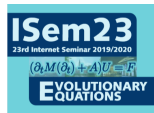


# Two scale convergence of Thermoelasticity

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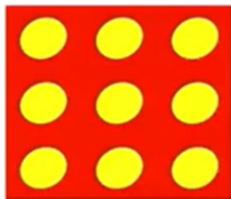
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22<sup>th</sup> – 26<sup>th</sup> 2020



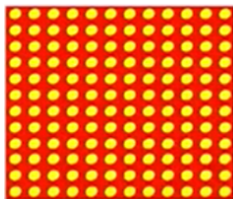
- 1 Motivation
  - What is Homogenization? Physical and Mathematical.
  - What is Thermoelasticity?
- 2 Problem Description
- 3 Two scale convergence
- 4 Conclusion and Open problems

## Motivation

# Composite material ?



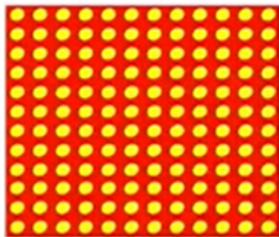
$$\varepsilon = 1$$



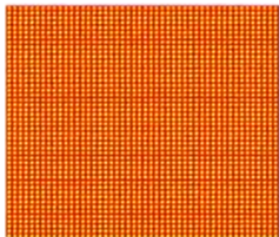
$$\varepsilon = \frac{1}{4}$$

Figure – Composite material

# Composite material ?



$$\varepsilon = \frac{1}{4}$$



$$\varepsilon = \frac{1}{16}$$

Figure – Composite material

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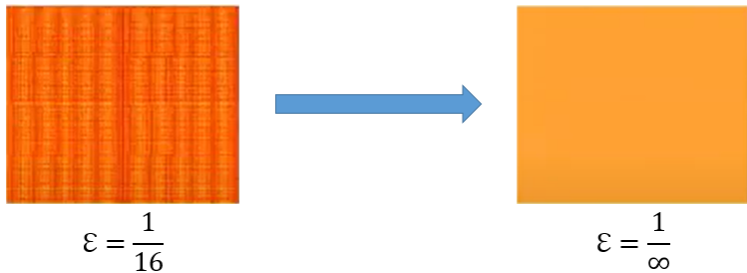


Figure – Composite material

# What is Homogenization (averaging) ?

- It is the study of microscopic and or/bulk behavior of solutions to PDEs or other type of equations posed on a heterogeneous domain/media, where the heterogeneities are present at the microscopic scale  $\varepsilon$ .

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- It is the study of microscopic and or/bulk behavior of solutions to PDEs or other type of equations posed on a heterogeneous domain/media, where the heterogeneities are present at the microscopic scale  $\varepsilon$ .
- The heterogeneities can be in the form of fine mixing of two or more materials with different physical properties in the domain or due to singularities in the domain in the form of pores, granules etc.

# Nature of a typical problem in homogenization

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- The problem posed on the heterogeneous domain has a unique solution.
- Heterogeneities cause the solution to develop high frequency oscillations.
- So a direct numerical study will not be able to capture either the bulk behaviour or the oscillations present in the solution which are at a microscopic level.

# Periodic Oscillations in Heterogeneities

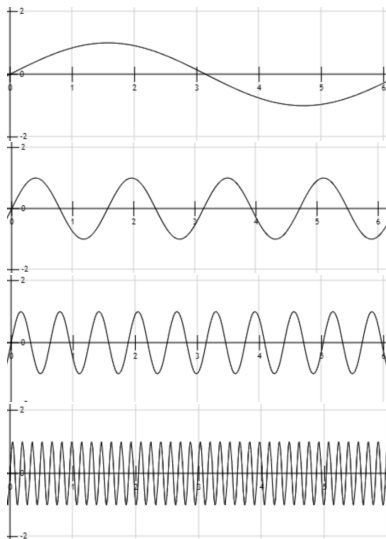


Figure –  $\sin(x)$ ,  $\sin(4x)$ ,  $\sin(10x)$ ,  $\sin(40x)$

# What is the way out ?

Do an asymptotic analysis as  $\varepsilon \rightarrow 0$ , that is we try to approximate the solution  $u_\varepsilon$  corresponding to the heterogeneous domain, for small  $\varepsilon > 0$ , by the solution of a homogenized problem in which the small parameter does not appear.

