditions (Lip) and (Hi-Yo) be satisfied.

$$\Rightarrow S^{n+1}(\cdot)x \in C^1([0, \infty), E), \quad \forall x \in \overline{D(A)}.$$

Define:

$$S^n(t)x := \frac{d}{dt}S^{n+1}(t)x \in \overline{D(A)}, \quad \forall t \geq 0, \forall x \in \overline{D(A)}.$$

Thanks to the Lipschitz condition (Lip) of S^{n+1} it follows that

$$\|S^n(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Order: $n + 1 \rightarrow n$

Let the equivalent conditions (Lip) and (Hi-Yo) be satisfied.

$$\Rightarrow S^{n+1}(\cdot)x \in C^1([0, \infty), E), \quad \forall x \in \overline{D(A)}.$$

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$$\|S^n(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

$\Rightarrow (S^n(t))_{t \geq 0} \subset \mathcal{L}(\overline{D(A)})$ is strongly continuous and exponentially bounded.

$$S^{n+1}(t): E \rightarrow E \quad \Rightarrow \quad S^n(t): \overline{D(A)} \rightarrow \overline{D(A)}$$

Generator of $(S^n(t))_{t \geq 0}$

Questions

1. Is the family $(S^n(t))_{t \geq 0}$ an n -times integrated semigroup?
2. Does it possess a generator?

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Consider the *part of A in $\overline{D(A)}$* , i.e.

$$D(A_{\overline{D(A)}}) := \{x \in D(A) : Ax \in \overline{D(A)}\}, \quad A_{\overline{D(A)}}x := Ax \quad (x \in D(A_{\overline{D(A)}})).$$

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It follows that

1. $\rho(A_{\overline{D(A)}}) \supset \rho(A) \supset (\omega, \infty)$.
2. $R(\lambda, A_{\overline{D(A)}}) = R(\lambda, A)$. (up to proper restrictions)

$$\Rightarrow \frac{R(\lambda, A_{\overline{D(A)}})}{\lambda^n} = \int_0^\infty \lambda e^{-\lambda t} S^{n+1}(t) dt = \int_0^\infty e^{-\lambda t} S^n(t) dt,$$

for all $\lambda > \omega$.

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Questions

1. Is the family $(S^n(t))_{t \geq 0}$ an n -times integrated semigroup? Yes.
2. Does it possess a generator? Yes, it's $A_{\overline{D(A)}}$.

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Questions

1. Is the family $(S^n(t))_{t \geq 0}$ an n -times integrated semigroup? Yes.
2. Does it possess a generator? Yes, it's $A_{\overline{D(A)}}$.

We have proved the following result:

Corollary

If A satisfies the equivalent conditions (Lip) and (Hi-Yo), then $A_{\overline{D(A)}}$ generates an n -times integrated semigroup $(S^n(t))_{t \geq 0}$ in $\overline{D(A)}$.

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Remark

If A is densely defined, then

$$A_{\overline{D(A)}} = A.$$

Corollary

If A satisfies the equivalent conditions (Lip) and (Hi-Yo), then $A_{\overline{D(A)}}$ generates an n -times integrated semigroup $(S^n(t))_{t \geq 0}$ in $\overline{D(A)}$.

Remark

If A is densely defined, then

$$A_{\overline{D(A)}} = A.$$

Result

If the equivalent conditions (Lip) and (Hi-Yo) hold and A is densely defined, then A generates an n -times integrated semigroup $(S^n(t))_{t \geq 0}$.

Generalisation of Hille-Yosida theorem

Theorem

Let A be a densely defined operator on E . Let $n \in \mathbb{N}_0$ and $M, \omega \geq 0$.

