Project D
Quantitative Homogenisation Theory

Presentation 1

Isem 23

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Dictionary : The study of the relationship between the elastic properties of a material and its temperature or between its thermal conductivity and its stresses.

Encyclopedia : The ability of a material to stretch in response to changes in temperature.

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Hetnarski R.B., Eslami M.R., Thermal stresses - advanced theory and applications, Springer, 2019.

W. Nowacki, Problems of thermoelasticity, Progress in Aerospace Sciences, Polskiej Akademie Nauk, Warszawa, 10, (1970), pp. 1–63.

V. D. KUPRADZE, Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, North-Holland, Amsterdam, 1979.

J. A. Walker, Dynamical Systems and Evolution Equations, ser. Mathematical Concepts and Methods in Science and Engineering.Plenum Press, 1980, vol. 20.

A. Day, Heat Conduction Within Linear Thermoelasticity. Springer-Verlag, 1985.

Hubble space Telescope launched into Earth orbit in 1990

"The transfer of energy between its mechanical form and heat generally has been ignored as a source of both structural damping and excitation in the vast literature on control of flexible structures. Only a few recent papers have considered control of thermoelastic structures. However, the thermally induced vibrations that hampered the recently launched Hubble space telescope have highlighted the coupling between mechanical vibration and heat transfer and the need to model and control thermoelastic phenomena in flexible structures."

J. S. GIBSON, I. G. ROSEN, AND G. TAO, Approximation in control of thermoelastic systems, SIAM J. Control. Optim., 30 (1992), pp. 1163-1189.

We are interested in the study of the following linear thermoelastic system on $\mathbb{R} \times \mathbb{R}^d$:

$$\begin{cases} \partial_t^2 u - \operatorname{div}(c\nabla u) + \gamma\theta = f \\ \partial_t \theta - \operatorname{div}(k\nabla\theta) - \gamma\partial_t u = q \end{cases} \tag{1}$$

- t : time on $\mathbb{R}$
- u : proportional to the displacement on $\mathbb{R} \times \mathbb{R}^d$
- θ : temperature on $\mathbb{R} \times \mathbb{R}^d$
- $\partial_t u$: first derivative of u with respect to t
- c : wave speed for a constant temperature state
- k : heat conductivity
- γ : the amount of thermal-mechanical coupling
- f : the external forces
- q : the heat flux

We shall study the thermoelastic system posed on $\mathbb{R}^d$ with rapidly varying periodic coefficients.

$\varepsilon > 0$
$c, \gamma, k : (0,1)^d \to \mathbb{R}$

$$\begin{cases} \partial_t^2 u_\varepsilon - \operatorname{div}\left(c\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon\right) + \gamma\left(\frac{\cdot}{\varepsilon}\right)\theta_\varepsilon = f \\ \partial_t \theta_\varepsilon - \operatorname{div}\left(k\left(\frac{\cdot}{\varepsilon}\right)\nabla \theta_\varepsilon\right) - \gamma\left(\frac{\cdot}{\varepsilon}\right)\partial_t u_\varepsilon = q \end{cases} \tag{2}$$

γ, k and c are bounded and positive.

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With

$$U = \begin{pmatrix} \partial_t u_\varepsilon \\ \theta_\varepsilon \\ c\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon \\ k\left(\frac{\cdot}{\varepsilon}\right)\nabla \theta_\varepsilon \end{pmatrix} \quad \textit{and} \quad F = \begin{pmatrix} f \\ q \\ 0 \\ 0 \end{pmatrix}$$

we can write equation 2 in the form of evolutionary equation.

$$(\partial_t M_\varepsilon + N_\varepsilon + A)\, U = F \tag{3}$$

$$M = \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{$\mathbf{0}$} \\ 0 & 1 & \\ \hline \multirow{2}{*}{$\mathbf{0}$} & & c^{-1}\left(\frac{\cdot}{\varepsilon}\right) \quad \\ & & \quad \mathbf{0} \end{array}\right), \quad N = \left(\begin{array}{cc|c} 0 & \gamma & \multirow{2}{*}{$\mathbf{0}$} \\ -\gamma & 0 & \\ \hline \multirow{2}{*}{$\mathbf{0}$} & & \mathbf{0} \quad \\ & & \quad k^{-1}\left(\frac{\cdot}{\varepsilon}\right) \end{array}\right)$$

$$A = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & -div & 0 \\ & & 0 & -div \\ \hline -grad & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 0 & -grad & & \end{array}\right)$$

grad denotes the gradient differential operator on function spaces of $L^2(\mathbb{R}^d)$ and $div = grad^*$.