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## The mathematical problem is :

- to identify the homogenized equation
- to prove the convergence of  $u_\varepsilon$  corresponding to the heterogeneous equation to that of the homogenized equation as  $\varepsilon \rightarrow 0$
- to capture the oscillations in  $u_\varepsilon$  (known as correctors in the literature)
- the solution together with the correctors can be used (and also can be computed numerically) to get good approximation to the original solution.

# The mathematical problem

The Problem of homogenization of evolution equations can be stated as follows :

$$\begin{cases} z'_\varepsilon(t) = \mathbf{A}_\varepsilon z_\varepsilon(t) & t > 0, \\ z_\varepsilon(0) = z, \end{cases}$$

---

**Question :**

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Assuming  $z_\varepsilon \rightarrow z_0$ . What's the evolution equation that  $z_0$  satisfies? and how to give an explicit analytical construction of the limiting evolution equation?

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$$\begin{cases} z'_0(t) = \mathbf{A}_0 z_0(t) & t > 0, \\ z_0(0) = z, \end{cases}$$

# Baby example

A **stationary** temperature field in a **non-homogeneous** rod with periodic structure :

$$-\frac{d}{dx} \left( a \left( \frac{x}{\varepsilon} \right) \frac{du_\varepsilon}{dx} \right) = f(x), \quad u_\varepsilon(0) = u_\varepsilon(1) = 0,$$



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## Assumptions

- 1 The rod is of periodic structure.
- 2 The thermal conductivity  $a_\varepsilon$  is periodic with period  $\varepsilon = \frac{1}{n}$ , where  $n$  is a large natural number.
- 3  $a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$  and  $a(y)$  is a 1-periodic function which is positive and differentiable.

$$-\frac{d}{dx} \left( a \left( \frac{x}{\varepsilon} \right) \frac{du_\varepsilon}{dx} \right) = f(x), \quad u_\varepsilon(0) = u_\varepsilon(1) = 0,$$

- Since we are considering a periodic structure at a microscopic scale like in composite materials, then  $\varepsilon$  will be very small causing rapid oscillations in the coefficients and hence in the solutions.

---

**Question : What is the homogenized equation ?**  
**Some Fundamental Difficulties**

# Baby example

- It can be seen that the periodic function  $a(\frac{x}{\varepsilon})$  converge to the average : In other words, in weak sense

$$a(\frac{x}{\varepsilon}) \rightharpoonup m(a) := \int_Y a(y)dy.$$

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- Does  $u$  satisfies the averaged equation ?

$$-\frac{d}{dx} \left( m(a) \frac{du}{dx} \right) = f(x), \quad u(0) = u(1) = 0,$$

- In general, it is not true. The limit problem is

$$-\frac{d}{dx} \left( \left[ m\left(\frac{1}{a}\right) \right]^{-1} \frac{du}{dx} \right) = f(x), \quad u(0) = u(1) = 0,$$

- What about  $N$ -dimension ( $a = A$ )?

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- What about  $N$ -dimension ( $a = A$ )?
- The difficulty is that weak convergence does not preserve non linearities. In particular

$$f_n \rightharpoonup f \text{ and } g_n \rightharpoonup g$$

does not imply  $g_n f_n \rightharpoonup gf$

For example :  $\sin(2\pi nx) \rightharpoonup 0$ , but  $\sin^2(2\pi nx) \rightharpoonup \frac{1}{2}$ .

# How to get Homogenized equation ?

- Multiscale asymptotic expansion method : Bensoussan, Lions and Papanicolaou, see also Keller and Larsen, and Sanchez-Palencia ;
- **Two scale method** : Nguetseng, Allaire.
- the  $G$ -convergence of Spagnolo well adapted to problem involving Green kernel ;
- the  $\Gamma$ -convergence of De Giorgi suitable for homogenization of optimization problem ;
- the  $H$ -convergence of Tartar and Murat (a generalization of the  $G$ -convergence method to non symmetric problems).