The following assertions are equivalent:

- A generates an n -times integrated semigroup $(S^n(t))_{t \geq 0}$ satisfying

$$\|S^n(t)\| \leq M e^{\omega t}, \quad \forall t \geq 0.$$

- $(\omega, \infty) \subset \rho(A)$ and

$$\left\| \frac{(\lambda - \omega)^{k+1}}{k!} \left[\frac{R(\lambda, A)}{\lambda^n} \right]^{(k)} \right\| \leq M, \quad \forall \lambda > \omega, \forall k \in \mathbb{N}_0. \quad (\text{Hi-Yo})$$

Banach lattices

Definition

A real ordered Banach space (E, \leq) is called a Banach lattice if

1. (E, \leq) is a vector lattice.
2. $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$.

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Examples

- If (Ω, μ) is a measure space, then

$$L^p(\Omega, \mu), \quad \text{for all } p \in [1, \infty],$$

is a Banach lattice.

- If K is a compact space, then

$$C(K) = \{K \rightarrow \mathbb{C} : f \text{ is continuous}\}$$

is a Banach lattice.

Resolvent positive operators

Theorem

Let E be a Banach lattice and A be an operator on E such that $(\omega, \infty) \subset \rho(A)$ for some $\omega \in \mathbb{R}$ and

$$R(\lambda, A) \geq 0, \quad \forall \lambda > \omega. \quad (A \text{ is resolvent positive})$$

Then A generates a 2-times integrated semigroup.

Moreover, if $D(A)$ is dense, A generates a once integrated semigroup.

Remark

If $E = L^p(\Omega)$ ($1 \leq p < \infty$) and A is a resolvent positive operator

\Rightarrow A generates a once integrated semigroup

Example

Let $E = C([-1, 0]) \times \mathbb{R}$ and

$$A: C^1([-1, 0]) \times \{0\} \subset E \rightarrow E, \quad A(f, 0) = (f', -f(0)).$$

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$$A: C^1([-1, 0]) \times \{0\} \subset E \rightarrow E, \quad A(f, 0) = (f', -f(0)).$$

Then

$$\rho(A) = \mathbb{C} \quad \text{and} \quad R(\lambda, A)(f, c) = \left(e^{\lambda x} \left[\int_x^0 e^{-\lambda y} f(y) dy + c \right], 0 \right).$$

A is resolvent positive.

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A is resolvent positive. Let $e_\lambda \in C([-1, 0])$ given by

$$e_\lambda(x) = e^{\lambda x}, \quad \text{with } \lambda > 0 \text{ and } x \in [-1, 0].$$

$$\Rightarrow \quad (e_\lambda, 0) = R(\lambda, A)(0, 1)$$

By simple calculations we have

$$e_\lambda = \int_0^\infty \lambda^2 e^{-\lambda t} k_t dt,$$

where $k_t \in C([-1, 0])$ is given by

$$k_t(x) = \begin{cases} 0 & \text{if } x + t \leq 0 \\ x + t & \text{if } x + t > 0 \end{cases} \quad \text{with } t \in \mathbb{R}^+.$$

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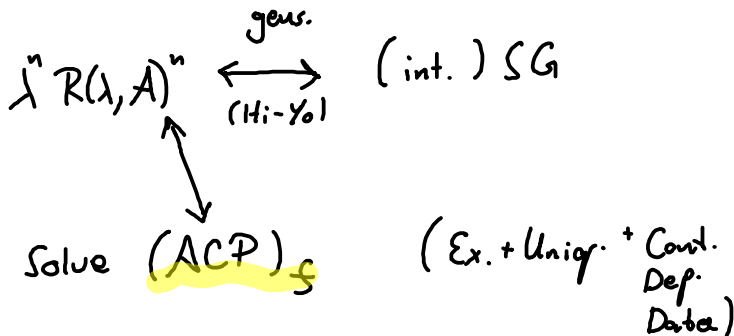
Suppose by contradiction that A generates a once integrated semigroup. Then

$$\frac{R(\lambda, A)}{\lambda}(0, 1) = \left(\frac{e_\lambda}{\lambda}, 0 \right) \text{ is a Laplace transform}$$