$$A : \operatorname{dom} A \subseteq H = L^2 \times (L^2)^d \to H$$

$$A \textit{ is skew self adjoint}$$

We shall prove that M_ε is a material law.

$$M_\varepsilon(z) = M_\varepsilon + z^{-1} N_\varepsilon \tag{4}$$

$$N_\varepsilon = \left(\begin{array}{cc|c} 0 & \gamma & \multirow{2}{*}{$\mathbf{0}$} \\ -\gamma & 0 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \begin{matrix} \mathbf{0} & \\ & k^{-1}\left(\frac{\cdot}{\varepsilon}\right) \end{matrix} \end{array} \right)$$

$$\mathrm{dom}\, M_\varepsilon = \mathbb{C} \setminus \{0\} \quad \text{open and } \mathrm{M}_\varepsilon(\mathrm{z}) \text{ holomorphic on } \mathrm{dom}\, \mathrm{M}_\varepsilon$$

For assumptions on γ, c and k, M_ε and N_ε are bounded and $M_\varepsilon(\cdot)$ defines a material law with the abscissa of boundedness $s_b(M) = \inf_{\nu\in\mathbb{R}}\{\|M_\varepsilon(z)\|_{\infty,\mathbb{C}_{Rez>\nu}} \text{ finite}\} = 0$

Let's take $\nu_0 > 0$ and verify that the required real part condition of Picard's theorem holds i.e. $\exists c > 0$

$$Re\left(\langle \phi, zM(z)\phi\rangle_H\right) \geq c\|\phi\|_H^2 \tag{5}$$

for $\phi \in H$ and $z \in \mathbb{C}_{Rez>\nu_0}$

$$zM_\varepsilon(z) = zM_\varepsilon + N_\varepsilon$$
$$\langle \phi, zM_\varepsilon(z)\phi\rangle_H = \langle \phi, zM_\varepsilon\phi\rangle_H + \langle \phi, N_\varepsilon\phi\rangle_H$$

$$\begin{aligned}\langle \phi, zM_\varepsilon\phi\rangle_H &= z\,\langle \phi, M_\varepsilon\phi\rangle_H \\ &= z\left\langle \phi, \left(\phi_1, \phi_2, c^{-1}\phi_3, 0\right)\right\rangle_H \\ &= z\left(\|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2 + \left\langle \phi_3, c^{-1}\phi_3\right\rangle_H\right)\end{aligned}$$

$$
\begin{aligned}
Re\left(\langle \phi, zM_\varepsilon \phi \rangle_H\right) &= \frac{1}{2}\left(\langle \phi, zM_\varepsilon \phi \rangle_H + \overline{\langle \phi, zM_\varepsilon \phi \rangle}_H\right) \\
&= \frac{1}{2}\left(\langle \phi, zM_\varepsilon \phi \rangle_H + \left\langle \phi \overline{zM_\varepsilon}, \phi \right\rangle_H\right) \\
&= Re(z)\left(\|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2 + \left\langle \phi_3, c^{-1}\phi_3 \right\rangle_H\right) \quad (6) \\
&\geq \nu_0\left(\|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2 + \alpha\|\phi_3\|_{L^2}^2\right) \\
&\geq \nu_0 \ \min(1,\alpha) \ \sum_{i=1}^{3} \|\phi_i\|_{L^2}^2
\end{aligned}
$$

$$\langle \phi, N_\varepsilon \phi \rangle_H = \left\langle \phi, \left(\gamma\phi_2, -\gamma\phi_1, 0, k^{-1}\phi_4\right)\right\rangle_H$$
$$= \langle \phi_1, \gamma\phi_2 \rangle_H + \langle \phi_2, -\gamma\phi_1 \rangle_H + \left\langle \phi_4, k^{-1}\phi_4 \right\rangle_H$$

$$\begin{aligned} Re\left(\langle \phi, N_\varepsilon \phi \rangle_H\right) &= Re\left(2i\gamma Im\left(\langle \phi_1, \phi_2 \rangle_H\right) + \left\langle \phi_4, k^{-1}\phi_4 \right\rangle_H\right) \\ &= Re\left(\left\langle \phi_4, k^{-1}\phi_4 \right\rangle_H\right) \\ &\geq \beta \|\phi_4\|_{L^2}^2 \end{aligned} \tag{7}$$

$$
\begin{aligned}
Re\left(\langle \phi, zM_\varepsilon(z)\phi\rangle_H\right) &= \langle \phi, zM_\varepsilon \phi\rangle_H + \langle \phi, N_\varepsilon \phi\rangle_H \\
&\geq \min(1,\nu_0)\ \min(1,\alpha,\beta)\ \sum_{i=1}^{4} \|\phi_i\|_{L^2}^2 \\
&\geq c\|\phi\|_H^2
\end{aligned}
$$