# Homogenized equation

$$-\frac{d}{dx} \left( a \left( \frac{x}{\varepsilon} \right) \frac{du_\varepsilon}{dx} \right) = f(x), \quad u_\varepsilon(0) = u_\varepsilon(1) = 0,$$

and the limit problem is

$$-a^{hom} \frac{d^2 u^0}{dx^2} = f(x), \quad u(0) = u(1) = 0,$$
$$a^{hom} = \left( \int_0^1 \frac{1}{a(y)} dy \right)^{-1}$$

# Some numerics

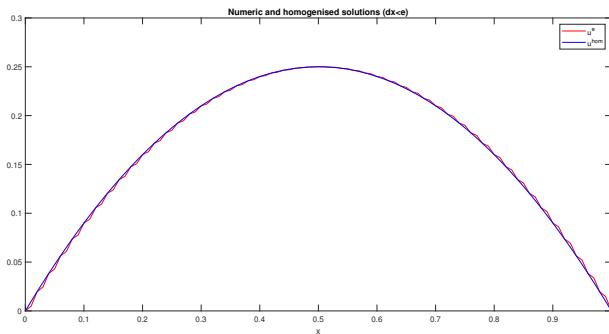
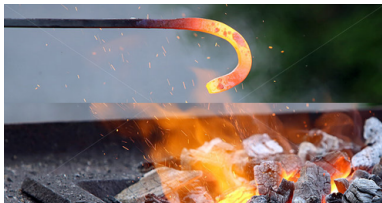


Figure –  $a(y) = (2 + \sin(2\pi y))^{-1}$ .  $\epsilon = 0.02$ ,  $f(x) \equiv 1$ .

# What is thermoelasticity ?

The study of the relationship between the elastic properties of a material and its temperature or between its thermal conductivity and its stresses.

# Thermoelasticity



$$u_{tt}(x, t) - u_{xx}(x, t) = 0$$

$$\theta_t(x, t) - \theta_{xx}(x, t) = 0$$

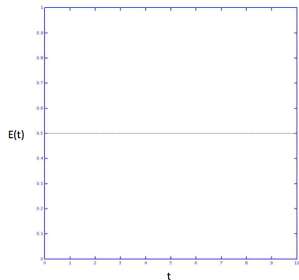
# Decay of coupled system : Heat vs Wave

$$u_{tt}(x, t) - \Delta u(x, t) = 0$$

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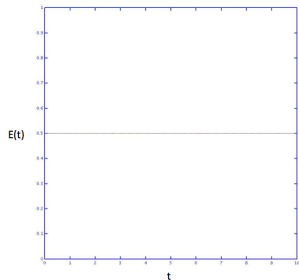
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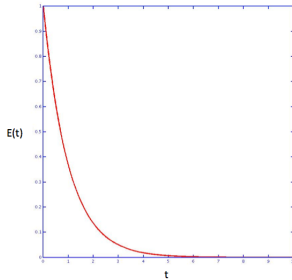
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$$u_{tt}(x, t) - \Delta u(x, t) = 0$$



$$\theta_t(x, t) - \Delta \theta(x, t) = 0$$





$$(S) \begin{cases} u_{tt} - u_{xx} + \gamma f(\theta, \theta_x) & = 0 \\ \theta_t - \theta_{xx} + \gamma g(u_t, u_{tx}) & = 0 \end{cases}$$

# Linear Thermoelasticity

$$\text{Strong coupling} \quad \begin{cases} u_{tt} - u_{xx} + \gamma \theta_x = 0, & \Omega \times (0, +\infty), \\ \theta_t - \theta_{xx} + \gamma u_{tx} = 0, & \Omega \times (0, +\infty), \end{cases}$$

$$\text{Weak coupling} \quad \begin{cases} u_{tt} - u_{xx} + \gamma \theta = 0, & \Omega \times (0, +\infty), \\ \theta_t - \theta_{xx} + \gamma u_t = 0, & \Omega \times (0, +\infty), \end{cases}$$

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**A. DAY**, Heat Conduction with Linear Thermoelasticity, Springer-Verlag, New York, 1985.



**V. D. KUPRADZE**, Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, North-Holland, Amsterdam, 1979.



**C. Dafermos**, On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity, Arch. Rational. Mech. Anal. 29, 241-271 (1968)

## Problem description

- $\Omega \subset \mathbb{R}^N$  bounded domain with boundary  $\partial\Omega$ .
- $Y = (0, 1)^N$  be the periodicity cell with Lebesgue measure  $dy$ .
- The symbol  $\#$  denotes  $Y$ -periodic functions.
- $C_0^\infty(\Omega)$  : the linear spaces of all (test) functions which are infinitely differentiable, and respectively compactly supported in domain  $\Omega$ .
- $C_\#^\infty(Y) = \{f \in C^\infty(Y) + f \text{ is } Y - \text{periodic}\}$
- $L_\#^p(Y) = \{f / f \in L_{loc}^p(Y) + f \text{ is } Y - \text{periodic}\}$ .
- $H_\#^1(Y) = \overline{C_\#^\infty(Y)}$
- $H_\#^{-1}(Y) = \left(H_\#^1(Y)\right)'$