Thanks to "Integrated Widder", we have that

$$k: \mathbb{R}^+ \rightarrow C([-1, 0]) \text{ is differentiable} \Rightarrow \text{Contradiction!}$$

Integrated Semigroups and Cauchy Problems



Integrated Semigroups and Cauchy Problems

Back to our *inhomogeneous Cauchy problem*

$$(\text{ACP}_f) \begin{cases} u'(t) = Au(t) + f(t) & (t \in [0, \tau]), \\ u(0) = x_0, \end{cases}$$

where $\tau > 0$, $f \in L^1([0, \tau], E)$, $x_0 \in E$.

A *classical solution* of (ACP_f) is a function

$$u \in C^1([0, \tau], E) \cap C([0, \tau], D(A))$$

satisfying (ACP_f) for all $t \in [0, \tau]$.

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A generator of a once integrated semigroup.

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satisfying (ACP_f) for ~~all~~^{a.a.} $t \in [0, \tau]$.

Existence of Solutions to (ACP_f)

(*init.* *inhom.*) + *const. dep. data*

Proposition (Representation Formula and Uniqueness)

Consider $v(t) := S(t)x_0 + \int_0^t S(s)f(t-s) ds$ ($x_0 \in E$) *Variation of Const*

If u is a class. sol. of (ACP_f) , then $v \in \underline{C^2}([0, \tau], E)$ and $u = \underline{v}$.

Theorem (Existence)

If $v \in \underline{C^2}([0, \tau], E)$, then $u := \underline{v}$ is a solution of (ACP_f) .

Existence of Solutions to (ACP_f)

Theorem (Existence)

If $\underline{v(\cdot)} = \underline{S(\cdot)x_0} + \underline{\int_0^{\cdot} S(s)f(t-s) ds} \in C^2([0, \tau], E)$,

then $u := v'$ is a solution of (ACP_f).

Proof. Recall $\left[\underline{A \int_0^t S(s)x_0 ds} = S(t)x_0 - tx_0 \quad (t \geq 0, x_0 \in E) \right]_{n=1}$

$$A \int_0^t v(s) ds = A \int_0^t S(s)x_0 ds + A \int_0^t \int_0^s S(r)f(s-r) dr ds$$

classical
 $= S(t)x_0 - tx_0 + \int_0^t S(t-s)f(s) - (t-s)f(s) ds$

Fubini
 $= v(t) - tx_0 + \int_0^t (t-s)f(s) ds$

Existence of Solutions to (ACP)_f

Theorem (Existence)

If $\underline{v}(\cdot) = S(\cdot)x_0 + \int_0^\cdot S(s)f(t-s)ds \in C^2([0, \tau], E)$, $\left\{ \begin{array}{l} \underline{v}(0) = S(0)x_0 = 0 \\ \underline{u}(0) = \underline{v}'(0) = x_0 \end{array} \right.$

then $u := \underline{v}'$ is a solution of (ACP)_f.

Proof. $\underline{A} \int_0^t \underline{v}(s) ds = \underline{v}(t) - t x_0 - \int_0^t (t-r)f(r) dr$ $| (0)' \quad \underline{v}(t) \in \mathcal{A}$

$\Rightarrow \underline{A} \underline{v}(t) = \underline{v}'(t) - \underline{x}_0 - \int_0^t f(r) dr$ $| (0)'' \quad \underline{v}'(t) \in \mathcal{D}(\underline{A})$

$\Rightarrow \underline{A} \underline{v}'(t) = \frac{\partial}{\partial t} \underline{v}'(t) - f(t)$

$\Rightarrow \underline{u} = \underline{v}'$ solves our (ACP)_f

Differentiability of $v(t) = \underbrace{S(t)x_0}_{\uparrow} + \int_0^t \underbrace{S(s)f(t-s)}_{\downarrow} ds$

Sufficient condition

- $f \in \underline{C^2}([0, \tau], E)$ $\not\checkmark$
- $x_0 \in D(A)$ \checkmark
- $x_1 := Ax_0 + f(0) \in D(A)$ \checkmark

Remark

The mere regularity of f is not enough!