$\rightarrow$ the real part condition of Picard's theorem holds.

For all $\varepsilon > 0$ and all $\nu \geq \nu_0$, the operator :

$$\partial_{t,\nu} M_\varepsilon + N_\varepsilon + A$$

is densely defined and closable in $L_{2,\nu}(\mathbb{R}, H)$.

The respective closure is continuously invertible with causal inverse being eventually independent of ν.

$$\overline{\partial_{t,\nu} M_\varepsilon + N_\varepsilon + A}^{\,-1} \in L(L_{2,\nu}(\mathbb{R}, H)) \tag{8}$$

For $\nu > 0$ we have

$$U \in \mathrm{dom}(\partial_{t,\nu} M_\varepsilon) \cap \mathrm{dom}(A)$$

and for a suitable $F \in H$ the thermoelastic system can be recovered from the evolutionary equation $(\partial_{t,\nu} M_\varepsilon + N_\varepsilon + A)U = F$:

$$U = \begin{pmatrix} \partial_t u_\varepsilon \\ \theta_\varepsilon \\ c\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon \\ k\left(\frac{\cdot}{\varepsilon}\right) \nabla \theta_\varepsilon \end{pmatrix} = \overline{\partial_{t,\nu} M_\varepsilon + N_\varepsilon + A}^{\,-1} \begin{pmatrix} \partial_{t,\nu}^{-1} f \\ q \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

We shall study the one space dimensional (1d) homogenisation of the thermoelastic system with rapidly varying periodic coefficients:

$$\begin{cases} \partial_t^2 u_\varepsilon - \partial_x \left(c\left(\frac{\cdot}{\varepsilon}\right)\partial_x u_\varepsilon \right) + \gamma\left(\frac{\cdot}{\varepsilon}\right)\theta_\varepsilon = f, \\ \partial_t \theta_\varepsilon - \partial_x \left(k\left(\frac{\cdot}{\varepsilon}\right)\partial_x \theta_\varepsilon \right) - \gamma\left(\frac{\cdot}{\varepsilon}\right)\partial_t u_\varepsilon = q, \end{cases} \quad t \in \mathbb{R}, x \in \mathbb{R}. \tag{10}$$

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Cooper, S., Waurick, M. (2019). Fibre homogenisation. Journal of Functional Analysis, 276(11), 3363-3405.

After change of spatial variables $y = \frac{x}{\varepsilon}$:

$$\tilde{u}_\varepsilon(t,y) = u_\varepsilon(t,\varepsilon y) \; and \; \tilde{\theta}_\varepsilon(t,y) = \theta_\varepsilon(t,\varepsilon y)$$

solves :

$$\begin{cases} \partial_t^2 \tilde{u}_\varepsilon - \varepsilon^{-2}\partial_y\left(c\partial_y \tilde{u}_\varepsilon\right) + \gamma\tilde{\theta}_\varepsilon = \tilde{f}_\varepsilon \\ \partial_t \tilde{\theta}_\varepsilon - \varepsilon^{-2}\partial_y\left(k\partial_y \tilde{\theta}_\varepsilon\right) - \gamma\partial_t \tilde{u}_\varepsilon = \tilde{q}_\varepsilon \end{cases} \quad t \in \mathbb{R}, y \in \mathbb{R} \qquad (11)$$

for $\tilde{f}_\varepsilon(t,y) = f(t,\varepsilon y) \; and \; \tilde{q}_\varepsilon(t,y) = q(t,\varepsilon y)$