# Problem description

$$(S) \left\{ \begin{array}{ll} c(\frac{x}{\varepsilon}) \frac{\partial^2 u_\varepsilon}{\partial t^2} = \operatorname{div}(a(\frac{x}{\varepsilon}) \nabla u_\varepsilon) - \gamma(\frac{x}{\varepsilon}) \theta_\varepsilon & \text{in } \Omega \times (0, T) \\ d(\frac{x}{\varepsilon}) \frac{\partial \theta_\varepsilon}{\partial t} = \operatorname{div}(b(\frac{x}{\varepsilon}) \nabla \theta_\varepsilon) + \gamma(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial t} & \text{in } \Omega \times (0, T) \\ u_\varepsilon = 0 = \theta_\varepsilon & \text{on } \partial\Omega \times (0, T) \\ u_\varepsilon(x, 0) = \tilde{u}_0(x), \frac{\partial u_\varepsilon}{\partial t}(x, 0) = \tilde{u}_1(x), \theta_\varepsilon(x, 0) = \tilde{\theta}_0(x) & \text{in } \Omega. \end{array} \right.$$

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$$E(t) = \frac{1}{2} \int_{\Omega} |(a_\varepsilon(x))^{\frac{1}{2}} \nabla u(x)|^2 + c_\varepsilon(x) |v(x)|^2 + d_\varepsilon(x) |\theta(x)|^2 dx$$

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- 1  $u_\varepsilon$  represents the displacement field.
- 2  $\theta_\varepsilon$  the temperature increment field.
- 3  $\gamma(\cdot)$  The thermomechanical coupling.
- 4  $a(\cdot)$  and  $b(\cdot)$  are respectively the diffusivity and propagation speed in the medium.
- 5  $c(\cdot)$  and  $d(\cdot)$  are positive.

# Some Hypothesis (H)

- $a(\cdot)$  and  $b(\cdot)$  are selfadjoint  $Y$ -periodic matrices.
- there exists  $\alpha > 0$  independent of  $\varepsilon$  such that for all  $\xi \in \mathbb{R}^N$  :

$$\alpha|\xi|^2 \leq \xi^T a(y)\xi \leq \frac{1}{\alpha}|\xi|^2.$$

$$\alpha|\xi|^2 \leq \xi^T b(y)\xi \leq \frac{1}{\alpha}|\xi|^2.$$

The functions  $c$ ,  $d$  and  $\gamma$  are bounded positive  $Y$ -periodic functions

- $0 < \alpha \leq c(y) \leq \alpha^{-1} < \infty \quad \forall y \in Y.$
- $0 < \alpha \leq d(y) \leq \alpha^{-1} < \infty \quad \forall y \in Y.$
- $0 < \gamma(y) < 1 \quad \forall y \in Y.$

$$A_\varepsilon w = -\operatorname{div}\left(a\left(\frac{\cdot}{\varepsilon}\right)\nabla w\right), \quad B_\varepsilon w = -\operatorname{div}\left(b\left(\frac{\cdot}{\varepsilon}\right)\nabla w\right).$$

Define the inner product

$$\left\langle (u, v, \theta), (\tilde{u}, \tilde{v}, \tilde{\theta}) \right\rangle_\varepsilon = \langle A_\varepsilon u, \tilde{u} \rangle_2 + \langle c_\varepsilon v, \tilde{v} \rangle_2 + \langle d_\varepsilon \theta, \tilde{\theta} \rangle_2$$

$$\|(u, v, \theta)\|_\varepsilon^2 = \int_\Omega |(a_\varepsilon(x))^{1/2} \nabla u(x)|^2 + c_\varepsilon(x)|v(x)|^2 + d_\varepsilon(x)|\theta(x)|^2 dx.$$