Recall $\underline{S(t)x} = tx + \int_0^t \underline{S(s)Ax} ds$ ($x \in D(A), t \geq 0$) \uparrow

$$v'(t) = \underline{x_0} + \underbrace{S(t)Ax_0 + S(t)f(0)}_{S(t)(Ax_0 + f(0))} + \int_0^t \underbrace{S(s)f'(t-s)}_{\downarrow} ds$$

Generalization

Let A be the generator of an n -times integrated semigroup.

Theorem (Existence and Uniqueness)

If $v(\cdot) := S(\cdot)x_0 + \int_0^\cdot S(s)f(t-s) ds \in C^{n+1}([0, \tau], E)$,

then $u := v^{(n)}$ is the unique solution of (ACP_f) .

Sufficient condition

- $f \in C^{n+1}([0, \tau], E)$
- $x_0 \in D(A)$
- $x_1 := Ax_0 + f(0) \in D(A)$
- ...
- $x_n := Ax_{n-1} + f^{(n-1)}(0) \in D(A)$

$$f \equiv 0$$

$$x_0 \in D(A^{n+1})$$

Trade: Less reg SG
More reg. on Data

Hille-Yosida Condition Gen. Res.

Question

What if A a priori only satisfies

A is not d.d.

$$\begin{aligned}
 & \text{(Hi-Yo)} \quad \left\| \frac{\lambda^{k+1}}{k!} \left[\frac{R(\lambda, A)}{\lambda} \right]^{(k)} \right\| = \left\| (\lambda R(\lambda, A))^k \right\| \leq 1 \quad (\lambda > 0, k \in \mathbb{N}_0) ? \\
 & \begin{matrix} \nwarrow \\ \downarrow \\ \swarrow \end{matrix} \\
 & \text{(LIP)} \quad \xrightarrow{\text{(RM)}} \text{D:ff.}
 \end{aligned}$$

A generates a once integrated but not a C_0 -semigroup in general.

Example

$$\begin{aligned}
 E &= C_0(-\infty, 0] := \{f \in C(-\infty, 0] : \lim_{x \rightarrow -\infty} f(x) = 0\} \\
 D(A) &= \{f \in E \cap C^1(-\infty, 0] : f(0) = 0, Af = f'\}
 \end{aligned}$$

$$\begin{aligned}
 & \text{(Hi-Yo)} \quad \lambda f - f' = g \in E \\
 & R(\lambda, A)g := f(x) = e^{-\lambda x} \int_x^0 e^{-\lambda s} g(s) ds \in E \quad (\lambda > 0)
 \end{aligned}$$

Example – where is the once integrated semigroup?

$$(R(\lambda, A)f)(x) = e^{\lambda x} \int_x^0 e^{-\lambda s} f(s) ds$$

int. by parts = $\int_0^{\infty} \lambda e^{-\lambda t} (S(t)f)(x) dt$

$$(S(s)f)(x) = \begin{cases} \int_x^{x+t} f(s) ds & t \leq -x \\ \int_x^0 f(s) ds & t > -x \end{cases}$$

$S(\cdot)f$ diff'bla iff $f \in \overline{D(A)}$

Hille-Yosida revisited

A still not d.d.

Theorem

Let

- E Banach space with Radon-Nikodým property
- A operator such that $(\omega, \infty) \subset \rho(A)$ for some $a \geq 0$
- $\|((\lambda - \omega)R(\lambda, A))^m\| \leq M \quad (\lambda > \omega, m = 0, 1, 2, \dots)$

Then

- exists a semigroup $(T(t))_{t \geq 0}$, strongly continuous for $t \geq 0$
- $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt$
- for the once integrated semigroup $(S(t))$ generated by A

$$S(t) = \int_0^t T(s) ds$$

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