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Using a 2^{nd} change of variables $U_\varepsilon = \begin{pmatrix} \partial_t \tilde{u}_\varepsilon \\ \tilde{\theta}_\varepsilon \\ \varepsilon^{-1}c\partial_y \tilde{u}_\varepsilon \\ \varepsilon^{-1}k\partial_y \tilde{\theta}_\varepsilon \end{pmatrix}$ we get the evolutionary equation :

$$\left(\partial_t M + N + \varepsilon^{-1}A\right) U_\varepsilon = F_\varepsilon$$

$$\left(\partial_t M + N + \varepsilon^{-1} A\right) U_\varepsilon = F_\varepsilon \tag{12}$$

with

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 0 & -\partial_y & 0 \\ 0 & 0 & 0 & -\partial_y \\ -\partial_y & 0 & 0 & 0 \\ 0 & -\partial_y & 0 & 0 \end{pmatrix}$$

∂_y is the derivative operator with respect to the spatial variable y.

$$\left(\partial_t M + N + \varepsilon^{-1} A\right) U_\varepsilon = F_\varepsilon$$

By the unitary Fourier-Laplace Transformation
$\mathcal{L}_v : L_{2,\nu}(\mathbb{R}; L^2(\mathbb{R})) \to L_2(\mathbb{R}; L^2(\mathbb{R}))$, we get for $\lambda = \nu + \mathrm{i}t, t \in \mathbb{R}$

$$(M_\lambda + \varepsilon^{-1} A) U_{\varepsilon,\lambda} = F_{\varepsilon,\lambda}, \quad y \in \mathbb{R} \tag{13}$$

with :

$$U_{\varepsilon,\lambda}(y) := \mathcal{L}_v U_\varepsilon(\lambda, y) \; and \; F_{\varepsilon,\lambda}(y) := \mathcal{L}_v F_\varepsilon(\lambda, y)$$

$$M_\lambda := \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k} \end{pmatrix} = \lambda M + N$$

Now we will reduce the problem from the whole space to a family of problems on the periodic cell $(0, 1)$

Let $\mathcal{U} := L^2(\mathbb{R}) \to L^2([-\pi, \pi) \times [0,1))$ be the unitary Floquet-Bloch-Gelfand transform. Then:

$$U_{\varepsilon,\theta,\lambda}(y) := \mathcal{U} U_{\varepsilon,\lambda}(\theta, y)$$

solves for *a.e.* $\theta \in [-\pi, \pi)$ the problem

$$(M_\lambda + \varepsilon^{-1} A_\theta) U_{\varepsilon,\theta,\lambda} = F_{\varepsilon,\theta,\lambda} \quad y \in [0,1) \tag{14}$$

where $F_{\varepsilon,\theta,\lambda}(y) := \mathcal{U} F_{\varepsilon,\lambda}(\theta, y)$ and

$$A_\theta = \begin{pmatrix} 0 & 0 & -\partial_\theta & 0 \\ 0 & 0 & 0 & -\partial_\theta \\ -\partial_\theta & 0 & 0 & 0 \\ 0 & -\partial_\theta & 0 & 0 \end{pmatrix}$$

∂_θ denotes the differential operator on θ-quasi-periodic functions, i.e. $D(\partial_\theta) = \{u \in H^1(0,1) \text{ where } u(1) = e^{i\theta} u(0)\}$

Now the GOAL has been reduced to studying the asymptotics of $U_{\varepsilon,\theta,\lambda}$ in ε uniformaly in θ, λ and R.H.S.

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Eigenvalues and vectors of A_θ satisfy

$$A_\theta U_n = \lambda_n U_n$$

for $U_n = (u_{1n}, u_{2n}, u_{3n}, u_{4n}) = B_n u_n(y)$ for some $B_n \in \mathbb{C}^4$.

$$\lambda_n^{\pm} = \pm i\,(\theta + 2\pi n) \quad n \in \mathbb{Z}$$

$$u_n^{\pm}(y) = e^{\lambda_n^{\pm} y}$$

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A_θ loses invertibility as θ goes to zero: $\lambda_0^\pm = \pm \mathrm{i}\theta$.