$$H = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega).$$

Remark :  $\|\cdot\|_H \sim \|\cdot\|_\varepsilon$

$$(S) \left\{ \begin{array}{l} c(\frac{x}{\varepsilon}) \frac{\partial^2 u_\varepsilon}{\partial t^2} = \operatorname{div}(a(\frac{x}{\varepsilon}) \nabla u_\varepsilon) - \gamma(\frac{x}{\varepsilon}) \theta_\varepsilon \\ d(\frac{x}{\varepsilon}) \frac{\partial \theta_\varepsilon}{\partial t} = \operatorname{div}(b(\frac{x}{\varepsilon}) \nabla \theta_\varepsilon) + \gamma(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial t} \\ u_\varepsilon = 0 = \theta_\varepsilon \\ u_\varepsilon(x, 0) = \tilde{u}_0(x), \quad \frac{\partial u_\varepsilon}{\partial t}(x, 0) = \tilde{u}_1(x), \quad \theta_\varepsilon(x, 0) = \tilde{\theta}_0(x) \end{array} \right.$$

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$$\Leftrightarrow \begin{cases} U'_\varepsilon(t) &= \mathbf{A}_\varepsilon U_\varepsilon(t) \\ U_\varepsilon(0) &= \tilde{U}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\theta}_0)^T, \end{cases}$$

$$U_\varepsilon = \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \\ \theta_\varepsilon \end{pmatrix}, \quad \mathbf{A}_\varepsilon = \begin{pmatrix} 0 & I & 0 \\ -\frac{1}{c_\varepsilon} A_\varepsilon & 0 & -\frac{\gamma_\varepsilon}{c_\varepsilon} I \\ 0 & \frac{\gamma_\varepsilon}{d_\varepsilon} I & -\frac{1}{d_\varepsilon} B_\varepsilon \end{pmatrix}.$$

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$$D(\mathbf{A}_\varepsilon) = D(A_\varepsilon) \times D(A_\varepsilon^{\frac{1}{2}}) \times D(B_\varepsilon)$$

$$\mathbf{A}_\varepsilon U = \left( v, -\frac{1}{c_\varepsilon} A_\varepsilon u - \frac{\gamma_\varepsilon}{c_\varepsilon} \theta, \frac{\gamma_\varepsilon}{d_\varepsilon} v - \frac{1}{d_\varepsilon} B_\varepsilon \theta \right)^T.$$

The family of operators  $\mathbf{A}_\varepsilon$  generates a family of contraction semigroups  $\mathbf{T}_\varepsilon(\cdot)$  on the Hilbert space  $H$  endowed with the norm  $\|\cdot\|_\varepsilon$ .

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## Sketch on the proof :

- $\mathbf{A}_\varepsilon$  is dissipative.
- $0 \in \rho(\mathbf{A}_\varepsilon)$

## Theorem 1

For  $\varepsilon > 0$ , the following a priori estimates hold :

$$\|c_\varepsilon^{\frac{1}{2}} u_\varepsilon\|_2 \leq C \|F\|_2$$

$$\|a_\varepsilon^{\frac{1}{2}} \nabla u_\varepsilon\|_2 \leq C \|F\|_2$$

$$\|d_\varepsilon^{\frac{1}{2}} \theta_\varepsilon\|_2 \leq C \|F\|_2$$

$$\|b_\varepsilon^{\frac{1}{2}} \nabla \theta_\varepsilon\|_2 \leq C \|F\|_2$$

with a constant  $C$  independent of  $\varepsilon$ .

# A priori estimates.

Sketch on the proof :

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Sketch on the proof :

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- $R(\lambda, \mathbf{A}_\varepsilon)F = U_\varepsilon$

$$\begin{aligned}v_\varepsilon &= \lambda u_\varepsilon - f \\(\lambda^2 c_\varepsilon - A_\varepsilon)u_\varepsilon + \gamma_\varepsilon \theta_\varepsilon &= c_\varepsilon g + \lambda c_\varepsilon f \\-\lambda \gamma_\varepsilon u_\varepsilon + (\lambda d_\varepsilon - B_\varepsilon)\theta_\varepsilon &= d_\varepsilon h - \gamma_\varepsilon f.\end{aligned}$$

- 

$$\begin{aligned}\lambda^3 \langle c_\varepsilon u_\varepsilon, u_\varepsilon \rangle - \lambda \langle A_\varepsilon u_\varepsilon, u_\varepsilon \rangle + \lambda \langle \gamma_\varepsilon \theta_\varepsilon, u_\varepsilon \rangle &= \langle c_\varepsilon g + \lambda c_\varepsilon f, \lambda u_\varepsilon \rangle \\ \langle -\lambda \gamma_\varepsilon u_\varepsilon, \theta_\varepsilon \rangle + \lambda \langle d_\varepsilon \theta_\varepsilon, \theta_\varepsilon \rangle - \langle B_\varepsilon \theta_\varepsilon, \theta_\varepsilon \rangle &= \langle d_\varepsilon h - \gamma_\varepsilon f, \theta_\varepsilon \rangle\end{aligned}$$

- 

$$\lambda^3 \|c_\varepsilon^{\frac{1}{2}} u_\varepsilon\|_2^2 + \lambda \|a_\varepsilon^{\frac{1}{2}} \nabla u_\varepsilon\|_2^2 + \lambda \|d_\varepsilon^{\frac{1}{2}} \theta_\varepsilon\|_2^2 + \|b_\varepsilon^{\frac{1}{2}} \nabla \theta_\varepsilon\|_2^2 \leq C \|F\|_2^2.$$

# Functional spaces for two-scale limits

The 'dual' space of admissible 'microscopic fluxes', of tensor fields on  $Y$ .

$$W_a := \left\{ \psi \in \left( L^2_{\#}(Y) \right)^N \mid \operatorname{div}_y \left( \left( a(y) \right)^{\frac{1}{2}} \psi(y) \right) = 0 \quad \text{in } H_{\#}^{-1}(Y) \right\}$$

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$$W_b := \left\{ \psi \in \left( L^2_{\#}(Y) \right)^N \mid \operatorname{div}_y \left( \left( b(y) \right)^{\frac{1}{2}} \psi(y) \right) = 0 \quad \text{in } H_{\#}^{-1}(Y) \right\}$$

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$W_a$  and  $W_b$  are closed linear subspaces of  $\left( L^2_{\#}(Y) \right)^N$ , and hence can themselves be regarded as Hilbert spaces with respective inherited  $L^2$  inner products.

## Two scale convergence

# Two scale convergence ?

## Definition of two scale convergence

A sequence of functions  $\{u_\varepsilon\}$  that is bounded in  $L^2(\Omega, dx)$  is said to be weakly two-scale convergent  $u_0(x, y) \in L^2(\Omega \times Y)$  (we write  $u_\varepsilon(x) \xrightarrow{2} u_0(x, y)$ ) if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \phi(x) b\left(\frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y u_0(x, y) \phi(x) b(y) dy dx,$$

for any  $\phi \in C_0^\infty(\Omega)$ ,  $b \in C_\#^\infty(Y)$ .

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## Definition 2

A sequence of functions  $\{u_\varepsilon\}$  that is bounded in  $L^2(\Omega, dx)$  is strongly two-scale convergent to  $u_0(x, y) \in L^2(\Omega \times Y)$  (we write  $u_\varepsilon(x) \xrightarrow{2} u_0(x, y)$ ) if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon(x) u_\varepsilon(x) dx = \int_{\Omega} \int_Y v(x, y) u_0(x, y) dy dx, \quad \text{for } v_\varepsilon(x) \xrightarrow{2} v(x, y).$$

# Two scale convergence ?

## Some remarks

- If  $b = 1$ , then  $u_\varepsilon(x) \xrightarrow{2} u_0$  implies  $u_\varepsilon \rightharpoonup u$ .

# Two scale convergence ?

## Some remarks

- If  $b = 1$ , then  $u_\varepsilon(x) \xrightarrow{2} u_0$  implies  $u_\varepsilon \rightharpoonup u$ .
- Strong two-scale convergence implies weak two-scale convergence.  
The mean value property implies that  $\phi(x)b(x/\varepsilon) \xrightarrow{2} \phi(x)b(y)$  for  $\phi \in C_0^\infty(\Omega)$ ,  $b \in C_\#^\infty(Y)$ .

## Weak two-scale Trotter Kato theorem

Let  $\mathbf{T}_\varepsilon(\cdot)$ ,  $\varepsilon > 0$ , be a contraction  $C_0$ -semigroup on the Hilbert space  $H$ , with generator  $\mathbf{A}_\varepsilon$ ,  $\varepsilon > 0$ . Suppose that for any  $\lambda > 0$ , we have

- (i) weak two-scale resolvent convergence :  $R(\lambda, \mathbf{A}_\varepsilon)f \xrightarrow{2} R(\lambda)f$  for some  $f \in H$  and
- (ii) the range of  $R(\lambda)$  is dense in  $H$ .