Considering :

$$U_{\varepsilon,\theta,\lambda}(y) = \sum_{n=-\infty}^{+\infty} b_n u_n(y) \quad \textit{for some } b_n \in \mathbb{C}^4$$

$$U_{\varepsilon,\theta,\lambda}(y) = b_{0\varepsilon} e^{i\theta y} + \sum_{\substack{-\infty \\ n\neq 0}}^{+\infty} b_n e^{\lambda_n y} = b_{0\varepsilon} e^{i\theta y} + w_{\varepsilon}$$

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From

$$(M_\lambda + \varepsilon^{-1} A_\theta) U_{\varepsilon,\theta,\lambda} = F_{\varepsilon,\theta,\lambda} \tag{15}$$

we see that w_ε satsifies :

$$\varepsilon^{-1} A_\theta w_\varepsilon = F_{\varepsilon,\theta,\lambda} - M_\lambda U_{\varepsilon,\theta,\lambda} - \varepsilon^{-1} A_\theta M_\lambda U_{\varepsilon,\theta,\lambda} \tag{16}$$

We shall prove (if time) that

$$\|w_\epsilon\|_{L^2(0,1)} \leq \frac{\varepsilon}{2\pi}\left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)\|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)} \qquad (17)$$

for all $\theta \in (-\pi, \pi)$ and $\lambda \in \{it + \nu : t \in \mathbb{R}\}$.

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that is

$$\|U_{\varepsilon,\theta,\lambda} - b_{0\varepsilon}e^{i\theta y}\|_{L^2(0,1)} \;\leq \frac{\varepsilon}{2\pi}\left(1+\frac{\|M_\lambda\|}{Re(M_\lambda)}\right)\|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)}$$

for all $\theta \in (-\pi,\pi)$ and $\lambda \in \{it+\nu : t \in \mathbb{R}\}$.

Now $b_{0\varepsilon} \in \mathbb{C}^4$ satisfies :

$$M_\lambda b_{0\varepsilon} e^{i\theta y} + \varepsilon^{-1} A_\theta b_{0\varepsilon} e^{i\theta y} = F_{\varepsilon,\theta,\lambda} - M_\lambda w_\varepsilon - \varepsilon^{-1} A_\theta w_\varepsilon \qquad (18)$$

We multiply both sides by $e^{i\theta y}$ and take in $L^2(0,1)$ the inner product to find (since $A_\theta w_\varepsilon \perp e^{i\theta y}$) :

$$\left\langle M_\lambda b_{0\varepsilon} e^{i\theta y}, e^{i\theta y} \right\rangle + \varepsilon^{-1} \left\langle A_\theta b_{0\varepsilon} e^{i\theta y}, e^{i\theta y} \right\rangle = \left\langle F_{\varepsilon,\theta,\lambda}, e^{i\theta y} \right\rangle - \left\langle M_\lambda w_\varepsilon, e^{i\theta y} \right\rangle$$

That is $b_{0\varepsilon}$ is the solution to the system :

$$(\tilde{M}_\lambda + \varepsilon^{-1}\tilde{A}_\theta)b_{0\varepsilon} = \left\langle F_{\varepsilon,\theta,\lambda}, e^{i\theta y} \right\rangle - \left\langle M_\lambda w_\varepsilon, e^{i\theta y} \right\rangle \qquad (19)$$

$$\tilde{M}_\lambda = \begin{pmatrix} \lambda & -\langle\gamma\rangle & 0 & 0 \\ \langle\gamma\rangle & \lambda & 0 & 0 \\ 0 & 0 & \left\langle c^{-1} \right\rangle & 0 \\ 0 & 0 & 0 & \left\langle k^{-1} \right\rangle \end{pmatrix}, \qquad \langle f \rangle = \int_0^1 f(y)dy$$

$$\tilde{A}_\theta = \begin{pmatrix} 0 & 0 & -i\theta & 0 \\ 0 & 0 & 0 & -i\theta \\ -i\theta & 0 & 0 & 0 \\ 0 & -i\theta & 0 & 0 \end{pmatrix}$$

Let $c_{0\varepsilon}$ solve :

$$(\tilde{M}_\lambda + \varepsilon^{-1}\tilde{A}_\theta)c_{0\varepsilon} = \left\langle F_{\varepsilon,\theta,\lambda}, e^{i\theta y} \right\rangle \tag{20}$$

which is well-posed as the 4×4 matrix $\tilde{M}_\lambda + \varepsilon^{-1}\tilde{A}_\theta$ is invertible.

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We show the inequality :

$$|b_{0\varepsilon} - c_{0\varepsilon}| \leq \frac{\varepsilon}{2\pi}\left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)^2 \|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)} \tag{21}$$

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Theorem

$$\|U_{\varepsilon,\theta,\lambda} - c_{0\varepsilon}e^{i\theta y}\|_{L^2(0,1)} \leq \frac{\varepsilon}{2\pi}\left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)^2 \|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)} \tag{22}$$

for all $\theta \in [-\pi, \pi)$ and $\lambda \in \{it + \nu : t \in \mathbb{R}\}$.