Then there exists a unique operator  $\mathbf{A}_0$  generating a contraction  $C_0$ -semigroup  $\mathbf{T}_0(\cdot)$  on the Hilbert space  $H$ , such that  $R(\lambda) := R(\lambda, \mathbf{A}_0)$ . Moreover,

$$\mathbf{T}_\varepsilon(t)f \xrightarrow{2} \mathbf{T}_0(t)f$$

for all  $t > 0$  and  $f \in H$ .

## Theorem 2

There exist  $u_0(x), \theta_0(x) \in L^2(\Omega)$ ,  $\xi_0(x, y) \in L^2(\Omega; W_a)$  and  $\chi_0(x, y) \in L^2(\Omega; W_b)$  such that, up to extracting a subsequence in  $\varepsilon$  which we do not relabel,

$$\begin{aligned}u_\varepsilon &\rightharpoonup u_0(x) \\ \varepsilon \nabla u_\varepsilon &\rightharpoonup 0 \\ \left( (a(x/\varepsilon)) \right)^{\frac{1}{2}} \nabla u_\varepsilon &\rightharpoonup \xi_0(x, y) \\ \theta_\varepsilon &\rightharpoonup \theta_0(x) \\ \varepsilon \nabla \theta_\varepsilon &\rightharpoonup 0 \\ \left( (b(x/\varepsilon)) \right)^{\frac{1}{2}} \nabla \theta_\varepsilon &\rightharpoonup \chi_0(x, y)\end{aligned}$$

## Theorem 3

There exists a unique function (up to an additive constant)  $u_1(x, y)$  in  $L^2(\Omega; H_{\#}^1(Y))$  solving the cell problem

$-div_y \left( a(y) [\nabla_y u_1(x, y) + \nabla_x u_0(x)] \right) = 0$ , such that

$$\xi_0(x, y) = (a(y))^{\frac{1}{2}} [\nabla_y u_1(x, y) + \nabla_x u_0(x)].$$

Similarly we can show that there exists a unique function (up to an additive constant)  $\theta_1(x, y)$  in  $L^2(\Omega; H_{\#}^1(Y))$  solving the cell problem

$-div_y \left( b(y) [\nabla_y \theta_1(x, y) + \nabla_x \theta_0(x)] \right) = 0$ , such that

$$\chi_0(x, y) = (b(y))^{\frac{1}{2}} [\nabla_y \theta_1(x, y) + \nabla_x \theta_0(x)].$$

## Sketch on the proof :

$$\xi_0(x, y) \in L^2(\Omega; W_a)$$

- $\xi_0 - \sqrt{a} \nabla_x u_0 \perp L^2(\Omega, W_a)$
- Using generalized Weyl's decomposition Lemma.

Lemma : (Generalised Weyl's decomposition.)

If  $\eta \in L^2(Y)$  and  $\eta \in W^\perp$ . Then there exists  $u_1 \in L^2(\Omega; H_{\#}^1)$  such that

$$\eta(y) = \sqrt{a(y)} \nabla_y u_1(x, y).$$

# The weak two-scale resolvent convergence

## Theorem 4

- (i) The strongly continuous contraction semigroups  $\mathbf{T}_\varepsilon(\cdot)$  associated with  $\mathbf{A}_\varepsilon$  weakly two-scale converge to the semigroup  $\mathbf{T}_0(\cdot)$  associated with  $\mathbf{A}_0$ , i.e. if  $f \in H$  then for all  $t \geq 0$ ,

$$\mathbf{T}_\varepsilon(t)f \xrightarrow{2} \mathbf{T}_0(t)f.$$

where

$$\mathbf{A}_0 = \begin{pmatrix} 0 & I & 0 \\ -\frac{1}{\langle c \rangle} A^{hom} & 0 & -\frac{\langle \gamma \rangle}{\langle c \rangle} I \\ 0 & \frac{\langle \gamma \rangle}{\langle d \rangle} I & -\frac{1}{\langle d \rangle} B^{hom} \end{pmatrix}.$$

$$A^{hom}w = -div(a^{hom}\nabla w), \quad B^{hom}w = -div(b^{hom}\nabla w)$$

$$a^{hom}\nabla w = \langle a^{\frac{1}{2}}\nabla\xi_0 \rangle \quad b^{hom}\nabla w = \langle a^{\frac{1}{2}}\nabla\chi_0 \rangle$$