Comparaison to the standard homogenisation

$$S_0 \;:\; \begin{cases} \partial_t^2 u_0 - \partial_x\left(c_0 \partial_x u_0\right) + \gamma_0 \theta_0 = f \\ \partial_t \theta_0 - \partial_x\left(k_0 \partial_x \theta_0\right) - \gamma_0 \partial_t u_0 = q \end{cases} \tag{23}$$

S_0 verifies all the assumptions before for : $c_0 = \left\langle c^{-1} \right\rangle^{-1}$, $k_0 = \left\langle k^{-1} \right\rangle^{-1}$ and $\gamma_0 = \langle \gamma \rangle$.

$$(\tilde{M}_\lambda + \varepsilon^{-1} A_\theta) U^{hom}_{\varepsilon,\theta,\lambda} = F_{\varepsilon,\theta,\lambda} \tag{24}$$

$$\tilde{M}_\lambda = \begin{pmatrix} \lambda & -\langle \gamma \rangle & 0 & 0 \\ \langle \gamma \rangle & \lambda & 0 & 0 \\ 0 & 0 & \left\langle c^{-1} \right\rangle & 0 \\ 0 & 0 & 0 & \left\langle k^{-1} \right\rangle \end{pmatrix}$$

$$(\tilde{M}_\lambda + \varepsilon^{-1} A_\theta) \left\langle U^{hom}_{\varepsilon,\theta,\lambda}, e^{i\theta y} \right\rangle = \left\langle F_{\varepsilon,\theta,\lambda}, e^{i\theta y} \right\rangle \tag{25}$$

Using our theorem, we get :

$$\| U^{hom}_{\varepsilon,\theta,\lambda} - c_{0\varepsilon} e^{i\theta y} \|_{L^2(0,1)} \leq \frac{\varepsilon}{2\pi} \left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)^2 \|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)} \quad (26)$$

remember we already have :

$$\| U_{\varepsilon,\theta,\lambda} - c_{0\varepsilon} e^{i\theta y} \|_{L^2(0,1)} \leq \frac{\varepsilon}{2\pi} \left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)^2 \|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)}$$

by triangle inequality we get :

Theorem

$$\| U_{\varepsilon,\theta,\lambda} - U^{hom}_{\varepsilon,\theta,\lambda} \|_{L^2(0,1)} \leq \frac{\varepsilon}{\pi} \left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)^2 \|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)}$$

for all $\theta \in [-\pi, \pi)$, $\lambda \in \{it + \nu : t \in \mathbb{R}\}$.

Applying inverse Gelfand transform, inverse Fourier-Laplace transform, inverse change of special variables and noting that :

$$\left(1+\frac{\|M_\lambda\|}{Re(M_\lambda)}\right) \leq C(1+|\lambda|)$$

$$\|U_\varepsilon - U_\varepsilon^{hom}\|_{L_{2,\nu}(\mathbb{R},H)} \leq c\varepsilon \|\mathcal{L}_\nu^*(1+|\lambda|)^2 \mathcal{L}_\nu F\|_{L_{2,\nu}(\mathbb{R},H)} \quad c>0 \quad (27)$$

U_ε and U_ε^{hom} solve respectively :

$$(\partial_t M_\varepsilon + N_\varepsilon + A)\, U_\varepsilon = F \quad (28)$$

$$\left(\partial_t \tilde{M} + \tilde{N} + A\right) U_\varepsilon^{hom} = F \quad (29)$$

Improvements of the final estimate

$$\|U_\varepsilon - U_\varepsilon^{hom}\|_{L_{2,\nu}(\mathbb{R},H)} \leq c\varepsilon \|\mathcal{L}_\nu^* (1+|\lambda|)^2 \mathcal{L}_\nu F\|_{L_{2,\nu}(\mathbb{R},H)} \qquad c > 0$$

- For wave equation the term in red is improved to $1 + |\lambda|$
- For heat equation the term in red is improved to 1
- For the coupled thermoelastic system we expect something in between.

Thank you for your attention !