(ii) For  $\varepsilon > 0$ , the family of Cauchy problems

$$\frac{dU_\varepsilon}{dt} = \mathbf{A}_\varepsilon U_\varepsilon, \quad U_\varepsilon(0) = \tilde{U}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\theta}_0)^T,$$

(weakly) two-scale converge to the limit Cauchy problem

$$\frac{dU_0}{dt} = \mathbf{A}_0 U_0, \quad U_0(0) = \tilde{U}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\theta}_0)^T,$$

## Conclusion and Open problems



$$(S) \left\{ \begin{array}{l} \frac{\partial^2 u_\varepsilon}{\partial t^2} = \operatorname{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon) - \gamma(\frac{x}{\varepsilon})\theta_\varepsilon \\ \frac{\partial \theta_\varepsilon}{\partial t} = \operatorname{div}(b(\frac{x}{\varepsilon})\nabla \theta_\varepsilon) + \gamma(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial t} \\ u_\varepsilon = 0 = \theta_\varepsilon \\ u_\varepsilon(x, 0) = \tilde{u}_0(x), \quad \frac{\partial u_\varepsilon}{\partial t}(x, 0) = \tilde{u}_1(x), \quad \theta_\varepsilon(x, 0) = \tilde{\theta}_0(x) \end{array} \right.$$



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- What about almost periodic case?

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- What about almost periodic case?
- What about non periodic case?



$$(S) \begin{cases} \frac{\partial^2 u_\varepsilon}{\partial t^2} = \operatorname{div}(a(\frac{x}{\varepsilon})\nabla u_\varepsilon) - \gamma(\frac{x}{\varepsilon})\theta_\varepsilon \\ \frac{\partial \theta_\varepsilon}{\partial t} = \operatorname{div}(b(\frac{x}{\varepsilon})\nabla \theta_\varepsilon) + \gamma(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial t} \\ u_\varepsilon = 0 = \theta_\varepsilon \\ u_\varepsilon(x, 0) = \tilde{u}_0(x), \frac{\partial u_\varepsilon}{\partial t}(x, 0) = \tilde{u}_1(x), \theta_\varepsilon(x, 0) = \tilde{\theta}_0(x) \end{cases}$$

- What about almost periodic case?
- What about non periodic case?
- Numerical schemes associated with respect to the step size  $h$  and  $\varepsilon$ ?







# Conclusion and Open problems



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- What about almost periodic case?
- What about non periodic case?
- Numerical schemes associated with respect to the step size  $h$  and  $\varepsilon$ ?
- Nonlinear case...

# References

-  **G. Allaire**, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., **23** (1992), 1482-1512.
-  **G. Nguetseng**, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), 608-623.
-  **I. V. Kamotski & V. P. Smyshlyaev**, *Two-scale homogenization for a general class of high contrast PDE systems with periodic coefficients*, *Applicable Analysis*, 2018.
-  **S. Cooper, M. Waurick**, *Fibre homogenisation*, *Journal of Functional Analysis*, **276**, 2019, p.3363–3405.
-  **S. Cooper, Savostianov, A.**, *Homogenisation with error estimates of attractors for damped semi-linear anisotropic wave equations*. *Advances in Nonlinear Analysis*, (2019) Volume 9, Issue 1.
-  **S. E. Pastukhova**, *On the Convergence of Hyperbolic Semigroups in Variable Hilbert Spaces*, *Journal of Mathematical Sciences*, **127**, 2005, 2263-2283.

-  **V. V. Zhikov**, *On Two-Scale Convergence*, Journal of Mathematical Sciences, **120**, 2004, 1328–1352.
-  **V. V. Zhikov and S. E. Pastukhova**, *On the Trotter–Kato Theorem in a Variable Space*, Functional Analysis and Its Applications, **41**, 2007, 264–270.
-  **A. Bensoussan, J. L. Lions, G. Papanicolaou**, *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.
-  **V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik**, *Homogenization of Differential Operators and Integral Functionals*, Springer, Berlin (1994).
-  **D. Cioranescu, P. Donato**, *An Introduction to Homogenization*, Oxford University Press, Oxford, 2000.
-  **E. Frénod**, *Two-Scale Approach to Oscillatory Singularly Perturbed Transport Equations*. Lecture Notes in Mathematics 2190, 124 p. 2017, Springer.
-  **Christian Seifert, Sascha Trostorff & Marcus Waurick**, 23rd Internet Seminar "Evolutionary Equations", 2020, 196 pages.

Thank you for your attention