Proof of inequality (17):

$$\|w_\epsilon\|_{L^2(0,1)} \leq \frac{\varepsilon}{2\pi}\left(1+\frac{\|M_\lambda\|}{Re(M_\lambda)}\right)\|F_{\varepsilon,\theta,\lambda}\|_{L^2(0,1)}$$

$$\varepsilon^{-1}A_\theta w_\varepsilon = F_{\varepsilon,\theta,\lambda} - M_\lambda U_{\varepsilon,\theta,\lambda} - \varepsilon^{-1}A_\theta b_{0\varepsilon}e^{i\theta y} \tag{30}$$

We multiply both sides by $A_\theta w_\epsilon$ and take in $L^2(0,1)$ the inner product to find :

$$\begin{aligned}\varepsilon^{-1}\langle A_\theta w_\varepsilon, A_\theta w_\varepsilon\rangle &= \langle F_{\varepsilon,\theta,\lambda}, A_\theta w_\varepsilon\rangle - \langle M_\lambda U_{\varepsilon,\theta,\lambda}, A_\theta w_\varepsilon\rangle \\ &\quad - \left\langle \varepsilon^{-1}A_\theta b_0 e^{i\theta y}, A_\theta w_\varepsilon\right\rangle \\ &= \langle F_{\varepsilon,\theta,\lambda}, A_\theta w_\varepsilon\rangle - \langle M_\lambda U_{\varepsilon,\theta,\lambda}, A_\theta w_\varepsilon\rangle\end{aligned}$$

$$\varepsilon^{-1}\|A_\theta w_\varepsilon\|^2 \leq \|F_{\varepsilon,\theta,\lambda} - M_\lambda U_{\varepsilon,\theta,\lambda}\|\|A_\theta w_\varepsilon\| \tag{31}$$

$$\|A_\theta w_\varepsilon\| \leq \varepsilon\|F_{\varepsilon,\theta,\lambda} - M_\lambda U_{\varepsilon,\theta,\lambda}\| \tag{32}$$

Knowing that $\|A_\theta w_\varepsilon\| = \|w'_\varepsilon\|$ and by Poincaré inequality we get :

$$\|w_\varepsilon\| \leq \frac{1}{|\ \theta + 2\pi\ |}\|w'_\varepsilon\| \leq \frac{1}{2\pi}\|w'_\varepsilon\|$$

$$\begin{aligned}\|w_\varepsilon\| &\leq \frac{\varepsilon}{2\pi}\|F_{\varepsilon,\theta,\lambda} - M_\lambda U_{\varepsilon,\theta,\lambda}\| \\ &\leq \frac{\varepsilon}{2\pi}\left(\|F_{\varepsilon,\theta,\lambda}\| + \|M_\lambda U_{\varepsilon,\theta,\lambda}\|\right) \\ &\leq \frac{\varepsilon}{2\pi}\left(\|F_{\varepsilon,\theta,\lambda}\| + \|M_\lambda\|\|U_{\varepsilon,\theta,\lambda}\|\right)\end{aligned} \tag{33}$$

Besides, we have :

$$\Re\left(\left\langle M_\lambda U_{\varepsilon,\theta,\lambda} + \varepsilon^{-1} A_\theta U_{\varepsilon,\theta,\lambda}, U_{\varepsilon,\theta,\lambda}\right\rangle\right) = \Re(\langle F_{\varepsilon,\theta,\lambda}, U_{\varepsilon,\theta,\lambda}\rangle) \quad (34)$$

Since A_θ is self skewadjoint, $\Re(\left\langle \varepsilon^{-1} A_\theta U_{\varepsilon,\theta,\lambda}, U_{\varepsilon,\theta,\lambda}\right\rangle) = 0$ and :

$$\Re\left\langle M_\lambda U_{\varepsilon,\theta,\lambda}, U_{\varepsilon,\theta,\lambda}\right\rangle = \Re\left\langle F_{\varepsilon,\theta,\lambda}, U_{\varepsilon,\theta,\lambda}\right\rangle \quad (35)$$

we can prove that :

$$\|U_{\varepsilon,\theta,\lambda}\| \leq \frac{\|F_{\varepsilon,\theta,\lambda}\|}{Re(M_\lambda)} \quad (36)$$

and then :

$$\|w_\varepsilon\| \ \leq \frac{\varepsilon}{2\pi}\left(1 + \frac{\|M_\lambda\|}{Re(M_\lambda)}\right)\|F_{\varepsilon,\theta,\lambda}\| \quad (37)$$